Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 7. Schedule

- May 18, (1) KP set theory I
- May 23, (2) KP set theory II
- May 25, (3) KP set theory III
- May 30, (4) KP set theory IV and α recursion theory
- Jun. 1, (5) Recursively large ordinals I
- Jun. 6, (6) Recursively large ordinals II and second order arithmetic

Today's topics

1. Recap
2. α recursion

Recap

$$
\begin{aligned}
\mathsf{KP} \;&:= \text{axioms of extensionality, pairing, union, empty set}\\
&+\ \Delta_0\text{-Sep, or } \Delta_1\text{-Sep}: \quad \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z)).\\
&+\ \Delta_0\text{-Coll, or } \Sigma_1\text{-Coll}: \quad \forall x(\forall y \in x\, \exists z \varphi(z) \rightarrow \exists u \forall y \in x \exists z \in u \varphi(z)).\\
&+\ \text{foundation}: \quad \forall x[\forall y \in x\, \varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x).\\
\mathsf{KP}\omega \;&:= \mathsf{KP} + \text{axiom of infinity}: \quad \exists x\{0 \in x \wedge \forall y \in x(y \cup \{y\} \in x)\}.
\end{aligned}
$$

Let $\varphi(\vec{x}, y)$ be a Σ_1 formula such that $\mathsf{KP} \vdash \forall \vec{x} \exists! y \varphi(\vec{x}, y)$. Then we can introduce a Σ_1 **operator** F by the following axiom: $(\mathrm{F}) \ : \ \forall \vec{x}\ (\mathrm{F}(\vec{x}) = y \leftrightarrow \varphi(\vec{x}, y))$.

Theorem (Σ_1 recursion)

Let G be a Σ_1 operator. There exists a Σ_1 operator F such that

$$\mathsf{KP} \vdash \mathrm{F}(x) = \mathrm{G}(x, \{(y, \mathrm{F}(y)) : y \in x\}).$$

Definition (Constructible sets)

A Σ_1 operater L_α on the ordinals are defined as follows.

$$\begin{cases} \mathrm{L}_0 := \varnothing \\ \mathrm{L}_{\alpha+1} := \mathrm{Def}(\mathrm{L}_\alpha) \\ \mathrm{L}_\alpha := \bigcup_{\beta<\alpha} \mathrm{L}_\beta \quad (\alpha \text{ is a limit ordinal}) \end{cases}$$

Let L denote a Σ_1 class $\bigcup_{\alpha \in \mathrm{Ord}} \mathrm{L}_\alpha$. The elements of L are called **constructible sets**.

Definition (Mostowski's Axiom β)

Axiom β asserts: for any well-founded relation $r \subset a \times a$ on any set a, there exists a function f whose domain is a and satisfies

$$f(x) = \{f(y) : y \in a \wedge (y, x) \in r\} \quad (\text{for all} x \in a).$$

Such an f is called a **collapsing function** of r.

- A collapsing function f of r has two roles. One is to express r as ϵ. So, moreover if r satisfies extensionality: $\forall b_1, b_2 \in a[\forall x \in a((x, b_1) \in r \leftrightarrow (x, b_2) \in r) \to b_1 = b_2]$, then $f\colon (a, r) \cong (\mathrm{range}(f), \in)$.
- The other role is to fold the relation by removing unnecessary elements and squeezing it into a smallest set. E.g., if $a = \{0, 2, \{1, 2\}\}$ with $r = \varepsilon$, then $f(a) = \{0, 1, \{1\}\}$.

Theorem (1)

Axiom β holds in KP $+\Sigma_1$-Sep.

Proof

- Let r be a well-founded relation on a set a. We write $x <_r y$ for $(x, y) \in r$.
- We define a Σ_1 operator F by Σ_1 recursion over ordinals as follows.

$$\mathrm{F}(\alpha) = \{x \in a : \forall y \in a(y <_r x \to \exists \beta \in \alpha\ y \in \mathrm{F}(\beta))\}.$$

- First, to show $\bigcup_\alpha \mathrm{F}(\alpha) = a$, we put by Σ_1-Sep,

$$a_0 := \{x \in a : \exists \alpha(x \in \mathrm{F}(\alpha))\}.$$

- Suppose that $a - a_0 \neq \varnothing$. Let x be a $<_r$-minimal element in this set. Then,
$$\forall y \in a(y <_r x \to \exists \beta\ y \in \mathrm{F}(\beta)).$$
- Using Σ_1-Coll, we can obtain α such that
$$\forall y \in a(y <_r x \to \exists \beta \in \alpha\ y \in \mathrm{F}(\beta)).$$
- Therefore, by the definition of $\mathrm{F}(\alpha)$, we have $x \in \mathrm{F}(\alpha)$, i.e., $x \in a_0$, which contradicts the choice of x. Therefore, $\bigcup_\alpha \mathrm{F}(\alpha) = a$.
- Then again by Σ_1-Coll, there exists some γ such that $\bigcup_{\alpha<\gamma} \mathrm{F}(\alpha) = a$.
- Next, for $\alpha < \gamma$, by Σ_1 recursion, we define a sequence of functions $\{f_\alpha\}$ as follows.
$$f_\alpha(x) := \{f_\beta(y) : \beta < \alpha \wedge y \in \mathrm{F}(\beta) \wedge y <_r x\} \ (\text{ for any } x \in \mathrm{F}(\alpha)).$$
It is easy to show that if $\beta \leq \alpha$ then $f_\beta \subset f_\alpha$. So $f = \bigcup_{\alpha<\gamma} f_\alpha$ is defined as a function, and we have
$$f(x) = \{f(y) : y \in a \wedge y <_r x\} \quad (\text{ for any } x \in a)$$
- Thus, Axiom β is shown.

- Axiom β is effective in downscaling the hierarchy of formulas.
- For example, "r is well-founded" can be expressed as "there is no infinite descending sequence" which is Π_1. But it can also be expressed as "there is a collapsing function" which is Σ_1. Thus, the well-foundedness is Δ_1.
- It is also useful to consider the relationship between analytical and set hierarchies.

Theorem (2)

In KP $+$ $\text{axiom } \beta$, if $\varphi(\xi)$ is a Σ^1_2 formula, then there is an equivalent Σ_1 formula, and vice versa.

Proof.

- It is relatively easy to represent a Σ^1_2 formula as a Σ_1 formula.
- A Π^1_1 formula can be expressed as "T^ξ is well-founded", and it is equivalent to $\exists f(f : (T^\xi, <) \cong (\text{range}(f), \in))$ as Σ_1. A Σ^1_2 formula has a function existential quantifier $\exists g \in {}^\omega\omega$ in front of a Π^1_1 formula. Since a function quantifier is a kind of set quantifier, a Σ^1_2 formula remains Σ_1.

- The reverse direction is a little more difficult.

- To treat formulas of set theory in second-order arithmetic, we use a method of coding countable models of set theory with real numbers. Such a model must be a well-founded ω-model satisfying the axiom of extensionality.

- Then, a set-theoretic Σ_1 formula $\varphi(\xi)$(where, $\xi \in {}^\omega\omega$) can be expressed as a formula $\psi(\xi)$ in second-order arithmetic as follows

$$\exists\eta[\eta \text{ is a well-founded } \omega \text{ model of extensionality } \wedge \xi \in \eta \wedge \eta \models \varphi(\xi)]. \qquad (1)$$

  Here, "η is a well-founded ω-model of extensionality" can be expressed as Π^1_1 as explained before. $\xi \in \eta$ means that ξ belongs to the model coded by η, which is expressed as Δ^1_1. Finally, $\eta \models \varphi(\xi)$ is Δ^1_1. So the whole $\psi(\xi)$ becomes Σ^1_2.

- Next we show that $\psi(\xi)$ in (1) is equivalent to $\varphi(\xi)$.

- First assume that a Σ_1 formula $\varphi(\xi) \equiv \exists x \theta(x, \xi)$ (where θ is Σ_0) holds. Then there exists a set ν such that $\theta(\nu, \xi)$.
- Now, consider the transitive closure $\mathrm{TC}(\{\nu, \xi\})$. This is a model of $\theta(\nu, \xi)$ and thus a model of $\varphi(\xi)$.
- Since ξ is countable, $\mathrm{TC}(\xi)$ is also countable. By the Löwenheim–Skolem theorem, there exists a countable elementary substructure of $\mathrm{TC}(\{\nu, \xi\})$ which contains $\mathrm{TC}(\xi)$ as it is.
- Finally, again by axiom β, we transform it to a transitive model (M, ϵ) of $\varphi(\xi)$ with $\xi \in M$.
- Therefore, $\psi(\xi)$ holds.

- Conversely, suppose that a Σ^1_2 formula $\psi(\xi)$ holds.
- By axiom β, a model η of $\psi(\xi)$ in (1) can be collapsed into a transitive set model $(M, \in)$. Since $\varphi(\xi)$ is a Σ_1 formula, $(M, \in) \models \varphi(f(\xi))$ implies $(V \models)\varphi(f(\xi))$. □

- This theorem can be generalized to the equivalence of Σ^1_{n+1} and Σ_n formulas under stronger assumptions.
- First, it is easy to see that Σ^1_{n+1} formula becomes Σ_n from the above proof. To show the converse, we use the following notion.
- We say that M is a β_k-**model** if it is an ω-model such that for any Σ^1_k formula $\psi(\xi)$ (including the elements of M as parameters), the following Δ^1_{n+1} formula holds

$$\forall \xi \in M(\psi(\xi) \leftrightarrow M \models \psi(\xi)).$$

- Then, a Σ_n formula $\varphi(\xi)$ is expressed by a Σ^1_{n+1} formula $\psi(\xi)$ as follows.

$$\exists \eta[\eta \text{ is a } \beta_n \text{ model} \land \xi \in \eta \land \eta \models \varphi(\xi)].$$

- Any well-founded model is a β_1-model, and its existence follows from axiom β.
- Constructing a β_k model requires even stronger axioms. See Sections 7.3 and 7.7 of Simpson [Sim] for details.

Theorem (Shoenfield absoluteness)

(In $\mathsf{KP}\omega + \Sigma_1$-Sep [i]) For any Σ^1_2 formula $\varphi(\xi)$, $\forall \xi \in \mathrm{L}\ (\varphi(\xi) \Leftrightarrow \mathrm{L} \models \varphi(\xi))$.

Proof.

- By theorem (2), any Σ^1_2 formula $\varphi(\xi)$ is Σ_1, so $\mathrm{L} \models \varphi(\xi) \Rightarrow (V \models)\varphi(\xi)$ is obvious.
- Suppose a Σ^1_2 formula $\varphi(\xi)$ is in the form $\exists \eta A(\xi, \eta)$ with $A \in \Pi^1_1$. By the Kondo-Addison theorem, let A^* be a Π^1_1 function uniformizing A.
- Then $\eta = \eta^* \in {}^\omega\omega$ that satisfies $A^*(\xi, \eta)$ can be expressed as the following $\Delta^1_2(\xi)$ formula

$$\eta^*(m) = n \leftrightarrow \exists \eta (A^*(\xi, \eta) \wedge \eta(m) = n) \leftrightarrow \forall \eta (A^*(\xi, \eta) \rightarrow \eta(m) = n).$$

- Then by theorem (2), $\eta^* \subset \omega \times \omega$ is $\Delta_1(\xi)$. So if $\xi \in \mathrm{L}$ then $\eta^* \in \mathrm{L}$.
- Since $A^*(\xi, \eta)$ is a Δ_1 relation, if $\xi, \eta^* \in \mathrm{L}$ and $(V \models)A^*(\xi, \eta^*)$ then $\mathrm{L} \models A^*(\xi, \eta^*)$.
- Therefore, we have $\mathrm{L} \models \varphi(\xi)$. □

[i] In fact, Axiom β and the Kondo-Addison theorem are sufficient for the theorem.

Introduction

- Among various challenges to extend the theory of computation over the natural numbers to more general structures, one of the most successful generalizations is α-recursion theory, that is, recursion theory over so-called admissble ordinals. Also, there are set versions of admissible recursion theory.
- Admissible ordinals inherit many properties of ω. In particular, they are closed under many operations such as primitive recursive operators.
- It is also possible to define α-computability by generalizing the length of the tape of a Turing machine to ordinal α.
- Let us begin with the definition of primitive recursive operators on ordinals.

Definition (PRO)

The **Primitive recursive ordinal operators** (PRO) on Ord are defined as follows.

1. Constant 0, Successor $\mathrm{S}(x) = x + 1$, projection $P^n_i(x_1, x_2, \ldots, x_n) = x_i$ $(1 \leq i \leq n)$, and **less-than operator** $C(x, y, u, v)$ are the initial PRO's, where

$$C(x, y, u, v) := \begin{cases} x & \text{if } u < v \\ y & \text{o.w.} \end{cases}$$

2. Operator composition. If $G_i : \mathrm{Ord}^n \to \mathrm{Ord}$ $(1 < i < m), H : \mathrm{Ord}^m \to \mathrm{Ord}$ are PRO, the operator $F = H(G_1, \ldots, G_m) : \mathrm{Ord}^n \to \mathrm{Ord}$ defined as follows is also PRO

$$F(x_1, \ldots, x_n) = H(G_1(x_1, \ldots, x_n), \ldots, G_m(x_1, \ldots, x_n)).$$

3. Primitive recursion. If $G : \mathrm{Ord}^{n+2} \to \mathrm{Ord}$ is PRO, the operator $F : \mathrm{Ord}^{n+1} \to \mathrm{Ord}$ defined as below is also PRO

$$F(y, x_1, \ldots, x_n) = G(\sup_{z<y} F(z, x_1, \ldots, x_n), y, x_1, \ldots, x_n)$$

In 3 above, $\sup_{z<y}$ is the same as $\bigcup_{z<y}$. If $y = 0$, it takes value 0.

Example

The following primitive recursive operators are the natural extensions of the corresponding operators of arithmetic.

$$x + y = \begin{cases} x & \text{if } y = 0 \\ \sup_{u<y}((x+u)+1) & \text{otherwise} \end{cases}$$

$$x \cdot y = \begin{cases} 0 & \text{if } y = 0 \\ \sup_{u<y}((x \cdot u)+x) & \text{otherwise} \end{cases}$$

Primitive recursive operators are defined over all the ordinals, but we often consider functions over an ordinal, i.e., an initial segment of the ordinals. Thus, the following definitions are important.

Definition

An ordinal α is said to be closed under PRO if for all PRO operators $F(\vec{x})$, $\forall \vec{\beta} < \alpha \ F(\vec{\beta}) < \alpha$.

Definition

The **Veblen** function $\varphi_\alpha(\beta)$ is defined as follows.

- $\varphi_0(\beta) = \omega^\beta$ for the β-th ordinal closed under addition.
- $\varphi_{\alpha+1}(\beta)$ for the β-th ordinal γ closed under $\varphi_\alpha(x)$, i.e., $\varphi_\alpha(\gamma) = \gamma$. In particular, $\varphi_1(0) = \varepsilon_0$.
- For limit ordinal λ, $\varphi_\lambda(\beta)$ is the β-th ordinal γ of $\bigcap_{\alpha<\lambda} \mathrm{Range}(\varphi_\alpha)$.

Oswald Veblen

Example

- $\varphi_\omega(0)$ is the next ordinal of ω that is closed under PRO.
- To show this, we claim that for any PRO $F(x_1, \ldots, x_n)$, there exists some $i \in \omega$ such that for all ordinals $\alpha_1, \ldots, \alpha_n$,

$$F(\alpha_1, \ldots, \alpha_n) < \varphi_i(\max\{\alpha_1, \ldots, \alpha_n\})$$

- This can be proved in a way similar to the proof that the Ackermann function is not primitive recursive (part 1 of of this course).
- The first ordinal ξ such that $\varphi_\xi(0) = \xi$ is called Γ_0.
- Although these definitions of ordinals by $\varphi_\alpha(\beta)$ seem to be highly non-constructive at first glance, all of them are recursive ordinals, and can be expressed in a generalized Cantor's normal form.

- From now on, we will arbitrarily fix an ordinal α that is closed under PRO and consider its subsets.
- An n-arity relation $R \subset \alpha^n$ is **primitive recursive** if its characteristic function χ_R with domain α^n is PRO.
- It can be shown that primitive recursive relations are closed under Boolean operations and bounded quantifiers, as is the case for primitive recursive relations on ω.

Definition

For an ordinal α closed under PRO, $A \subset \alpha$ is **α recursively enumerable** (α-RE), if there exist a primitive recursive relation $R(x, y, z)$ and parameters $\gamma < \alpha$ such that

$$A = \{x < \alpha : \exists y < \alpha \, R(x, y, \gamma)\}.$$

We say that $A \subset \alpha$ is **α-recursive** if both A and its complement $\alpha - A$ are α-RE.

"ω-RE" is the ordinary RE or CE . Also, many of the theorems in recursion theory over ω can be generalized to ordinals closed under PRO. We give two basic lemmas without proofs.

Lemma

There exists an α-RE relation $W(e,x)$ such that any α-RE set X can be expressed as

$$X = W_e = \{x < \alpha : W(e,x)\} \qquad \text{for some } e < \alpha$$

Lemma

There are two α-RE sets that cannot be separated by an α-recursive set.

- Many-to-one reducibility can also be defined naturally. However, there are some variations in the formulation of the notion that "A is α-recursive in oracle B", and it becomes difficult to treat only under the condition that α is closed under PRO.
- In order to mimic arguments of recursion theory on ω, we need some stronger closure conditions, which needs KP set theory again.

Definition

An ordinal α is said to be **admissible** if $\mathrm{L}_\alpha \models$ KP holds.

Example: The first admissible ordinal is ω

- It is easy to see that L_ω is the set of x such that $\mathrm{TC}(x)$ is finite, and it satisfies the axiom of KP.

Example: The next admissible ordinal ω^+ after ω is ω_1^{CK}

- $\mathrm{L}_{\omega+1}$ contains all definable sets on ω, in particular, all recursive trees.
- Since L_{ω^+} satisfies KP, by the Σ_1 recursion, the ordinal $||T||$ of every well-founded tree T exists in L_{ω^+}.
- All that is left is to show that $\mathrm{L}_{\omega_1^{\mathrm{CK}}}$ is a model of KP.
- "ξ is an ω-model of KPω" can be expressed by Δ_1^1 formula $\varphi(\xi)$. Assume $\exists\xi\varphi(\xi)$. By Gandy's basis theorem (page 420 of [Rog]), we can take ξ such that $\omega_1^{\xi} = \omega_1^{\mathrm{CK}}$ and $\varphi(\xi)$.
- If M is the well-founded part of the ω model represented by ξ, then M is also a model of KPω (Truncation Lemma, [Bar] p.73).
- Since $\omega_1^{\xi} = \omega_1^{\mathrm{CK}}$, it is clear that for any ordinal $\alpha \in M$, $\alpha < \omega_1^{\mathrm{CK}}$.
- Conversely, all $\alpha < \omega_1^{\mathrm{CK}}$ are included in M. Since $x = \mathrm{L}_\alpha$ is Δ_1, $(\mathrm{L}_\alpha)^M = \mathrm{L}_\alpha$.
- So $M = \mathrm{L}_{\omega_1^{\mathrm{CK}}}$ and ω_1^{CK} is admissible.

Homework

What is the third admissible ordinal number?

Lemma

$\alpha \in \mathrm{Ord}$ is admissible $\Leftrightarrow \mathrm{L}_\alpha \models \Delta_0\text{-}\mathrm{Coll}$.

Proof

$\Rightarrow$ is obvious.

To show $\Leftarrow$.

Assume $\mathrm{L}_\alpha \models \Delta_0\text{-}\mathrm{Coll}$. Then α is a limit ordinal and by the definition of L_α, it is easy to see that all the axioms of KP hold in L_α. □

In KP set theory, we define

Definition

For an admissible ordinal α,

(1) $A \subset \alpha$ is α-**finite** $\Leftrightarrow A \in \mathrm{L}_\alpha$,

(2) $A \subset \alpha$ is α-**recursively enumerable**(α-RE) $\Leftrightarrow A$ is $\Sigma_1(\mathrm{L}_\alpha)$,

(3) $A \subset \alpha$ is α-**recursive** $\Leftrightarrow A$ is $\Delta_1(\mathrm{L}_\alpha)$,

(4) $f : \alpha \to \alpha$ is α-**recursive** $\Leftrightarrow$ the graph of f is $\Delta_1(\mathrm{L}_\alpha)$.

- Here, $A \subset \alpha$ is said to be $\Sigma_1(\mathrm{L}_\alpha)$ if there exists a Σ_1 formula $\varphi(x)$ such that $A = \{\beta < \alpha : \mathrm{L}_\alpha \models \varphi(\beta)\}$.
  Similarly for $\Delta_1(\mathrm{L}_\alpha)$.
- We defined α-REfor ordinals closed under PRO on page 18, which can be shown to be equivalent to the above definition on admissible ordinals. But to prove their equivalence, we have to do something similar to what we did for the equivalence on ω in lecture03-06.
- Now, we discard the old definition, and adopt the new one.

Thank you for your attention!