Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 7. Schedule

- May 18, (1) KP set theory I
- May 23, (2) KP set theory II
- May 25, (3) α recursion theory
- May 30, (4) Recursively large ordinals I
- Jun. 1, (5) Recursively large ordinals II
- Jun. 6, (6) Second-order arithmetic and reverse mathematics

Today's topics

1. Recap
2. Some properties of KP
3. Σ_1 operator
4. Σ_1 recursion
5. Ordinals

Recap

- In set theory, the **Lévi hierarchy** is introduced by imitating the arithmetic hierarchy.
  - $\Sigma_0(=\Pi_0=\Delta_0)$ formula: all quantifiers are bounded, i.e., $\exists x\in y, \forall x\in y$.
  - A Σ_{n+1} formula is $\exists x\varphi$ with $\varphi\in\Pi_n$. A Π_{n+1} formula is $\forall x\varphi$ with $\varphi\in\Sigma_n$.
  - A Δ_n formula is a Π_n (or Σ_n) formula equivalent to a Σ_n (or Π_n) formula.

Definition

For a set Γ of formulas, the axioms of Γ-**separation** and Γ-**collection** are defined as

$$\begin{array}{lll} \Gamma\text{-Sep}: & \forall x\exists y\forall z(z\in y\leftrightarrow z\in x\wedge\varphi(z)) & \text{for any } \varphi(z)\in\Gamma. \\ \Gamma\text{-Coll}: & \forall x(\forall y\in x\,\exists z\varphi(z)\to\exists u\forall y\in x\exists z\in u\varphi(z)) & \text{for any } \varphi(z)\in\Gamma. \end{array}$$

- The axiom of Γ-separation asserts the existence of set $y=\{z\in x:\varphi(z)\}$. So, it is also called the subset axiom or the comprehension axiom.
- The axiom of Γ-collection can be viewed as a version of axiom of replacement, but also treated as a kind of reflection principle.

Definition (Axioms of KP)

KP is a first-order theory in the language $\{\in\}$ consisting of the following axioms.

$$
\begin{aligned}
\mathsf{KP} \; := \; & \text{axiom of extensionality}: \quad \forall z(z \in x \leftrightarrow z \in y) \to x = y \\
+ \; & \text{axiom of pairing}: \quad \forall x \forall y \exists z (x \in z \wedge y \in z) \\
+ \; & \text{axiom of union}: \quad \forall w \exists z \forall x \forall y (x \in y \wedge y \in w \to x \in z) \\
+ \; & \text{axiom of empty set}: \quad \exists y \forall x (x \notin y) \\
+ \; & \Delta_0\text{-Sep} \; + \; \Delta_0\text{-Coll} \\
+ \; & \text{axiom of foundation}: \quad \forall x [\forall y \in x\, \varphi(y) \to \varphi(x)] \to \forall x \varphi(x). \\
\mathsf{KP}\omega := \mathsf{KP} \; & + \text{axiom of infinity}: \quad \exists x \{0 \in x \wedge \forall y \in x (y \cup \{y\} \in x)\}.
\end{aligned}
$$

- ZF is KPω + power set $\forall x \exists z \forall y (y \subset x \to y \in z)$ + axiom of unrestricted separation and collection (or replacement). For ZF, the axiom of regularity: $x \neq \varnothing \to \exists y \in x (y \cap x = \varnothing)$ is often used instead of the axiom of foundation.

- In KP, the Σ_1 formulas are closed under bounded quantifiers. For instance, by using Δ_0-Coll, we have

$$\forall x \in y \exists z \varphi \leftrightarrow \exists u \forall x \in y \exists z \in u \varphi \quad (\varphi \in \Delta_0).$$

- In KP, the consecutive unbounded quantifiers in a formula can be combined into one by the axiom of pairing as follows

$$\exists x \exists y \varphi \leftrightarrow \exists u \exists x \in u \exists y \in u \varphi.$$

- Let Σ denote the smallest class of formulas containing Σ_1 formulas and is closed under $\wedge$, $\vee$, $\exists x \in y$, $\forall x \in y$, $\exists x$. In KP, the classes Σ and Σ_1 are essentially the same.
- One of Platek's original axioms is Σ **reflection principle**, stating that any Σ formula φ is equivalent to a special Σ_1 formula $\exists u \varphi^u$, where φ^u is obtained from φ by replacing all unbounded quantifiers $\exists x$, $\forall x$ by $\exists x \in u$ and $\forall x \in u$, respectively.

Theorem (Σ reflection principle)

KP $\vdash \varphi \leftrightarrow \exists u \varphi^u$ for any $\varphi \in \Sigma$.

- Note that for a Σ formula φ, KP proves $\varphi^u \wedge u \subset v \rightarrow \varphi^v$.

Some properties of KP

By the axiom of pairing, we can show the existence of **ordered pair**$(x, y) := \{\{x\}, \{x, y\}\}$. In addition, there exists the **direct product** $a \times b = \{(x, y) : x \in a, y \in b\}$ as follows.

Lemma

KP $\vdash \forall a, b \exists ! c (c = a \times b)$.

Proof.

- From the axiom of pairing and Δ_0-Sep, $\forall x \forall y \exists z [z = (x, y)]$.
- Δ_0-Coll gives $\forall x \exists w \forall y \in b \exists z \in w [z = (x, y)]$ and again by Δ_0-Coll, there exists d such that $\forall x \in a \exists w \in d \forall y \in b \exists z \in w [z = (x, y)]$.
- Now, by the axiom of union, letting $c_1 = \cup d$, we have $\forall x \in a \forall y \in b\ (x, y) \in c_1$.
- By Σ_0-Sep, there exists $c = \{z \in c_1 : \exists x \in a \exists y \in b [z = (x, y)]\}$.
- The uniqueness follows from the axiom of extensionality. □

Theorem

KP $\vdash \Sigma_1$-Coll.

Proof. Let φ be a Σ_1 formula. We want to show the following

$$\forall x(\forall y \in x \exists z \varphi \to \exists u \forall y \in x \exists z \in u \varphi).$$

We may assume φ is in the form $\exists w \theta$ (θ is Σ_0). By the axiom of pairing, we get the following.

$$\exists z\, \varphi = \exists z \exists w\, \theta \to \exists v \exists z \in v \exists w \in v\, \theta.$$

Then, by Δ_0-Coll,

$$\forall y \in x \exists z \varphi \to \exists u \forall y \in x \exists v \in u \exists z \in v \exists w \in v\ \theta.$$

Furthermore, setting $s = \bigcup u = \{z : \exists v \in u(z \in v)\}$, we have $\exists s \forall y \in x \exists z \in s\ \exists w\ \theta$. □

Theorem

KP $\vdash \Delta_1\text{-Sep}$.

Proof. Let a Σ_1 formula $\exists w\, \psi(w,x)$ (where ψ is Σ_0) and a Π_1 formula $\forall v\, \theta(v,x)$ (θ is Σ_0) be given so that $\forall x[\forall v\theta(v,x) \leftrightarrow \exists w\psi(w,x)]$ holds. That is, either is a Δ_1 formula. Then,

$$\forall x\exists w[\neg\theta(w,x) \vee \psi(w,x)].$$

By $\Delta_0\text{-Coll}$, for any y, there exists z such that

$$\forall x\in y\exists w\in z[\neg\theta(w,x) \vee \psi(w,x)].$$

Since $\forall x[\forall v\theta(v,x) \leftrightarrow \exists w\psi(w,x)]$, we have $\exists w\psi(w,x) \to \forall v \in z\theta(v,x) \to \exists w \in z\psi(w,x)$. So, $\{x\in y : \exists w\psi(w,x)\} = \{x\in y : \exists w\in z\psi(w,x)\}$ exists by $\Delta_0\text{-Sep}$. Therefore, $\Delta_1\text{-Sep}$ holds. □

Definition (KP $+(\mathrm{F})$)

Let $\varphi(\vec{x}, y)$ be a Σ_1 formula such that KP $\vdash \forall\vec{x}\exists! y\varphi(\vec{x}, y)$. Then we introduce a functional (operator) symbol F and call it a Σ_1 **operator** if the following axiom (F) holds.

$(\mathrm{F}) \ : \ \forall\vec{x}(\mathrm{F}(\vec{x}) = y \leftrightarrow \varphi(\vec{x}, y))$.

- KP $+(\mathrm{F})$ is a conservative extension of KP (i.e., the provability of formulas without F does not change in both systems).
- Axiom (F) is nothing but a definition. Strictly, its conservation is derived from the completeness theorem of first-order logic.
- Note that F is a second-order (meta-mathematical) object, called "class" or "functional", and so its existence cannot be argued in KP.
- It is easy to see that $\mathrm{F}(\vec{x}) = y$ is Δ_1 .
  - From the axiom (F), it is Σ_1.
  - Furthermore, F is Π_1 since $\mathrm{F}(\vec{x}) \neq y \leftrightarrow \exists z(\varphi(\vec{x}, z) \land z \neq y)$.

Lemma

Let F be a Σ_1 operater. The following sets exist in KP: for any set u,

$$\mathrm{F}\restriction u := \{(x, \mathrm{F}(x)) : x \in u\}, \quad \mathrm{F}``u := \{\mathrm{F}(x) : x \in u\}$$

Proof

- By Σ_1-Coll, there exists v such that $\forall x \in u \exists y \in v \mathrm{F}(x) = y$, and thus $\mathrm{F}\restriction u \subset u \times v$. Since F is Δ_1, $\mathrm{F}\restriction u$ exists by Δ_1-Sep.
- Similarly, the existence of $\mathrm{F}``u = \{y \in v : \exists x \in u\, \mathrm{F}(x) = y\}$ follows from Δ_1-Sep. □

Theorem (Σ_1 recursion)

Let G be a Σ_1 operater. There exists a Σ_1 operater F such that

$$\mathsf{KP} \vdash \mathrm{F}(x) = \mathrm{G}(x, \{(y, \mathrm{F}(y)) : y \in x\}).$$

Proof.

- First, we define a relation $\Phi(f)$ as follows.

  $\Phi(f) \equiv$ "f is a function" $\wedge$ "$\mathrm{dom}(f)$ is transitive" $\wedge \forall x \in \mathrm{dom}(f)(f(x) = \mathrm{G}(x, f \restriction x))$.

  Here $\Phi(f)$ roughly means that f is a function $\mathrm{F} \restriction \mathrm{dom} f$.

- "f is a function" is expressed as $\forall (x, y_1) \in f\ \forall (x, y_2) \in f\ (y_1 = y_2)$, which is Δ_0.
- "$\mathrm{dom}(f)$ is transitive" is $\forall y \in \mathrm{dom}(f)\ \forall z \in y\ (z \in \mathrm{dom}(f))$, which is also Δ_0.
- Since G is a Σ_1 operater, $\forall x \in \mathrm{dom}(f)(f(x) = \mathrm{G}(x, f \restriction x))$ is Δ_1.
- Thus, $\Phi(f)$ is also Δ_1.

- Then, $\mathrm{F}(x) = y$ can be expressed by the following Σ_1 formula Ψ.

$$\Psi(x, y) \equiv \exists f(\Phi(f) \wedge f(x, y))$$

- To show that F is a Σ_1 operater, we need to prove KP $\vdash \forall x \exists ! y \ \Psi(x, y)$

- First, we prove $\forall x \exists y \Psi(x, y)$ by way of contradiction. Assume that x exists such that $\neg \exists y \Psi(x, y)$.

- Then, if we choose a $\in$-minimal such x by the axiom of foundation, we get $\forall x' \in x \exists y \Psi(x', y)$, i.e.,

$$\forall x' \in x \exists f(\Phi(f) \wedge x' \in \mathrm{dom}(f)).$$

  Then, by Σ_1-Coll, there exists v such that

$$\forall x' \in x \exists f \in v(\Phi(f) \wedge x' \in \mathrm{dom}(f)).$$

- Now, let $w = \{f \in v | \Phi(f)\}$ by Δ_1-Sep. And let $u = \bigcup w$, by the axiom of union.
- We can show that u is a function. Otherwise, there exists f_1, f_2 such that $\Phi(f_1), \Phi(f_2)$ and there is $z \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$, $f_1(z) \neq f_2(z)$.
- By the axiom of foundation, we choose a $\in$-minimal such z. But then, $f_1(z) = G(x, f_1 \upharpoonright z) = G(x, f_2 \upharpoonright z) = f_2(z)$ from the definition of $\Phi(f)$, which contradicts our assumption.
- Then we have $\Phi(u)$.
- In addition, if $u' = u \cup \{(x, \mathrm{G}(x, u \upharpoonright x))\}$, then $\Phi(u')$ and $x \in \mathrm{dom}(u')$, and so $\exists y \Psi(x, y)$, which contradicts the choice of x.
- Finally, KP $\vdash \forall x \exists ! y \Psi(x, y)$ can be shown in the same way that we proved that u is a function as above.
- So F is a Σ_1 operater. □

- There are many applications of Σ_1 recursion. Let's look at a simple example.

Definition (Transitive closure)

For any set x, its **transitive closure** $\mathrm{TC}(x)$ is defined as follows.

$$\mathrm{TC}(x) := x \cup \bigcup \{\mathrm{TC}(y) : y \in x\}.$$

- $\mathrm{TC}(x)$ is well-defined as a Σ_1 operator.
- The property that x is **transitive**, denoted as $\mathrm{Tran}(x)$, is defined by

$$\forall y \in x \forall z \in y (z \in x).$$

Then, $\mathrm{TC}(x)$ is the smallest transitive set containing x.

The Σ_1 recursion works best on ordinal recursion.

Definition (Ordinal)

Define a Δ_0 predicate $\mathrm{Ord}(x)$ that expresses that x is an **ordinal** as follows.

$$\mathrm{Ord}(x) \equiv \mathrm{Tran}(x) \wedge \forall y \in x\ \mathrm{Tran}(y)$$

In addition, the relation $\alpha < \beta$ on the ordinals is defined as $\alpha \in \beta$.

The following facts can be easily shown in KP.

- $0 = \varnothing$ is the smallest ordinal.
- The successor order $\alpha + 1$ of ordinal α is $\alpha \cup \{\alpha\}$. In particular, the finite ordinal $n + 1$ is $\{0, 1, \ldots, n\}$.
- Each element of an ordinal is an ordinal.
- For a set of ordinals x, $\cup x = \sup x$ is an ordinal.
- $\leq$ on an ordinal is a total (linear) order.

By Σ_1-recursion, we can introduces various operaters on ordinals. E.g., the addition $+$:

$$\alpha + \beta = \alpha \cup \sup\{(\alpha + \gamma) + 1 : \gamma < \beta\}.$$

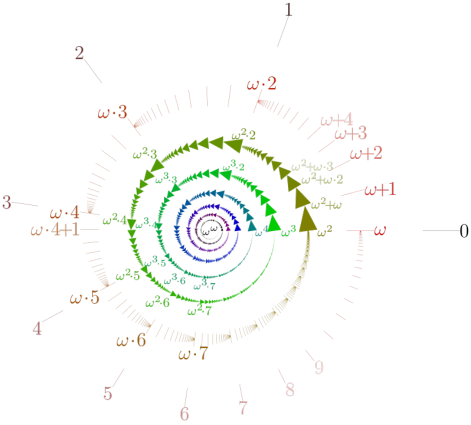
Homework

For ordinal addition $+$, show $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

- The smallest infinite ordinal is the set of all finite ordinals, written as ω, whose existence requires a system KPω containing the axiom of infinity.
- If we regard finite ordinals as natural numbers, ω and $\mathbb{N}$ are the same.
- In KPω, arithmetical quantifiers can be treated quantifiers bounded by ω (e.g., $\forall n \in \omega$), and so the arithmetical hierarchy has little effect on the set-theoretic hierarchy.

- Now we overview the ordinal numbers after ω.
- The successor of ω is $\omega + 1 = \omega \cup \{\omega\}$, and its successor is $(\omega + 1) + 1 = \omega + 2$.
- After that, there are infinite ordinals such as $\omega + 3$, $\omega + 4$, ... with the same **order type** as ω, and their limit is denoted by $\omega + \omega$ or $\omega \cdot 2$.
- The next similar limit ordinal is $\omega + \omega + \omega$ $(\omega \cdot 3)$, then $\omega + \omega + \omega + \omega$ $(\omega \cdot 4)$, etc.

- Let ω^2 be the limit after arranging the limit numbers like this. This is the next ordinal of ω closed under addition $+$. [a]
- Similarly, let ω^3 be the third ordinal closed with $+$, then ω^4, ω^5, . . . , and so on. Let those limit be denoted by ω^ω, which is also closed under addition $+$.
- In general, let ω^α be the α-th ordinal closed under addition $+$.[b]



source: wikipedia

[a] $\forall x, y < \omega^2 (x + y < \omega^2)$ or $\forall x < \omega^2 (x + \omega^2 = \omega^2)$.
[b] $\omega^0 = 1$ is considered the first (0-th) such ordinal.

- Then, we can also consider an ordinal α closed under ω^α, which is called a ε **number**.[i]
- The first ε number is called ε_0. An ordinal α smaller than ε_0 can be expressed uniquely as follows

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n} (\text{ However } \varepsilon_0 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0),$$

  which is called the **Cantor normal form**.
- Although ε_0 looks very large, the admissible ordinals that we will deal with later are much larger and are closed under all recursive functions.

[i] An ε number satisfies $\omega^\varepsilon = \varepsilon$.

Thank you for your attention!