Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 7. Schedule

- May 18, (1) KP set theory I
- May 23, (2) KP set theory II
- May 25, (3) KP set theory III
- May 30, (4) α recursion theory
- Jun. 1, (5) Recursively large ordinals I
- Jun. 6, (6) Recursively large ordinals II
- Jun. 8, (7) Second-order arithmetic and reverse mathematics

Today's topics

1. Recap
2. Constructible sets
3. Set theory in second-order arithmetic

Recap

- In set theory, the **Lévi hierarchy** is introduced by imitating the arithmetic hierarchy.
  - $\Sigma_0 (= \Pi_0 = \Delta_0)$ formula: all quantifiers are bounded, i.e., $\exists x \in y, \forall x \in y$.
  - A Σ_{n+1} formula is $\exists x \varphi$ with $\varphi \in \Pi_n$. A Π_{n+1} formula is $\forall x \varphi$ with $\varphi \in \Sigma_n$.
  - A Δ_n formula is a Π_n (or Σ_n) formula equivalent to a Σ_n (or Π_n) formula.

KP is a first-order theory in the language $\{\in\}$

$$\begin{aligned}
\text{KP} \; &:= \text{axioms of extensionality, pairing, union, empty set} \\
&+ \; \Delta_0\text{-Sep:} \quad \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z)) \qquad \text{for } \varphi(z) \in \Delta_0. \\
&+ \; \Delta_0\text{-Coll} : \quad \forall x (\forall y \in x \, \exists z \varphi(z) \to \exists u \forall y \in x \exists z \in u \varphi(z)) \qquad \text{for } \varphi(z) \in \Delta_0. \\
&+ \; \text{foundation} : \quad \forall x [\forall y \in x \, \varphi(y) \to \varphi(x)] \to \forall x \varphi(x). \\
\text{KP}\omega \; &:= \text{KP} + \text{axiom of infinity} : \quad \exists x \{0 \in x \wedge \forall y \in x (y \cup \{y\} \in x)\}.
\end{aligned}$$

- $\text{KP} \vdash \Sigma_1\text{-Coll}$.
- $\text{KP} \vdash \Delta_1\text{-Sep}$.

Definition (KP $+(\mathrm{F})$)

Let $\varphi(\vec{x}, y)$ be a Σ_1 formula such that KP $\vdash \forall \vec{x} \exists! y \varphi(\vec{x}, y)$. Then we can introduce a Σ_1 **operator** F by the following axiom: $\quad (\mathrm{F})\ :\ \forall \vec{x}\ (\mathrm{F}(\vec{x}) = y \leftrightarrow \varphi(\vec{x}, y))$.

Theorem (Σ_1 recursion)

Let G be a Σ_1 operator. There exists a Σ_1 operator F such that

$$\mathsf{KP} \vdash \mathrm{F}(x) = \mathrm{G}(x, \{(y, \mathrm{F}(y)) : y \in x\}).$$

Definition (Ordinals)

A Δ_0 predicate $\mathrm{Ord}(x) \equiv \mathrm{Tran}(x) \wedge \forall y \in x\ \mathrm{Tran}(y)$ expresses that x is an **ordinal**. The relation $\alpha < \beta$ on the ordinals is defined by $\alpha \in \beta$.

- By Σ_1-recursion, we can introduces various operators on ordinals. E.g., the addition $+$:

$$\alpha + \beta = \alpha \cup \sup\{(\alpha + \gamma) + 1 : \gamma < \beta\}.$$

Constructible sets

- By Σ_1 recursion, we define the class of **constructible sets**, which Gödel introduced as a model of the continuum hypothesis and the axiom of choice.
- We first introduce the definability predicate $\mathrm{Def}(x)$.
- Roughly speaking, $a \in \mathrm{Def}(u)$ means that a is a subset of u defined by some formula $\varphi(x)$. More precisely, there is a formula $\varphi(x, \vec{b})$ with parameters $\vec{b}$ from u s.t.

$$a = \{c \in u : u \models \varphi(c, \vec{b})\} = \{c \in u : \varphi(c, \vec{b})^u\}.$$

  Here, $\varphi(c, \vec{b})^u$ is a Δ_0 formula if a formula φ is fixed. However, since φ is treated as a variable by its Gödel number, the use of satisfaction relation $\models$ is inevitable.
- Then, we can rewrite it again as the following Δ_1 formula,

$$a \in \mathrm{Def}(u) \leftrightarrow \exists \varphi \in \omega \exists \vec{b} \in u (\forall c (c \in a \leftrightarrow c \in u \land u \models \varphi(c, \vec{b}))),$$

  where $u \models \varphi$ is a Δ_1 relation defined by recursion on the construction of formula φ.

Definition (Constructible sets)

A Σ_1 operater L_α on the ordinals are defined as follows.

$$\begin{cases} \mathrm{L}_0 := \varnothing \\ \mathrm{L}_{\alpha+1} := \mathrm{Def}(\mathrm{L}_\alpha) \\ \mathrm{L}_\alpha := \bigcup_{\beta<\alpha} \mathrm{L}_\beta \quad (\alpha \text{ is a limit ordinal}) \end{cases}$$

Let L denote a Σ_1 class $\bigcup_{\alpha\in\mathrm{Ord}} \mathrm{L}_\alpha$. The elements of L are called **constructible sets**.

- In KP, we can easily show from the definition of L_α that

$$\alpha \leq \beta \Leftrightarrow \mathrm{L}_\alpha \subset \mathrm{L}_\beta, \quad \alpha < \beta \Leftrightarrow \mathrm{L}_\alpha \in \mathrm{L}_\beta.$$

- Therefore, for any α, L_α is transitive, hence it is Δ_0-absolute. Moreover, if $\mathrm{L}_\alpha \models \mathsf{KP}$, it is Δ_1^{KP}-absolute. [i]

[i] For a set of formulas Γ, we say that a structure M is Γ-**absolute** if for any formula in Γ, it is true in the structure iff it is true (in V). Δ_1^T represents a formula shown to be Δ_1 in theory T.

In Gödel's original paper (1940), an operator $F : \mathrm{Ord} \to V$ is first introduced, and then L is defined as its range. His operator F can be characterized as follows.

Lemma

There is a Δ_1 (primitive recursive) bijection $F : \mathrm{Ord} \to \mathrm{L}$ such that for any ε number α (or ω), $F``\alpha = \mathrm{L}_\alpha$.

Proof idea.

- Suppose $F``\alpha = \mathrm{L}_\alpha$ and $\alpha \geq \omega$.
- Each element of $\mathrm{L}_{\alpha+1}$ is determined by a formula and parameters from L_α. Since there are countably many formulas and parameters are from $\mathrm{L}_\alpha = F``\alpha$, $\mathrm{L}_{\alpha+1}$ can be coded by $\alpha^{<\omega}$. Similarly, $\mathrm{L}_{\alpha+2}$ can be coded by $(\alpha^{<\omega})^{<\omega} = \alpha^{<\omega}$. Then, $\mathrm{L}_{\alpha+\omega}$ by α^ω and $\mathrm{L}_{\alpha+\omega+\omega}$ by $\alpha^{\omega+\omega}$. Finally, for the next ε number $\beta > \alpha$, L_β is coded by $\alpha^\beta = \beta$.
- All that remains is to convert this correspondence into an injection and guarantee that it is Δ_1 (primitively recursive). □

- In KPω, we can define a well-order $<_\mathrm{L}$ on L as a Δ_1 relation. From this, the axiom of choice (AC) holds in L. For more advanced topics, refer to standard textbooks on set theory (such as Jech [Jec]).

Set theory in second-order arithmetic

- We consider a model of set theory from the standpoint of second-order arithmetic.
- By the Löwenheim-Skolem theorem, for any set model $(V, \in)$, there exists a countable model $(\tilde{V}, \tilde{\in})$ which is elementary equivalent to $(V, \in)$. We can rewrite it as (ω, E) (with $E \subset \omega \times \omega$).
- Furthermore, when $\xi \in {}^{\omega}\omega$ expresses $E = \{(n, m) : \xi(n, m) \geq 1\}$, the formula $\xi \models \varphi$ denotes that φ is true in the structure (ω, E).
- The relation $\xi \models \varphi$ can be defined as a Δ^1_1 relation by the usual Tarski clause.
- Therefore, $\{\xi : \xi \models \text{ZFC}\}$ is also Δ^1_1.

- When "ω" in the structure expressed by ξ is isomorphic to the real ω, ξ is said to be an ω-**model**.
- More strictly, we argue as follows. First, "x is a finite ordinal" can be expressed as

$$\varphi(x) \equiv \forall y(0 \in y \land \forall z \in y(z \cup \{z\} \ \in y) \to x \in y).$$

- Then, ξ becomes an ω-model, if any n such that $\xi \models \varphi(n)$ is finite. That is, the following Δ^1_1 formula expresses that ξ is an ω-model.

$$\forall n(\xi \models \varphi(n) \to \exists (s_0, s_1, \ldots, s_k)(\xi \models s_0 = 0 \land \\ \forall i < k(\xi \models s_{i+1} = s_i \cup \{s_i\}) \land \xi \models s_k = n).$$

- Furthermore, "ξ is a **well-founded** model" can be expressed by the following Π^1_1 formula

$$\forall \eta \, \exists n \; [\xi \models \eta(n+1) \notin \eta(n)].$$

To further consider models of set theory, we introduce the following axiom called β. Here β stands for "well-order (French: bon ordre)" or "well-founded (bien fondée)".

Definition (Mostowski's Axiom β)

Axiom β asserts: for any well-founded relation $r \subset a \times a$ on any set a, there exists a function f whose domain is a and satisfies

$$f(x) = \{f(y) : y \in a \wedge (y, x) \in r\} \quad (\text{for all} x \in a).$$

Such an f is called a **collapsing function** of r.

- A collapsing function f of r has two roles. One is to express r as ϵ. So, moreover if r satisfies extensionality: $\forall b_1, b_2 \in a[\forall x \in a((x, b_1) \in r \leftrightarrow (x, b_2) \in r) \rightarrow b_1 = b_2]$, then $f\colon (a, r) \cong (\mathrm{range}(f), \in)$.
- The other role is to fold the relation by removing unnecessary elements and squeezing it into a smallest set. E.g., if $a = \{0, 2, \{1, 2\}\}$ with $r = \varepsilon$, then $f(a) = \{0, 1, \{1\}\}$.

- Remark that while Σ_1 recursion only defines a Σ_1 operator F, Axiom β asserts the existence of a collapsing function f.
- Axiom β is a very strong assertion, because the well-foundedness of r depends not only on a but also on the outside of a.

Theorem (1)

Axiom β holds in KP $+\Sigma_1\text{-Sep}$.

Proof

- Let r be a well-founded relation on a set a. We write $x <_r y$ for $(x, y) \in r$.
- We define a Σ_1 operator F by Σ_1 recursion over ordinals as follows.

$$\mathrm{F}(\alpha) = \{x \in a : \forall y \in a (y <_r x \to \exists \beta \in \alpha \; y \in \mathrm{F}(\beta))\}.$$

- First, to show $\bigcup_\alpha \mathrm{F}(\alpha) = a$, we put by Σ_1-Sep,

$$a_0 := \{x \in a : \exists \alpha (x \in \mathrm{F}(\alpha))\}.$$

- Suppose that $a - a_0 \neq \varnothing$. Let x be a $<_r$-minimal element in this set. Then,

$$\forall y \in a(y <_r x \to \exists \beta \; y \in \mathrm{F}(\beta)).$$

- Using Σ_1-Coll, we can obtain α such that

$$\forall y \in a(y <_r x \to \exists \beta {\in} \alpha \; y \in \mathrm{F}(\beta)).$$

- Therefore, by the definition of $\mathrm{F}(\alpha)$, we have $x \in \mathrm{F}(\alpha)$, i.e., $x \in a_0$, which contradicts the choice of x. Therefore, $\bigcup_\alpha \mathrm{F}(\alpha) = a$.
- Then again by Σ_1-Coll, there exists some γ such that $\bigcup_{\alpha<\gamma} \mathrm{F}(\alpha) = a$.
- Next, for $\alpha < \gamma$, by Σ_1 recursion, we define a sequence of functions $\{f_\alpha\}$ as follows.

$$f_\alpha(x) := \{f_\beta(y) : \beta < \alpha \wedge y \in \mathrm{F}(\beta) \wedge y <_r x\} \text{ (for any } x \in \mathrm{F}(\alpha)).$$

It is easy to show that if $\beta \leq \alpha$ then $f_\beta \subset f_\alpha$. So $f = \bigcup_{\alpha<\gamma} f_\alpha$ is defined as a function, and we have

$$f(x) = \{f(y) : y \in a \wedge y <_r x\} \quad \text{(for any } x \in a)$$

- Thus, Axiom β is shown.

Thank you for your attention!