Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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June 6, 2023

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 7. Schedule

- May 18, (1) KP set theory I
- May 23, (2) KP set theory II
- May 25, (3) KP set theory III
- May 30, (4) KP set theory IV and α recursion theory
- Jun. 1, (5) Recursively large ordinals I
- Jun. 6, (6) Recursively large ordinals II and second order arithmetic

Today's topics

1. Recap
2. Admissible recursion theory
3. Recursive analogues of large cardinals
4. Reflecting ordinals
5. Inductive definitions

$$
\begin{aligned}
\mathsf{KP} \quad &:= \text{axioms of extensionality, pairing, union, empty set} \\
&+ \ \Delta_0\text{-Sep, or } \Delta_1\text{-Sep} : \quad \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \land \varphi(z)). \\
&+ \ \Delta_0\text{-Coll, or } \Sigma_1\text{-Coll} : \quad \forall x (\forall y \in x \, \exists z \varphi(z) \to \exists u \forall y \in x \exists z \in u \varphi(z)). \\
&+ \ \text{foundation} : \quad \forall x [\forall y \in x \, \varphi(y) \to \varphi(x)] \to \forall x \varphi(x). \\
\mathsf{KP}\omega &:= \mathsf{KP} + \text{axiom of infinity} : \quad \exists x \{0 \in x \land \forall y \in x (y \cup \{y\} \in x)\}.
\end{aligned}
$$

Definition (Constructible sets)

A Σ_1 operater L_α on the ordinals are defined as follows.

$$
\begin{cases}
\mathrm{L}_0 := \varnothing \\
\mathrm{L}_{\alpha+1} := \mathrm{Def}(\mathrm{L}_\alpha) \\
\mathrm{L}_\alpha := \bigcup_{\beta<\alpha} \mathrm{L}_\beta \quad (\alpha \text{ is a limit ordinal})
\end{cases}
$$

Let L denote a Σ_1 class $\bigcup_{\alpha \in \mathrm{Ord}} \mathrm{L}_\alpha$. The elements of L are called **constructible sets**.

Definition

An ordinal α is said to be **admissible** if $\mathrm{L}_\alpha \models \mathsf{KP}$ holds.

The first admissible ordinal is ω

- L_ω is the set of x such that $\mathsf{TC}(x)$ is finite.

The second admissible ordinal ω^+ is ω_1^{CK}

- $\mathrm{L}_{\omega_1^{\mathrm{CK}}} \bigcap \mathcal{P}(\omega) = \Delta_1^1 = \mathrm{Hyp}$, the set of hyperarithmetical sets (of natural numbers) (Slide 06-06, p.18).
- $\mathrm{L}_{\omega_1^{\mathrm{CK}}} \bigcap \mathrm{Ord}$ = the set of recursive ordinals = the set of hyp ordinals.

The third admissible ordinal ω^{++} is ω_1^{CK} in oracle $\mathcal{O}$

- $\mathrm{L}_{\omega^{++}} \bigcap \mathcal{P}(\omega) = (\Delta_1^1)^{\mathcal{O}}$

Lemma

$\alpha \in \mathrm{Ord}$ is admissible $\Leftrightarrow \mathrm{L}_\alpha \models \Delta_0\text{-Coll}$.

Definition

For an admissible ordinal α,

(1) $A \subset \alpha$ is α-**finite** $\Leftrightarrow A \in \mathrm{L}_\alpha$,

(2) $A \subset \alpha$ is α-**recursively enumerable**(α-RE) $\Leftrightarrow A$ is $\Sigma_1(\mathrm{L}_\alpha)$,

(3) $A \subset \alpha$ is α-**recursive** $\Leftrightarrow A$ is $\Delta_1(\mathrm{L}_\alpha)$,

(4) $f : \alpha \to \alpha$ is α-**recursive** $\Leftrightarrow$ the graph of f is $\Delta_1(\mathrm{L}_\alpha)$.

- $A \subset \alpha$ is said to be $\Sigma_1(\mathrm{L}_\alpha)$ if there exists a Σ_1 formula $\varphi(x)$ such that $A = \{\beta < \alpha : \mathrm{L}_\alpha \models \varphi(\beta)\}$.

Homework

Suppose $A \subset \omega$. Prove that

A is ω_1^{CK}-recursive $\Leftrightarrow$ A is ω_1^{CK}-finite $\Leftrightarrow$ $A \subset \omega$ is hyperarithmetic (i.e., Δ_1^1).

Lemma (Spector Gandy)

The following are equivalent.

(1) $A \subset \omega$ is ω_1^{CK}-RE.

(2) $A \subset \omega$ is Π_1^1.

Proof. $(2) \Rightarrow (1)$

- By Corollary(1) of Lecture06-06, we have for any Π_1^1 set A, there exists a recursive tree T such that $n \in A \Leftrightarrow T_n = \{t : n^\wedge t \in T\} \in \mathrm{WF}$.
- By the theorem in P.18 of Lecture06-05, "$T_n \in \mathrm{WF}$" is equivalent to "there exist an ordinal $\sigma < \omega_1^{\mathrm{CK}}$ and an order-preserving function $f : T_n \to \sigma + 1$" which is $\Sigma_1(\mathrm{L}_{\omega_1^{\mathrm{CK}}})$.

$(1) \Rightarrow (2)$

- Note that $\mathrm{L}_{\omega_1^{\mathrm{CK}}}$ is the minimal model of KPω. This was shown as an example.
- Then, for any Σ_1 formula $\varphi(x)$,

$$\mathrm{L}_{\omega_1^{\mathrm{CK}}} \models \varphi(n) \quad \Leftrightarrow \quad \varphi(n) \text{ holds in any model } \xi \text{ of } \mathrm{KP}\omega,$$

where the latter can be expressed by a Π_1^1 formula. □

Lemma

α is admissible $\Leftrightarrow$ there is no cofinal (unbounded) $\Delta_1(\mathrm{L}_\alpha)$ function from $\beta < \alpha$ to α.

Proof. ($\Rightarrow$) Assume there is a cofinal function f from $\beta < \alpha$ to α, whose graph is represented by a $\Delta_1(\mathrm{L}_\alpha)$ formula θ. Then since $\forall u < \beta \exists v\ \theta$ holds in L_α, there is a $\gamma < \alpha$ s.t. $\forall u < \beta \exists v < \gamma\ \theta$ by $\Delta_1 - \mathrm{Coll}$, which contradicts with the cofinality of f.

($\Leftarrow$) Let α be a limit. To show $\mathrm{L}_\alpha \models \Delta_0\text{-}\mathrm{Coll}$, let θ be Δ_0 and assume $\forall u \in a \exists v \theta$ in L_α.

- Take β such that $a \subset \mathrm{L}_\beta$. Consider a $\Delta_1(\mathrm{L}_\alpha)$ increasing function $f : \beta \to \alpha$ such that for each $\beta' < \beta$, $f(\beta')$ is the smallest γ such that $\forall u \in a \cap \mathrm{L}_{\beta'}\ \exists v \in \mathrm{L}_\gamma \theta$.
- If there is $\beta' < \beta$ such that the function $f : \beta' \to \alpha$ is cofinal, which contradicts with the assumption.
- If there is no such β', f is a $\Delta_1(\mathrm{L}_\alpha)$ function from β to α, and thus it is not cofinal by the assumption, and so there exists a $\gamma < \alpha$ such that $\forall u \in a \exists v \in \mathrm{L}_\gamma \theta$. $\square$

- From the last lemma, admissible ordinals can be viewed as recursive analogues of regular cardinals in set theory, and hence they are also called **recursively regular**.
- It is natural to consider what are the recursive versions of large cardinals, such as regular limit cardinals (weakly inaccessible cardinals).
- First, let us introduce a relatively small analogue of large cardinals.

Definition (Recursive analogues of large cardinals)

(1) Ordinal α is **recursively inaccessible** $\Leftrightarrow$ α is admissible and is a limit of admissibles (For any $\beta < \alpha$, there exists an admissible ordinal γ such that $\beta < \gamma < \alpha$).

(2) Ordinal α is **recursively Mahlo** $\Leftrightarrow$ α is admissible and for any α-recursive function $f : \alpha \to \alpha$, there exists an admissible $\beta < \alpha$ such that $\forall \gamma < \beta \ f(\gamma) < \beta$.

(3) Ordinal α is **recursively hyper-Mahlo** $\Leftrightarrow$ α is admissible and for any α-recursive function $f : \alpha \to \alpha$, there exists a recursively Mahlo $\beta < \alpha$ such that $\forall \gamma < \beta \ f(\gamma) < \beta$.

Homework

Denote the α-th admissible ordinal by τ_α. Show the following.

(1) $\cup_{n\in\omega}\tau_n$ is not admissible.

(2) $\alpha = \tau_\alpha \Leftrightarrow \tau_\alpha$ is recursively inaccessible.

(3) Recursive Mahlo ordinals are recursively inaccessible.

Homework

Show the following.

(1) Let γ be the limit of admissible ordinals, then L_γ satisfies Axiom β.

(2) Let α be recursively inaccessible. Then L_α satisfies (Δ^1_2-CA) which asserts the existence of any Δ^1_2 subset of ω.

We introduce the following key notion to characterize even larger admissible ordinals.

Definition (Reflecting ordinals)

Let Γ be a set of formulas. We say that α is Γ-**reflecting** if for any $\varphi \in \Gamma$ (containing elements of L_α as parameters),

$$\mathrm{L}_\alpha \models \varphi \Rightarrow \exists \beta < \alpha\ \mathrm{L}_\beta \models \varphi,$$

where the parameters included in φ are in L_β.

Lemma

Ordinal α is Σ_{n+1}-reflecting $\Leftrightarrow$ α is Π_n-reflecting.

Proof. ($\Rightarrow$) is obvious.
To show ($\Leftarrow$), let $\exists y\ \varphi(y)$ be a Σ_{n+1} formula, where φ is Π_n with parameters in L_α.

$$\begin{aligned} \mathrm{L}_\alpha \models \exists y \varphi(y) &\Rightarrow \mathrm{L}_\alpha \models \varphi(u) \text{ for some } u \in \mathrm{L}_\alpha \\ &\Rightarrow \text{by the } \Pi_n\text{-reflecting, there exists } \beta < \alpha \text{ such that } \mathrm{L}_\beta \models \varphi(u) \\ &\Rightarrow \mathrm{L}_\beta \models \exists y \varphi(y) \end{aligned}$$

Theorem

$\alpha(>\omega)$ is admissible $\Leftrightarrow$ α is Π_2-reflecting.

Proof.

$(\Rightarrow)$

- Suppose α is admissible and a Π_2 formula $\forall x \exists y \theta$ holds in L_α.
- Then, by Δ_0-Coll, for any $\beta < \alpha$, there exists $\gamma < \alpha$ such that $\forall x \in \mathrm{L}_\beta \exists y \in \mathrm{L}_\gamma \theta$. Thus, we define an α-recursive function f by taking $f(\beta)$ as the smallest such γ.
- If there is a β such that $\beta \geq f(\beta)$, then $\forall x \exists y \theta$ holds in L_β.
- Otherwise, define an ω-sequence $0 = \beta_0 < \beta_1 < \beta_2 < \ldots$ by $\beta_{n+1} = f(\beta_n)$. Then by the lemma on page 8, $\beta = \sup_n \beta_n < \alpha$. Thus, $f(\beta) = \beta < \alpha$, a contradiction.

$(\Leftarrow)$

- Let α be Π_2-reflecting. If Π_2 formula $\forall x \in a \exists y \theta$ holds in L_α, it also holds in L_β for some $\beta < \alpha$. That is, $\forall x \in a \exists y \in \mathrm{L}_\beta \theta$ holds in L_α.
- Therefore, L_α satisfies Δ_0-Coll, and so α is admissible. □

Lemma

For each $n \geq 1$, there exists a Π_{n+1} sentence θ_n such that for any limit ordinal α,

$$\alpha \text{ is } \Pi_n\text{-reflecting} \iff \mathrm{L}_\alpha \models \theta_n.$$

In particular, θ_2, representing admissibility, is a Π_3 sentence.

Proof Sketch.
Let $\varphi(e,x)$ be a universal Π_n formula. Then "α is Π_n-reflecting" is expressed as follows.

$$\mathrm{L}_\alpha \models \underbrace{\forall x \forall e < \alpha(\underbrace{\varphi(e,x) \to \exists \beta(x \in \mathrm{L}_\beta \wedge \varphi^{\mathrm{L}_\beta}(e,x))}_{\neg\Pi_n \vee \Sigma_1 \text{ i.e. } \Sigma_n})}_{\Pi_{n+1}}$$

So, letting θ_n be the above Π_{n+1} sentence (after $\mathrm{L}_\alpha \models$), the first half of the lemma holds. The second half also follows from the last theorem. □

Theorem

A Π_3-reflecting ordinal is recursively Mahlo.

Proof.

- Let α be a Π_3-reflecting ordinal and take any α-recursive function $f : \alpha \to \alpha$.
- We want to show that there exists an admissible $\beta < \alpha$ such that $\forall \gamma < \beta \; f(\gamma) < \beta$.
- The graph of f is represented by a $\Delta_1(\mathrm{L}_\alpha)$ formula $\varphi(x, y)$ (with parameters in L_α).
- Let θ_2 be the Π_3 sentence of admissibility, and define

$$\psi \;\equiv\; \forall x \in \mathrm{Ord}\; \exists y \in \mathrm{Ord}\; \varphi(x, y) \;\wedge\; \theta_2.$$

  Then ψ is Π_3.

- Since ψ holds L_α, by Π_3-reflecting, there exists a $\beta < \alpha$ such that ψ holds in L_β.
- Then since L_β satisfies θ_2, β is admissible. Since L_β satisfies $\forall x \exists y \varphi(x, y)$, $\forall \gamma < \beta \; f(\gamma) < \beta$.

□

- "α is recursively Mahlo" iff $\mathrm{L}_\alpha \models \theta_2 \wedge \forall f : \mathrm{Ord} \to \mathrm{Ord}\ \exists \beta\ (\theta_2^{\mathrm{L}_\beta} \wedge \forall \gamma < \beta\ f(\gamma) < \beta)$.
- Thus, "α is recursively Mahlo" can also be expressed by a Π_3 formula π_3.
- Then, we can also show that any Π_3-reflecting ordinal is a recursively hyper-Mahlo in the same way as the proof above.
- Moreover, it is also a recursively hyper-hyper-Mahlo, etc.
- In summary,

  the smallest Π_3-reflecting ordinal
  $>$ the smallest recursively hyper-hyper-Mahlo ordinal
  $>$ the smallest recursively hyper-Mahlo ordinal
  $>$ the smallest recursively Mahlo ordinal
  $>$ the smallest recursively inaccessible
  $> \omega_1^{\mathrm{CK}}$.

Inductive definitions

- The smallest Π_3-reflecting ordinal is the smallest ordinal that cannot be described by a Π^1_1 formula in second-order set theory, which is thus called a **recursively weakly compact ordinal**.
- It seems tremendously large, but from another point of view, it's not so big. We now consider the relationship with the **inductive definition**.
- Inductive definitions appear everywhere, both in mathematics and computer science. For example, a set of terms, a set of formulas, a set of theorems are defined by induction.
- However, in most cases, an inductive operator is finitary, and so defined objects are obtained in finite steps. We here consider infinitary operators, which define objects in transfinite steps. The construction of the Borel sets is a typical example.

In the following, we only deal with inductive definitions for sets of natural numbers. $\mathcal{P}(\omega)$ denotes the set of all subsets of ω.

Definition (Inductive definition)

- Given a function (also called an **operator**) $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, we define
$$\Gamma^\alpha = \bigcup\{\Gamma(\Gamma^\beta) : \beta < \alpha\}$$
as the transfinite increasing sequence $\{\Gamma^\alpha : \alpha \in \mathrm{Ord}\}$ of subsets of ω.
- Then, write $|\Gamma|$ for the first ordinal α such that $\Gamma^\alpha = \Gamma^{\alpha+1}$, which is called the **closure ordinal** of operator Γ.
- $\Gamma^{|\Gamma|}$ is called the set determined by **inductive definition** of Γ and also denoted by Γ^∞.

- An operator Γ is said to be **monotone**, if for any $X \subset Y \subset \omega$, $\Gamma(X) \subset \Gamma(Y)$.
- For a monotone Γ, $\Gamma^\infty = \bigcap\{X : \Gamma(X) \subset X\}$.
- An operator Γ is Σ^i_n (or Π^i_n) if $\{(x, X) \in \omega \times \mathcal{P}(\omega) : x \in \Gamma(X)\}$ is Σ^i_n (or Π^i_n).
- $|\Sigma^i_n| = \sup\{|\Gamma| : \Gamma \in \Sigma^i_n\}$ and $|\mathrm{mon}\Sigma^i_n| = \sup\{|\Gamma| : \Gamma \in \Sigma^i_n \text{and monotone}\}$.
- $|\Pi^i_n|$ and $|\mathrm{mon}\Pi^i_n|$ can be defined similarly.

- Our next goal is to show $|\Pi^0_n|$ is the smallest Π_{n+1}-reflecting ordinal.
- If this holds, then $|\Pi^0_1|$ is ω_1^{CK} and $|\Pi^0_2|$ is recursively weakly compact.
- This result is a bit surprising, since Π^0_1 and Π^0_2 are lowest levels of arithmetic formulas.
- We first review some basics. Σ^i_n has a universal formula $\varphi(e,x,X)$. Hence, any Σ^i_n operator is also denoted as Γ_e with index e if $x \in \Gamma_e(X) \Leftrightarrow \varphi(e,x,X)$.
- On the other hand, if Γ is defined by $(e,x) \in \Gamma(X) \Leftrightarrow \varphi(e,x,X_e)$, $(\Gamma_e)^\alpha = (\Gamma^\alpha)_e$ for each e. Hence, $|\Sigma^i_n|$ is $|\Gamma|$. Similarly for Π^i_n.

Homework

Show $|\Pi^0_n| = |\Sigma^0_{n+1}|$.

Lemma

Let Γ be universal Π^0_n $(n > 0)$ and $\alpha = |\Gamma|$. For any Π^0_n formula $\varphi(X)$, if $\varphi(\Gamma^\infty)$ then $\exists\beta < \alpha\varphi(\Gamma^\beta)$ holds.

Proof.

- Assume $\varphi(\Gamma^\infty)$, where Γ is a universal Π^0_n and $\varphi(X)$ is a Π^0_n formula.
- By way of contradiction, assume $\forall\beta < \alpha\ \neg\varphi(\Gamma^\beta)$.
- Now, let $X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}$ and a Π^0_n operator Γ' is defined as follows

$$\Gamma'(X \oplus Y) = \{2n : n \in \Gamma(X)\} \cup \{2n+1 : n \in \Gamma(Y) \wedge \varphi(X)\}.$$

- Starting with $Y = \varnothing$, Γ' mimics $\Gamma(X)$. Then, when it reaches a fixed point Γ^∞ that satisfies $\varphi(X)$, then fix $X = \Gamma^\infty$ and start mimicing $\Gamma(Y)$.
- Then $|\Gamma'| > |\Gamma|$ is obvious, which conflicts with the universality of Γ. $\square$

- Recall the following lemma

Lemma (revisited)

There is a primitive recursive bijection $F : \mathrm{Ord} \to \mathrm{L}$ such that if α is ω or an ε number then $F``\alpha = \mathrm{L}_\alpha$.

- For admissible $\alpha = |\Gamma|$, it is even easier to construct a α-recursive bijection $G : \alpha \to \Gamma^\alpha$ such that $G``\beta = \Gamma^\beta$ for any limit ordinal $\beta < \alpha$.
- Thus, $H = F \circ G^{-1}$ is a α-recursive bijection from Γ^α to L_α such that for an ε number $\beta < \alpha$, $H``\Gamma^\beta = \mathrm{L}_\beta$.

Now we are ready to show

Theorem

For any $n > 0$, $|\Pi^0_n|$ is the smallest Π_{n+1}-reflecting ordinal.

Proof Sketch.

- We only consider the case $n = 2$. Other cases can be treated similarly.
- Let Γ be a universal Π^0_2 operator with admissible $\alpha = |\Gamma|$.
- As already mentioned, there exists an α-recursive bijection $H : \Gamma^\alpha \to \mathrm{L}_\alpha$.
- Then $\Gamma^\alpha \notin \mathrm{L}_\alpha$, and Γ^α is $\Sigma_1(\mathrm{L}_\alpha)$.
- Moreover, Γ^α is m-complete. That is, any $\Sigma_1(\mathrm{L}_\alpha)$ set of natural numbers is m-reducible to Γ^α.
  $\because$ Let $\varphi(n)$ be a Σ_1 formula. Then, there exists an ε number $\beta < \alpha$ such that $\mathrm{L}_\beta \models \varphi(n)$ whenever $\mathrm{L}_\alpha \models \varphi(n)$. Also, we have $H``\Gamma^\beta = \mathrm{L}_\beta$. We now define a relation $m \mathrel{\tilde{\in}} l$ by $\mathrm{L}_\beta \models H(m) \in H(l)$, and then it is recursive in Γ^β. Hence, a $\Sigma_1(\mathrm{L}_\beta)$ set is arithmetic in Γ^β and so m-reducible to Γ^α.

- Now, for a Σ_1 formula $\exists w \neg\psi(u,v,w)$, where $\psi(u,v,w)$ is a Δ_0 formula with parameters in L_α, there exists a recursive function $g : \omega \times \omega \to \omega$ such that for every $m, n \in \omega$

$$g(m,n) \in \Gamma^\alpha \Leftrightarrow m, n \in \Gamma^\alpha \wedge \mathrm{L}_\alpha \models \exists w \neg\psi(H(m), H(n), w).$$

- Suppose $\mathrm{L}_\alpha \models \forall u \exists v \forall w \; \psi(u,v,w)$. That is,

$$\forall m \in \Gamma^\alpha \exists n \in \Gamma^\alpha g(m,n) \notin \Gamma^\alpha.$$

- Since the above is in the form $\Pi^0_2(\Gamma^\alpha)$, by the last lemma there exists a $\beta < \alpha$ such that

$$\forall m \in \Gamma^\beta \exists n \in \Gamma^\beta g(m,n) \notin \Gamma^\beta.$$

  We may assume that β is a ε number, by adding some conditions to the formula.

- Then, by using H again, we get

$$\mathrm{L}_\beta \models \forall v \exists v \forall w \; \psi(u,v,w).$$

- Thus, α is a Π_3-reflecting ordinal.

- Finally, for a contradiction, we assume that there exists a Π_3-reflecting ordinal β below α.
- Since $\beta < \alpha$, there exists $x \in \Gamma(\Gamma^\beta) - \Gamma^\beta$. Since Γ is Π^0_2, there is a recursive R s.t.,

$$x \in \Gamma(\Gamma^\beta) \Leftrightarrow \forall m \exists n R(m, n, \Gamma^\beta).$$

- Now we consider how to express $\forall m \exists n R(m, n, \Gamma^\beta)$ in L_β.
- Since Γ^β is $\Sigma_1(\mathrm{L}_\beta)$, $R(m, n, \Gamma^\beta)$ is $\Delta_2(\mathrm{L}_\beta)$.
- Although m, n in $R(m, n, \Gamma^\beta)$ range over natural numbers, they turn into set variables in the corresponding $\Delta_2(\mathrm{L}_\beta)$ formula.
- Thus, the interpretation of $\forall m \exists n R(m, n, \Gamma^\beta)$ over L_β is a Π_3 formula.
- For the sake of convenience, if we express this with the same formula, by the Π_3-reflexivity, there is a $\gamma < \beta (\mathrm{L}_\gamma \models \forall m \exists n R(m, n, \Gamma^\gamma)$.
- Therefore, $x \in \Gamma(\Gamma^\gamma) \subset \Gamma^\beta$, which contradicts with the choice of x.
- Thus α is the smallest Π_3-reflecting ordinal. □

Thank you for your attention!