Logic and Computation II

Part 6. Recursion-theoretic hierarchies

Kazuyuki Tanaka

BIMSA

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北京雁栖湖应用数学研究院
YANQI LAKE BEIJING INSTITUTE OF MATHEMATICAL SCIENCES AND APPLICATIONS

Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 7. Schedule

- May 18, (1) KP set theory I
- May 23, (2) KP set theory II
- May 25, (3) α recursion theory
- May 30, (4) Recursively large ordinals I
- Jun. 1, (5) Recursively large ordinals II
- Jun. 6, (6) Second-order arithmetic and reverse mathematics

Recap

- $T \subset {}^{\omega}\omega$ is said to be a **tree** if it is closed under initial segment, i.e.

$$\forall s \in T\ \forall t(t \subset s \to t \in T).$$

- A **path** P through T is a subtree with no branching, i.e., $\forall s, t \in P(t \subset s \vee s \subset t)$.
- We consider a partial order $\leq$ on ${}^{\omega}\omega$, defined by $t \leq s \Leftrightarrow s \subseteq t$. Then, in a tree, an infinite path $\varnothing = s_0 \subset s_1 \subset s_2 \subset \cdots$ is an infinite descending sequence. A tree with no infinite paths is said to be **well-founded**.

Theorem

A tree T is well-founded $\Longleftrightarrow$ there exists an ordinal number σ and a function $f : T \to \sigma + 1$ such that f is order-preserving $(s \subsetneq t \Leftrightarrow t < s \Leftrightarrow f(t) < f(s))$.

- Such an order-preserving function f is denoted as $f : T \xrightarrow{\text{o.p.}} \sigma + 1$ or $T \xrightarrow{f} \sigma + 1$.
- The **height** of T is the smallest ordinal number σ such that there exists $f : T \xrightarrow{\text{o.p.}} \sigma + 1$, represented by $||T||$.

Corollary

$||S|| \leq ||T||$ is Σ^1_1.
If T is a well-founded tree, then $\{S : ||S|| \leq ||T||\}$ is Δ^1_1.

- A tree $T \subset {}^{\omega}({}^{2}\omega)$ consisting of a finite sequence of ordered pairs of natural numbers is called a **tree of pairs**.
- It can also be viewed as a set of pairs of sequences s, t of the same length.
- Define the set of paths in a tree T of pairs

$$[T] := \{(\xi, \eta) \in {}^{2}({}^{\omega}\omega) : \forall m (\xi \restriction m, \eta \restriction m) \in T\}.$$

Corollary

For any Σ^1_1 formula $\varphi(\xi)$ there exists a primitive recursive pair-tree T such that

$$\varphi(\xi) \Leftrightarrow T^{\xi} \notin \mathrm{WF},$$

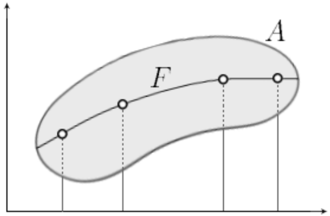
where $T^{\xi} := \{t \in {}^{\omega}\omega : (\xi \restriction \mathrm{leng}(t), t) \in T\}$.

Kondo's theorem

The final topic in this part is Kondo's theorem (1938). This result was appreciated by von Neumann and Gödel in their personal correspondence.
Kondo's original proof was very difficult, but Addison used Kleene's hierarchy to reformulate the statement and gave a concise proof.

Theorem (Addison's uniformization theorem)

If $A \subset {}^\omega\omega \times {}^\omega\omega$ is a Π^1_1 relation, then there exists a Π^1_1 function $F \subset A$ with the same domain, i.e., $\exists \eta A(\xi, \eta) \leftrightarrow \xi \in \mathrm{dom} F$. Such an F is said to **unifomize** A.



Proof(1/3).

- Let $A \subset {}^{\omega}\omega \times {}^{\omega}\omega$ be a Π^1_1 relation. Then there exists a computable tree T such that

$$(\xi, \eta) \in A \leftrightarrow \forall\gamma\exists n(\xi \restriction n, \eta \restriction n, \gamma \restriction n) \notin T \leftrightarrow T^{\xi,\eta} \in \mathrm{WF}.$$

- For finite sequences $s, t, \in {}^{n}\omega$, we define the following finite tree

$$T^{s,t} := \{u < n : (s \restriction \mathrm{leng}(u), t \restriction \mathrm{leng}(u), u) \in T\}.$$

  We here note that a finite sequence u is identified with its natural number code. We may assume that $\mathrm{leng}(u) \leq u (< n)$, and so $s \restriction \mathrm{leng}(u)$ and $t \restriction \mathrm{leng}(u)$ are well-defined.

- Then the following are obvious.

$$T^{\xi,\eta} \cap n = T^{\xi\restriction n, \eta\restriction n}, \ T^{\xi,\eta} = \bigcup_{n<\omega} T^{\xi\restriction n, \eta\restriction n}.$$

Proof(2/3).

- Next, we define a relation $R(s,t,v)$ which holds iff there exists n s.t. $s,t \in {}^n\omega$ and $v \in {}^n(\omega_1^{\mathrm{CK}})$ is an order-preserving function on $T^{s,t}$ and takes value 0 on the outside. So far, we do not claim that $R(s,t,v)$ is recursive, since it includes ω_1^{CK}.
  Now we have

$$
\begin{aligned}
(\xi,\eta) \in A \quad &\leftrightarrow \quad T^{\xi,\eta} \in \mathrm{WF} \ \leftrightarrow \ \exists f : T^{\xi,\eta} \xrightarrow{\text{o.p.}} \omega_1^{\mathrm{CK}} \\
&\leftrightarrow \quad \exists f \forall n f\restriction n : T^{\xi\restriction n,\eta\restriction n} \xrightarrow{\text{o.p.}} \omega_1^{\mathrm{CK}} \\
&\leftrightarrow \quad \exists f \forall n R(\xi\restriction n, \eta\restriction n, f\restriction n).
\end{aligned}
$$

- Fix a ξ and suppose that R^ξ has multiple paths (η,γ), i.e., $\forall n R(\xi\restriction n, \eta\restriction n, \gamma\restriction n)$.
- The key point of the proof is how to select $\eta = F(\xi)$ such that $(\xi,\eta) \in A$. We first select the leftmost path η_0 such that $R^\xi(\eta_0,\gamma)$ for some γ. Then, select the leftmost path γ_0 such that $R^\xi(\eta_0,\gamma_0)$. Noticing that η_0 is still the leftmost path η such that $R^\xi(\eta,\gamma_0)$, we can show F is Π^1_1.
- Thus, F uniformizes A.

Proof(3/3).

- Next we show F is Π^1_1. Assume $F(\xi)=\eta$. Since $(\xi,\eta)\in A$, a function $f : T^{\xi,\eta} \xrightarrow{\text{o.p.}} \omega_1^{\text{CK}}$ exists. Without loss of generality, we may assume

$$f(u) = ||T^{\xi,\eta}_u|| \text{ if } u \in T^{\xi,\eta},$$

$$f(u) = ||T^{\xi,\eta}_u|| = 0, \text{ otherwise.}$$

- Then f is Δ^1_1 in ξ,η, and it is the leftmost such that $R(\xi,\eta,f)$.
- Finally, the selection of the leftmost path η is expressed as follows:

$$\begin{aligned} F(\xi)=\eta \quad \Leftrightarrow \quad & T^{\xi,\eta} \in \mathrm{WF} \wedge \\ & \forall \eta' \forall n \{[\eta\restriction n = \eta'\restriction n \wedge \forall k < n ||T^{\xi,\eta}_k|| = ||T^{\xi,\eta'}_k||] \\ & \to [\eta(n) < \eta'(n) \vee (\eta(n) = \eta'(n) \wedge ||T^{\xi,\eta}_n|| \leq ||T^{\xi,\eta'}_n||)]\}. \end{aligned}$$

□

Corollary (Kondo)

A $\underset{\sim}{\Pi}^1_1$ set $A \subset {}^\omega\omega \times {}^\omega\omega$ can be uniformized (by a $\underset{\sim}{\Pi}^1_1$ function).

Homework

For two Σ^1_1 sets A and B that $A \cap B = \varnothing$ in Baire spaces, show that there exists a Δ^1_1 set C that separates them, i.e., $A \subset C \wedge B \cap C = \varnothing$.

Homework

(1) Show that there is a Σ^1_1 set that cannot be uniformized.
(2) Show that the Σ^1_2 set can be uniformized (by the Σ^1_2 function).

Further Reading

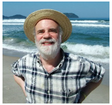
- H. Rogers, *Theory of Recursive Functions and Effective Computability*, The MIT Press, 5th edition, 1987

Today's topics

1. Recap
2. Introduction
3. Lévi hierarchy
4. Separation and collection
5. KP
6. Σ reflection

Introduction

- KP **set theory**, introduced by Kripke and Platek, is a generalization of Kreisel and Sacks' recursion theory on ω_1^{CK} (meta-recursion theory).
- It is an extension to the theory of computational structures or constructive properties on arbitrary ordinals and sets.
- The sets subject to the theory are called **admissible sets**, and the KP set theory that describes the world is obtained from the well-known Zermelo-Frenkel set theory (ZF set theory) by removing non-constructive axioms.
- In other words, KP removes the axiom of infinity and power set axioms from ZF, and further restricts the separation axiom schema and replacement axiom schema to logical expressions whose quantifiers are bounded.
- Without the axiom of infinity, we can only guarantee the existence of a finite set, so KPω with an axiom of infinity is often used.


S. Kripke


R. Platek


G. Kreisel


G.E. Sacks

To state precisely the axioms of KP set theory, we first define a hierarchy of formulas in set theory. Both KP and ZF are first-order theories in the language consisting only of relational symbol $\in$, and various set concepts are introduced by definition.

A set theory hierarchy, called the **Lévi hierarchy**, is introduced by imitating the arithmetic hierarchy.

Azriel Lévy

Since the same symbols Σ_n and Π_n are used for both the hierarchies, we will always use Σ^0_n and Π^0_n for the arithmetic hierarchy from now on. [i]

Definition (Lévy hierarchy)

- $\Sigma_0 (= \Pi_0 = \Delta_0)$ formula: all quantifiers are bounded, i.e., $\exists x \in y, \forall x \in y$.
- A Σ_{n+1} formula is in the form of $\exists x \varphi$ with φ a Π_n formula. A Π_{n+1} formula is in the form of $\forall x \varphi$ with φ a Σ_n formula.
- A Δ_n formula is a Π_n formula that is equivalent to Σ_n or a Σ_n formula that is equivalent to a Π_n formula.

[i] Set theory also handles second-order hierarchy Σ^1_n or Π^1_n sometimes. In such a case, Levy's hierarchy may be expressed as Σ^0_n or Π^0_n.

Definition

For a set Γ of formulas, the axioms of Γ-**separation** and Γ-**collection** are defined as follows.

$$\begin{array}{lll} \Gamma\text{-Sep}: & \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \land \varphi(z)) & \text{for any } \varphi(z) \in \Gamma. \\ \Gamma\text{-Coll}: & \forall x (\forall y \in x \, \exists z \varphi(z) \to \exists u \forall y \in x \exists z \in u \varphi(z)) & \text{for any } \varphi(z) \in \Gamma. \end{array}$$

- The axiom of Γ-separation asserts the existence of set $y = \{z \in x : \varphi(z)\}$. From this, it is easy to see that for any set a, b, there exists an **intersection** of them

$$a \cap b = \{x \in a : x \in b\}.$$

- The axiom of Γ-collection can be regarded as a weak version of the axiom of replacement, but it is often treated as a kind of reflection principle, which will be discussed later.

Definition (Axioms of KP)

KP is a first-order theory of the language with only relational symbols $\in$, and consists of the following axioms.

$$
\begin{aligned}
\mathsf{KP} \quad := &\ \text{axiom of extensionality}: \quad \forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y \\
+ &\ \text{axiom of pairing}: \quad \forall x \forall y \exists z(x \in z \wedge y \in z) \\
+ &\ \text{axiom of union}: \quad \forall w \exists z \forall x \forall y(x \in y \wedge y \in w \rightarrow x \in z) \\
+ &\ \text{axiom of empty set}: \quad \exists y \forall x(x \notin y) \\
+ &\ \Delta_0\text{-Sep} + \Delta_0\text{-Coll} \\
+ &\ \text{axiom of foundation}: \quad \forall x[\forall y \in x\, \varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x \varphi(x). \\
\mathsf{KP}\omega := &\ \mathsf{KP} + \text{axiom of infinity}: \quad \exists x\{0 \in x \wedge \forall y \in x(y \cup \{y\} \in x)\}.
\end{aligned}
$$

- The axiom of pairing asserts the existence of a set z such that $\{x, y\} \subset z$. Then, by using Δ_0-Sep, there exists

$$\mathbf{pair}\{x, y\} := \{w \in z : w = x \vee w = y\}.$$

The uniqueness of this set follows from the axiom of extensionality.

- The axiom of union asserts that there exists a set z such that

$$\cup w := \{x : \exists y (x \in y \wedge y \in w)\} \subset z.$$

Using Δ_0-Sep and the axiom of extensionality, **union** $\cup w$ uniquely exists.

- ZF is KPω + power set $\forall x \exists z \forall y (y \subset x \to y \in z)$ + axiom of unrestricted separation and collection (or replacement).

- For ZF, the axiom of regularity: $x \neq \varnothing \to \exists y \in x (y \cap x = \varnothing)$ is often used instead of the axiom of foundation. Note that the axiom of regularity is equivalent to the axiom of foundation for quantifier-free $\varphi(x)$, but also equivalent to the unrestricted foundation with help of the unrestricted separation.

Σ reflection

Lemma (1)

"The Σ_1 formulas are closed under bounded quantifiers" is provable in KP. The consecutive unbounded quantifiers in front of a Σ_1 formula (a Π_1 formula) can be combined into one.

Proof.

- The essential step in the first half is

$$\forall x \in y \exists z \varphi \leftrightarrow \exists u \forall x \in y \exists z \in u \varphi \quad (\varphi \in \Delta_0),$$

  which is obvious from Δ_0-Coll.

- For the second half, we can use the axiom of pairing to combine the consecutive unbounded quatifiers of the same kind into one as follows

$$\exists x \exists y \varphi \leftrightarrow \exists u \exists x \in u \exists y \in u \varphi.$$

- Let Σ denote the smallest class of formulas containing Σ_1 formulas and is closed under $\wedge$, $\vee$, $\exists x \in y$, $\forall x \in y$, $\exists x$.
- Lemma (1) shows that in KP, the classes Σ and Σ_1 are essentially the same. One of Platek's original axioms of KP is Σ **reflection principle**, stating that any Σ formula φ is equivalent to a special Σ_1 formula $\exists u \varphi^u$, where φ^u is obtained from φ by replacing all unbounded quantifiers $\exists x$, $\forall x$ with $\exists x \in u$ and $\forall x \in u$, respectively.
- Also, φ^u is often denoted as $u \models \varphi$, though free variables of φ may not be evaluated by elements of u.

Theorem (Σ reflection principle)

KP $\vdash \varphi \leftrightarrow \exists u \varphi^u$ for any $\varphi \in \Sigma$.

- First, note that for a Σ formula φ, KP proves $\varphi^u \wedge u \subset v \to \varphi^v$. This can be shown by induction on the construction of formulas. Since only the difference between φ^u and φ^v is that $\exists x \in u$ in φ^u is changed to $\exists x \in v$ in φ^v. Obviously, the key induction step $\exists x \in u \theta^u \wedge u \subset v \to \exists x \in v \theta^v$ holds.

Proof.

- By induction on the construction of formula φ.
- We may consider the following induction steps.
  - The case $\varphi = \forall x \in y \psi$:
    By the induction hypothesis $\psi \leftrightarrow \exists v \psi^v$, so
    $\varphi \leftrightarrow \forall x \in y \exists v \psi^v \leftrightarrow \exists w \forall x \in y \exists v \in w \psi^v$ holds in KP.
    Let $u = \cup w$. $v \in w$ means $v \subset u$. So $\exists w \forall x \in y \exists v \in w \psi^v \to \exists u \forall x \in y \psi^u$.
    On the other hand, $\exists u \forall x \in y \psi^u \to \forall x \in y \exists v \psi^v$ is obvious, so $\varphi \leftrightarrow \exists u \varphi^u$.
  - The case $\varphi = \exists x \psi$:
    By the induction hypothesis, $\psi \leftrightarrow \exists v \psi^v$, so if we set $u = v \cup \{x\}$,
    $\exists x \psi \to \exists u \exists x \in u \psi^u$.
    Conversely, $\exists u \exists x \in u \psi^u \to \exists x \exists u \psi^u \to \exists x \psi$. □

Thank you for your attention!