Logic and Computation II

Part 6. Recursion-theoretic hierarchies

Kazuyuki Tanaka

BIMSA

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北京雁栖湖应用数学研究院
YANQI LAKE BEIJING INSTITUTE OF MATHEMATICAL SCIENCES AND APPLICATIONS

Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 7. Schedule

- May 18, (1) KP set theory I
- May 23, (2) KP set theory II
- May 25, (3) KP set theory III
- May 30, (4) KP set theory IV and α recursion theory
- Jun. 1, (5) Recursively large ordinals I
- Jun. 6, (6) Recursively large ordinals II and second order arithmetic

Today's topics

1. Recap
2. Stable ordinals
3. Projectible ordinals
4. Admissible ordinals and second-order arithmetic

Recap

$$
\begin{aligned}
\mathsf{KP} \quad &:= \text{axioms of extensionality, pairing, union, empty set} \\
&+ \ \Delta_0\text{-Sep, or } \Delta_1\text{-Sep}: \quad \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z)). \\
&+ \ \Delta_0\text{-Coll, or } \Sigma_1\text{-Coll}: \quad \forall x (\forall y \in x\, \exists z \varphi(z) \to \exists u \forall y \in x \exists z \in u \varphi(z)). \\
&+ \ \text{foundation}: \quad \forall x [\forall y \in x\, \varphi(y) \to \varphi(x)] \to \forall x \varphi(x). \\
\mathsf{KP}\omega &:= \mathsf{KP} + \text{axiom of infinity}: \quad \exists x \{0 \in x \wedge \forall y \in x (y \cup \{y\} \in x)\}.
\end{aligned}
$$

Definition (Constructible sets)

A Σ_1 operator L_α on the ordinals is defined as follows.

$$
\begin{cases}
\mathrm{L}_0 := \varnothing \\
\mathrm{L}_{\alpha+1} := \mathrm{Def}(\mathrm{L}_\alpha) \\
\mathrm{L}_\alpha := \bigcup_{\beta<\alpha} \mathrm{L}_\beta \quad (\alpha \text{ is a limit ordinal})
\end{cases}
$$

Let L denote a Σ_1 class $\bigcup_{\alpha \in \mathrm{Ord}} \mathrm{L}_\alpha$. The elements of L are called **constructible sets**.

Definition

An ordinal α is said to be **admissible** if $\mathrm{L}_\alpha \models$ KP holds, i.e., $\mathrm{L}_\alpha \models \Delta_0\text{-Coll}$

Definition

For an admissible ordinal α,

(2) $A \subset \alpha$ is **α-recursively enumerable**(α-RE) $\Leftrightarrow$ A is $\Sigma_1(\mathrm{L}_\alpha)$,

(4) $f : \alpha \to \alpha$ is **α-recursive** $\Leftrightarrow$ the graph of f is $\Delta_1(\mathrm{L}_\alpha)$.

Lemma

α is admissible $\Leftrightarrow$ there is no cofinal (unbounded) $\Delta_1(\mathrm{L}_\alpha)$ function from $\beta < \alpha$ to α.

(1) Ordinal α is **recursively inaccessible** $\Leftrightarrow$ α is admissible and is a limit of admissibles (For any $\beta < \alpha$, there exists an admissible ordinal γ such that $\beta < \gamma < \alpha$).

(2) Ordinal α is **recursively Mahlo** $\Leftrightarrow$ α is admissible and for any α-recursive function $f : \alpha \to \alpha$, there exists an admissible $\beta < \alpha$ such that $\forall \gamma < \beta\ f(\gamma) < \beta$.

Definition (Reflecting ordinals)

For a set Γ of formulas, α is called Γ-**reflecting** if for any $\varphi \in \Gamma$ with parameters in L_α,

$$\mathrm{L}_\alpha \models \varphi \Rightarrow \exists \beta < \alpha \ \mathrm{L}_\beta \models \varphi.$$

- Ordinal α is Σ_{n+1}-reflecting $\Leftrightarrow$ α is Π_n-reflecting.
- For each $n \geq 1$, there exists a Π_{n+1} sentence θ_n such that for any limit ordinal α,

$$\alpha \text{ is } \Pi_n\text{-reflecting} \iff \mathrm{L}_\alpha \models \theta_n.$$

Theorem

- $\alpha (> \omega)$ is admissible $\Leftrightarrow$ α is Π_2-reflecting.
- A Π_3-reflecting ordinal is recursively Mahlo and so recursively inaccessible.

smallest Π_3-reflecting $>$ smallest recursively Mahlo $>$ smallest recursively inaccess. $> \omega_1^{\mathrm{CK}}$.

$\mathcal{P}(\omega)$ denotes the set of all subsets of ω.

Definition (Inductive definition)

- Given an **operator** $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, we define a transfinite increasing sequence $\{\Gamma^\alpha : \alpha \in \mathrm{Ord}\}$ of subsets of ω by $\Gamma^\alpha = \bigcup\{\Gamma(\Gamma^\beta) : \beta < \alpha\}$.
- Then, write $|\Gamma|$ for the first ordinal α such that $\Gamma^\alpha = \Gamma^{\alpha+1}$, which is called the **closure ordinal** of operator Γ.
- $\Gamma^{|\Gamma|}$, also denote Γ^∞, is the set determined by **inductive definition** of Γ.

- An operator Γ is said to be **monotone**, if for any $X \subset Y \subset \omega$, $\Gamma(X) \subset \Gamma(Y)$.
- For a monotone Γ, $\Gamma^\infty = \bigcap\{X : \Gamma(X) \subset X\}$.
- An operator Γ is Σ^i_n (or Π^i_n) if $\{(x, X) \in \omega \times \mathcal{P}(\omega) : x \in \Gamma(X)\}$ is Σ^i_n (or Π^i_n).
- $|\Sigma^i_n| = \sup\{|\Gamma| : \Gamma \in \Sigma^i_n\}$ and $|\mathrm{mon}\Sigma^i_n| = \sup\{|\Gamma| : \Gamma \in \Sigma^i_n$ and monotone$\}$.
- $|\Pi^i_n|$ and $|\mathrm{mon}\Pi^i_n|$ can be defined similarly.

- There is a universal Σ^i_n formula $\varphi(e, x, X)$, hence also a universal Σ^i_n operator Γ. Thus, $|\Sigma^i_n| = |\Gamma|$. Similarly for Π^i_n.

Lemma

Let Γ be universal Π^0_n $(n > 0)$ and $\alpha = |\Gamma|$. For any Π^0_n formula $\varphi(X)$, if $\varphi(\Gamma^\infty)$ then $\exists\beta < \alpha\varphi(\Gamma^\beta)$ holds.

Lemma (revisited)

There is a primitive recursive bijection $F : \mathrm{Ord} \to \mathrm{L}$ such that if α is ω or an ε number then $F``\alpha = \mathrm{L}_\alpha$.

- For admissible $\alpha = |\Gamma|$, it is even easier to construct a α-recursive bijection $G : \alpha \to \Gamma^\alpha$ such that $G``\beta = \Gamma^\beta$ for any limit ordinal $\beta < \alpha$.
- Thus, $H = F \circ G^{-1}$ is an α-recursive bijection from Γ^α to L_α such that for an ε number $\beta < \alpha$, $H``\Gamma^\beta = \mathrm{L}_\beta$. Moreover, a relation $m \mathrel{\tilde{\in}} l$ defined by $\mathrm{L}_\beta \models H(m) \in H(l)$ is recursive in Γ^β.

Theorem

For any $n > 0$, $|\Pi^0_n|$ is the smallest Π_{n+1}-reflecting ordinal.

Proof Sketch.

- We only consider the case $n = 2$. Other cases can be treated similarly.
- Let Γ be a universal Π^0_2 operator. We may assume that $|\Gamma|$ is admissible, denoted as α.
- As already mentioned, there exists an α-recursive bijection $H : \Gamma^\alpha \to \mathrm{L}_\alpha$.
- Then $\Gamma^\alpha \notin \mathrm{L}_\alpha$, and Γ^α is $\Sigma_1(\mathrm{L}_\alpha)$.
- Moreover, Γ^α is m-complete. That is, any $\Sigma_1(\mathrm{L}_\alpha)$ set of natural numbers is m-reducible to Γ^α.
  $\because$ Let $\varphi(n)$ be a Σ_1 formula. Then, there exists an ε number $\beta < \alpha$ such that $\mathrm{L}_\beta \models \varphi(n)$ for all n such that $\mathrm{L}_\alpha \models \varphi(n)$. Also, we have $H``\Gamma^\beta = \mathrm{L}_\beta$. Since $\mathrm{L}_\beta \models H(m) \in H(l)$ is recursive in Γ^β, a $\Sigma_1(\mathrm{L}_\beta)$ set is arithmetic in Γ^β and so m-reducible to Γ^α.

- Now, for a Σ_1 formula $\exists w \neg\psi(u,v,w)$, where $\psi(u,v,w)$ is a Δ_0 formula with parameters in L_α, there exists a recursive function $g : \omega \times \omega \to \omega$ such that for every $m, n \in \omega$

$$g(m,n) \in \Gamma^\alpha \Leftrightarrow m, n \in \Gamma^\alpha \wedge \mathrm{L}_\alpha \models \exists w \neg\psi(H(m), H(n), w).$$

- Suppose $\mathrm{L}_\alpha \models \forall u \exists v \forall w \ \psi(u,v,w)$. That is,

$$\forall m \in \Gamma^\alpha \exists n \in \Gamma^\alpha g(m,n) \notin \Gamma^\alpha.$$

- Since the above is in the form $\Pi^0_2(\Gamma^\alpha)$, by the last lemma there exists a $\beta < \alpha$ such that

$$\forall m \in \Gamma^\beta \exists n \in \Gamma^\beta g(m,n) \notin \Gamma^\beta.$$

  We may assume that β is a ε number, by adding some conditions to the formula.

- Then, by using H again, we get

$$\mathrm{L}_\beta \models \forall v \exists v \forall w \ \psi(u,v,w).$$

- Thus, α is a Π_3-reflecting ordinal.

- Finally, for a contradiction, we assume that there exists a Π_3-reflecting ordinal β below α.
- Since $\beta < \alpha$, there exists $x \in \Gamma(\Gamma^\beta) - \Gamma^\beta$. Since Γ is Π^0_2, there is a recursive R s.t.,

$$x \in \Gamma(\Gamma^\beta) \Leftrightarrow \forall m \exists n R(m, n, \Gamma^\beta).$$

- Now we consider how to express $\forall m \exists n R(m, n, \Gamma^\beta)$ in L_β.
- Since Γ^β is $\Sigma_1(\mathrm{L}_\beta)$, $R(m, n, \Gamma^\beta)$ is $\Delta_2(\mathrm{L}_\beta)$.
- Although m, n in $R(m, n, \Gamma^\beta)$ range over natural numbers, they turn into set variables in the corresponding $\Delta_2(\mathrm{L}_\beta)$ formula.
- Thus, the interpretation of $\forall m \exists n R(m, n, \Gamma^\beta)$ over L_β is a Π_3 formula.
- For the sake of convenience, if we express this with the same formula, by the Π_3-reflexivity, there is a $\gamma < \beta(\mathrm{L}_\gamma \models \forall m \exists n R(m, n, \Gamma^\gamma)$.
- Therefore, $x \in \Gamma(\Gamma^\gamma) \subset \Gamma^\beta$, which contradicts with the choice of x.
- Thus α is the smallest Π_3-reflecting ordinal. $\square$

- Using the hierarchy of second-order set theory, we extend the theorem as follows.

Theorem

$|\Pi^1_1|$ is the smallest Π^1_1 reflecting ordinal, and $|\Sigma^1_1|$ is the smallest Σ^1_1 reflecting ordinal.

- Note that the hierarchy of second-order set theory is represented by the same symbol as the analytical hierarchy of second-order arithmetic.
- We omit the details.

There are many other ways to characterize large ordinals.

Definition (Stability)

An ordinal α is β-**stable** if $\alpha < \beta$ and $\mathrm{L}_\alpha \prec_1 \mathrm{L}_\beta$.
Here, $\mathrm{L}_\alpha \prec_1 \mathrm{L}_\beta$ means that if a Σ_1 formula (with parameters in L_α) holds in L_β, it also holds in L_α. The converse is trivial.

Show the following.

(1) α is admissible if α is β-stable for some β.

(2) If α is β-stable and β is admissible, then α is recursively inaccessible.

Theorem

For a countable ordinal α, the following are equivalent.

(1) α is Π_n-reflecting for all n

(2) α is $(\alpha + 1)$-stable.

Proof sketch.

(1) $\implies$ (2)

- Suppose α is Π_n-reflecting for all n.
- Now, let $\exists x \varphi(x)$ be a Σ_1 formula, where $\varphi(x) \in \Delta_0$ with parameters in L_α.
- First, an atomic formula $x \in v$ appearing in $\varphi(x)$ is replaced by the equivalent Δ_0 formula $\exists y \in v(\forall z(z \in y \leftrightarrow z \in x))$.
- Then all atomic formulas involving x are only of the form $u \in x$. Without loss of generality, assume $\varphi(x)$ is such a Δ_0 formula.

- Assume $\mathrm{L}_{\alpha+1} \models \exists x \varphi(x)$. There exists $a \in \mathrm{L}_{\alpha+1}$ and $\mathrm{L}_{\alpha+1} \models \varphi(a)$.
- Then there exists a formula $\theta(x)$ such that $a = \{b : \mathrm{L}_\alpha \models \theta(b)\}$.
- If $\varphi(\theta)$ is the formula obtained from $\varphi(a)$ by replacing $u \in a$ by $\theta(u)$, then $\mathrm{L}_\alpha \models \varphi(\theta)$ by induction on the construction of Δ_0 formula $\varphi(x)$.
- Since α is reflecting, there exists $\beta < \alpha$ and $\mathrm{L}_\beta \models \varphi(\theta)$.
- Now, if we set $a' = \{b : \mathrm{L}_\beta \models \theta(b)\}$, again by induction on the construction of $\varphi(x)$ $\mathrm{L}_{\beta+1} \models \varphi(a')$ and so $\mathrm{L}_\alpha \models \varphi(a')$.
- Thus $\mathrm{L}_\alpha \models \exists x \varphi(x)$, and hence α is $(\alpha+1)$-stable.

(2) $\implies$ (1)

- If α is $(\alpha+1)$-stable, then α is admissible.
- Now, let ψ be a Π_n formula, and assume $\mathrm{L}_\alpha \models \psi$.
- Then, the proof roughly goes as follows. We have $\mathrm{L}_{\alpha+1} \models \exists\beta\psi^{\mathrm{L}_\beta}$, and so by stability, $\mathrm{L}_\alpha \models \exists\beta\psi^{\mathrm{L}_\beta}$ and thus $\mathrm{L}_\beta \models \psi$.
- The problem here is that since $\mathrm{L}_{\alpha+1}$ is not a model of KP, L_β can not be defined as a Σ_1 operator.
- So instead of using L_β, we state that there exists a transitive model W of KP that also satisfies a Π_2 condition $V = L$. We omit the details.
- Thus, α is Π_n-reflecting for all n. □

Among various characterizations of second-order reflecting properties, the following theorem is particularly elegant. We state it without proof.

Theorem

A countable ordinal α is Π^1_1-reflecting iff it is α^+ stable, where α^+ is the next admissible ordinal after α.

- "Σ^1_1-reflecting" requires a stronger stability condition.
- If $\alpha^+ + 1$ is stable, it is Σ^1_1-reflecting, but the converse is not true.
- The smallest Π^1_1-reflecting ordinal is less than the smallest Σ^1_1-reflecting ordinal, *i.e.*, $|\Pi^1_1| < |\Sigma^1_1|$.

Finally, we introduce another important notion on ordinals.

Definition (Projectability)

- An admissible ordinal α is **projectible** onto β if there is an α-recursive injection from α to β.
- The smallest ordinal β onto which α is projectible is called the **projectum** of α, denoted by α^*.
- α is called **projectible** if $\alpha^* < \alpha$.
- α is called **non-projectible** if $\alpha^* = \alpha$.

Theorem

An admissible ordinal α is not projectible $\Leftrightarrow \mathrm{L}_\alpha \models \Sigma_1{-}\mathrm{Sep}$.

Proof sketch $(\Rightarrow)$

- Let α be a non-projectible admissible ordinal.
- In L_α, to show Σ_1-Sep, we arbitrarily choose $a \in \mathrm{L}_\alpha$ and $\varphi(x) \in \Sigma_1(\mathrm{L}_\alpha)$. Then we want to show $A = \{u \in a : \mathrm{L}_\alpha \models \varphi(u)\} \in \mathrm{L}_\alpha$.
- Since there is an α-recursive bijection between L_α and α, by using Σ_1 recursion, we can enumerates the elements of $\Sigma_1(\mathrm{L}_\alpha)$ set A by ordinals $(< \alpha)$.
- If this enumeration exhausts α in the middle, it conflicts with the non-projectiveness since $a \in \mathrm{L}_\alpha$ can be enumerated by ordinals smaller than α.
- If A is enumerated by an ordinal β smaller than α, then $\beta \in \mathrm{L}_\alpha$ and there is an α-recursive bijection between A and β, which implies $A \in \mathrm{L}_\alpha$.

($\Leftarrow$)

- For contraposition, suppose α is a projectible admissible ordinal.
- Then, there is $\beta < \alpha$ and an α-recursive injection F from α to β.
- Since $F``\alpha \subset \beta$ is $\Sigma_1(\mathrm{L}_\alpha)$,

$$\mathrm{L}_\alpha \models \Sigma_1\text{-Sep} \implies F``\alpha \in \mathrm{L}_\alpha.$$

- Since F is a α-recursive injection from α to $F``\alpha$, we have $\alpha \in \mathrm{L}_\alpha$, which is a contradiction. $\square$

All projectible ordinals smaller than the first non-projectible ordinal are projectible to ω. This is a crucial condition in order to develop α recursion theory. For more details, please refer to the following books.

Further reading

- G.E. Sacks, *Higher Recursion Theory*, Springer 1990.
- C.T. Chong and L. Yu, *Recursion Theory: Computational Aspects of Definability*, De Gruyter 2015.

Admissible ordinals and the subsystems of second-order arithmetic

Recall:

Definition (The system of Recursive Comprehension Axioms)

RCA_0 consists of the following axioms.
(1) Basic Axioms of Arithmetic: Same as $\mathsf{Q}_<$.
(2) Δ^0_1 comprehension axiom (Δ^0_1-CA): for any $\varphi(x) \in \Sigma^0_1$ and $\psi(x) \in \Pi^0_1$,

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x)).$$

(3) Σ^0_1 induction: for any $\varphi(x) \in \Sigma^0_1$, $\varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$.

Definition (Subsystems of Second Order Arithmetic)

Γ-CA_0 is obtained from RCA_0 by adding $\exists X \forall x(x \in X \leftrightarrow \varphi(x))$, for any $\varphi(x) \in \Gamma$.

The correspondence between the property of admissible ordinal α and a subsystem of second order arithmetic for $\mathrm{L}_\alpha \cap \mathcal{P}(\omega)$ is summarized in the following table.

ordinal α	admissible	limit of admissibles	recursively inaccessible	non-projective
Second order arithmetic T	Δ^1_1-CA_0	Π^1_1-CA_0	Δ^1_2-CA_0	Π^1_2- CA_0

These relationships were already described in Kripke's UCLA lecture notes [Kri][i] in 1967.

[i]S. Kripke, "Transfinite Recursion, Constructible Sets, and Analogues of Cardinals", *Summaries of Talks Prepared in Connection with the Summer Institute on Axiomatic Set Theory, U.C.L.A.*, American Mathematical Society 1967(https://saulkripkecenter.org)

the smallest Σ_2 admissible ordinal ($L_\alpha \models \Delta_2$-Sep)

the smallest non-projective admissible ordinal ($L_\alpha \models \Sigma_1$-Sep)

$|\Sigma^1_1| = |\text{mon } \Sigma^1_1|$ = the smallest Σ^1_1-reflecting

$|\Pi^1_1|$ = the smallest Π^1_1-reflecting

$|\Sigma^1_0|$ = the smallest Σ^1_0-reflecting = $(+1)$ stable

$|\Pi^0_2| = |\Sigma^0_3|$ = the smallest Π_3-reflecting

the smallest recursively Mahlo

the smallest recursively inaccessible

$\omega_1^{\text{CK}} = |\Pi^0_1| = |\Sigma^0_2| = |\text{mon}\Pi^0_1| = |\text{mon}\Pi^1_1|$ =the smallest Π_2-reflecting

$\omega \quad = |\Sigma^0_1| = |\text{mon}\Sigma^0_1|$

0

Reverse Mathematics Program

H. Friedman, S. Simpson, etc

Reverse Mathematics

Which axioms are needed to prove a theorem?

Big Five subsystems in order of increasing strength:

RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0

- $\mathsf{WKL}_0 = \mathsf{RCA}_0 + \overbrace{\text{any infinite binary tree has an infinite path}}^{\text{Weak König Lemma}}$
  $= \mathsf{RCA}_0 + \Sigma^0_1$-SP

Σ^0_1-SP (Σ^0_1 *separation*):

$$\neg\exists x(\varphi_0(x) \wedge \varphi_1(x)) \to \exists X \forall x((\varphi_0(x) \to x \in X) \wedge (\varphi_1(x) \to x \notin X)),$$

where $\varphi_0(x)$ and $\varphi_1(x)$ are Σ^0_1 formulas.

- ACA$_0$ $= \mathsf{RCA}_0 + \overbrace{\exists X \forall n \big(n \in X \leftrightarrow \varphi(n)\big) \text{ for all arithmetical } \varphi(n)}^{\text{Arithmetical Comprehension}}$
  $= \mathsf{RCA}_0 + \Sigma^0_1\text{-CA}$

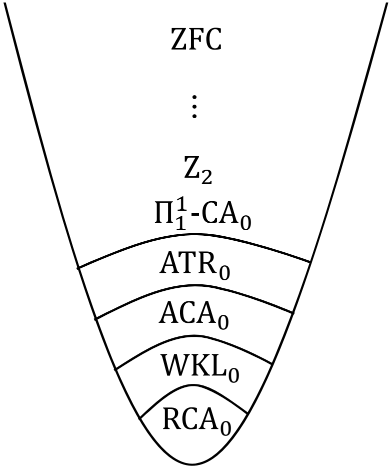
- ATR$_0$ $= \mathsf{RCA}_0 + \overbrace{\text{the existence of a transfinite hierarchy produced}}^{\text{Arithmetical Transfinite Recursion}}$
  interating arithemetic comprehension
  along a given well order

$\mathsf{WKL}_0 \leftrightarrow$ the maximum principle
$\leftrightarrow$ the Cauchy-Peano theorem
$\leftrightarrow$ Brouwer's fixed point theorem

$\mathsf{ACA}_0 \leftrightarrow$ the Bolzano-Weierstrass theorem
$\leftrightarrow$ the Ascoli-Arzela lemma

$\mathsf{ATR}_0 \leftrightarrow$ the Luzin separation theorem
$\leftrightarrow \Sigma^0_1$-determinacy

Π^1_1-$\mathsf{CA}_0 \leftrightarrow$ the Cantor-Bendixson theorem
$\leftrightarrow \Sigma^0_1 \wedge \Pi^0_1$-determinacy



Thank you for your attention!

In this summer break, I will organize seminars as extension of this lecture.
Please check our WeChat at least once a week.

Lecture in the next semester

Logic and Foundations I

- **Introduction** This is an advanced undergraduate and graduate-level course in mathematical logic and foundations of mathematics. It is almost complementary to my last courses "Logic and Computation I and II." So, completion of them is recommended but not required. If not, please self-study with the slides.
- **Topics** to be presented in the first semester include: theory of equations, Birkhoff's completeness theorem, Boolean algebras, Gentzen-Tait proof sysytem, Goedel's completeness theorem, basic model theory, ultra-products, non-standard analysis, subsystems of first order arithmetic, Presburger arithmetic, non-standard models of arithmetic, saturated models, etc.
- In the second semester, we will move on to theory of real closed fields, second order arithmetic and reverse mathematics.
- **Reference** [1] K. Tanaka, Logical Foundations of Mathematics (in Japanese), Shokabo 2019.
`https://www.shokabo.co.jp/mybooks/ISBN978-4-7853-1575-7.htm`

Reference book for next semester

Logical Foundations of Mathematics: a logical approach to