Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Apr.25, (1) Oracle computation and relativization
- Apr.27, (2) m-reducibility and simple sets
- May 4, (3) T-reducibility and Post's problem
- May 9, (4) Arithmetical hierarchy and polynomial-time hierarchy
- May 11, (5) Analytical hierarchy and descriptive set theory I
- May 16, (6) Analytical hierarchy and descriptive set theory II

Recap

- We identify a natural number n with the set $\{0, 1, \ldots, n-1\}$, and denote the set of natural numbers by $\omega = \{0, 1, \ldots\}$. By $X, Y, \ldots$, we will usually denote subsets of ω.
- Let ${}^X Y$ denote the set of functions from X to Y, read as "Y-pre-X".
- Moreover, we define
$$ {}^{\underline{\omega}}X := X^{<\omega} = \bigcup_{n \in \omega} {}^n X. $$
- For $\xi \in {}^\omega X$ or $\xi \in {}^n X (n \geq m)$, a sequence $(\xi(0), \xi(1), \ldots, \xi(m-1))$, denoted $\xi \restriction m$ or $\xi[m]$, is called an **initial segment** of ξ (with length m).
- For $s \in {}^{\underline{\omega}}\omega$, let $[s] = \{\xi \in {}^\omega \omega : s \subset \xi\}$. $\{[s] : s \in {}^{\underline{\omega}}\omega\}$ is an **open base** of the Baire space ${}^\omega \omega$.
- A set $G \subset {}^\omega \omega$ is **open** if there exists some $A \subset {}^{\underline{\omega}}\omega$ such that $G = \bigcup_{s \in A} [s]$.
- The complement of an open set is called **closed**.

- We say that a set $G \subset {}^\omega\omega$ is $\Sigma^0_1(\xi)$ if there exists a ξ-CE set A s.t. $G = \bigcup_{s \in A}[s]$, or equivalently there exists a Σ^0_1 formula φ s.t. $G = \{\eta \in {}^\omega\omega : \varphi(\eta, \xi)\}$.
- The class $\mathcal{G}$ of open sets coincides with $\bigcup_\xi \Sigma^0_1(\xi)$, which is denoted as $\mathbf{\Sigma}^0_1$ or $\underset{\sim}{\Sigma}^0_1$.
- A set $F \subset {}^\omega\omega$ is $\Pi^0_1(\xi)$ if its complement is $\Sigma^0_1(\xi)$. The class of closed sets $\mathcal{F} = \bigcup_\xi \Pi^0_1(\xi)$ is denoted as $\mathbf{\Pi}^0_1$ or $\underset{\sim}{\Pi}^0_1$.
- Also, the class of countable unions of closed sets $\mathcal{F}_\sigma = \bigcup_\xi \Sigma^0_2(\xi)$ is $\mathbf{\Sigma}^0_2$ or $\underset{\sim}{\Sigma}^0_2$.
- An **analytic set** is obtained as a projection of a a closed set, the class of such sets $\mathcal{A} = \bigcup_\xi \Sigma^1_1(\xi)$ is denoted as $\mathbf{\Sigma}^1_1$ or $\underset{\sim}{\Sigma}^1_1$.
- The class of **co-analytic set** $\mathcal{CA} = \bigcup_\xi \Pi^1_1(\xi)$ is denoted as $\mathbf{\Pi}^1_1$ or $\underset{\sim}{\Pi}^1_1$.
- The class of projections of co-analytic set is $\mathcal{PCA} = \bigcup_\xi \Sigma^1_2(\xi)$ is written as $\mathbf{\Sigma}^1_2$ or $\underset{\sim}{\Sigma}^1_2$.
- The finite hierarchy of such **projective sets** corresponds with the analytical hierarchy (with arbitrary oracles).

Today's topics

1. Recap
2. Linear orders and well-orders
3. Kleene's $\mathcal{O}$

- A pair (m, n) is identified with its code $\frac{(m+n)(m+n+1)}{2} + m$. We say that $\xi (\in {}^{\omega}\omega)$ is a **linear order** (abbreviated as LO) if

$$\{(m, n) : \xi(m, n) \geq 1\} \text{ is a linear ordering on } \mathbb{N}.$$

We denote $\xi(m, n) \geq 1$ by $m \leq_\xi n$ or simply $m \leq n$.

- A linear order with no infinite descending sequence is called a **well-order** (abbreviated as WO). By using $\leq$, we rewrite it as

$$\mathrm{WO}(\leq) \Leftrightarrow \mathrm{LO}(\leq) \wedge \forall \eta \exists n(\eta(n) \leq \eta(n+1)).$$

Note that this expression is Π^1_1.

- A finite sequence $s = (s_0, s_1, \ldots, s_{n-1}) \in {}^{\underset{\smile}{\omega}}\omega$ can also be identified with a code. Then, the concatenation of two sequences $s * t$ is a binary operation of natural numbers. A relation $t \subset s$, defined by $\exists u(t * u = s)$, represents "t is an initial segment of s".

- $T \subset {}^{\omega}\omega$ is said to be a **tree** if it is closed under initial segment, i.e.

$$\forall s \in T\ \forall t(t \subset s \to t \in T)$$

- A **path** P through T is a subtree with no branching, i.e., $\forall s, t \in P(t \subset s \vee s \subset t)$.
- We consider a partial order $\leq$ on ${}^{\omega}\omega$, defined by $t \leq s \Leftrightarrow s \subseteq t$. Then, in a tree, an infinite path $\varnothing = s_0 \subset s_1 \subset s_2 \subset \cdots$ is an infinite descending sequence. So, a tree with no infinite paths is said to be **well-founded**.
- The well-foundedness of a tree T can be expressed by the following Π^1_1 formula,

$$\mathrm{WF}(T) \Leftrightarrow \neg\exists f \forall n(f(n) \in T \wedge f(n) \subset f(n+1))$$

- In a tree T, a node $s^\wedge k = s * (k) \in T$ is called a **child** of s.
- A tree T is said to be **finitely branching** if every $s \in T$ has only finitely many children.
- For $s \in T$, the subtree rooted at s is written as $T_s = \{t : s * t \in T\}$.

Theorem (König's lemma)

Any finitely branching infinite tree T has an infinite path.

Recap: Ordinals and well-founded trees

- Ordinals can be considered as an extension of the natural numbers to enumarate infinite sets. The natural numbers (the finite ordinals) are defined as:

$$0 := \varnothing,\ 1 := \{0\},\ 2 := \{0,1\}, \ldots, \text{ in general, } n+1 := n \cup \{n\}.$$

- Any ordinal σ is the set of ordinals less than σ. So, $<$ on the ordinals is expressed as $\in$.
- An ordinal $\sigma + 1$ (the **successor** of σ) is defined as $\{0, 1, \ldots, \sigma\}$. An ordinal which is not a successor is called a **limit** ordinal.
- The first limit ordinal (except for 0) is the set of finite ordinals, denoted ω or ω_0. The second limit ordinal is the limit of $\omega + n$, denoted $\omega + \omega = \omega \cdot 2$. After that, come $\omega \cdot 3$, $\ldots$, $\omega \cdot \omega = \omega^2$, ω^3, $\ldots$, ω^ω, etc.

Theorem

A tree T is well-founded $\Longleftrightarrow$ there exists an ordinal number σ and a function $f : T \to \sigma + 1$ such that f is order-preserving $(s \subsetneq t \Leftrightarrow t < s \Leftrightarrow f(t) < f(s))$.

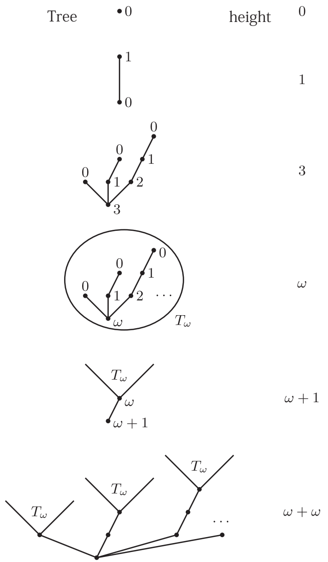
Such an order-preserving function f is denoted as $f : T \xrightarrow{\text{o.p.}} \sigma + 1$ or $T \xrightarrow{f} \sigma + 1$.

Definition

- The **height** of T is the smallest ordinal number σ such that there exists $f : T \xrightarrow{\text{o.p.}} \sigma + 1$, represented by $||T||$.
- If T is a recursive well-founded tree, $||T||$ is said to be **computable**.
- In addition, set $||T|| = -1$ when T is empty, and set $||T|| = \infty$ when T is not well-founded.

Example

Right-hand-ride is a typical well-founded tree T. Each vertex has an order-preserving ordinal, and its height $||T||$ is shown on the right side.



If f is an order-preserving function of a tree T, then $f_s(t) = f(s * t)$ is an order-preserving function of subtree $T_s = \{t : s * t \in T\}$, and $f(s) = f_s(\varepsilon) \geq ||T_s||$.

Theorem

For any T, $||T|| = \sup_{s \neq \varepsilon}(||T_s|| + 1)$.

Proof.

- If T is not well-founded, then both sides are $+\infty$. Therefore, we assume T is well-founded, and suppose $\sigma = ||T||$ and $f : T \xrightarrow{\text{o.p.}} \sigma + 1$.
- If $s \neq \varepsilon$, then $||T_s|| \leq f(s) < f(\varepsilon) = ||T||$. So, $\sup(||T_s|| + 1) \leq ||T||$.
- Suppose $\sigma = \sup(||T_s|| + 1) < ||T||$.
- We define a function $h : T \to \sigma + 1$ as

$$h(s) = \begin{cases} ||T_s|| & \text{if } s \neq \varepsilon \\ \sigma & \text{if } s = \varepsilon. \end{cases}$$

- Then h is order-preserving, and so $||T|| \leq \sigma$, which is a contradiction.
- Hence, $||T|| = \sigma$.

□

Theorem

Let S, T be trees. $||S|| \leq ||T|| \Leftrightarrow$ there exists an order-preserving function f from S to T.

Proof.
($\Leftarrow$) It is clear when $||T|| = \infty$. When $||T|| < \infty$, there exists h such that $T \xrightarrow{h} ||T|| + 1$. Moreover, by $S \xrightarrow{f} T$, we have $h \circ f : S \xrightarrow{\text{o.p.}} ||T|| + 1$, which implies $||S|| \leq ||T||$.

($\Rightarrow$) First, consider the case $||T|| = \infty$. Then, let $s_0 \subset s_1 \subset \cdots = \xi$ be an infinite path through T. We enumerate the elements of ${}^{\omega}2$ as $\{b_i\}$ such that the length of $b_i \leq$ the length of b_j if $i < j$. Then, define a function $h : {}^{\omega}2 \to \{s_i\}$ by $h(b_i) = s_i$ for all i. Obviously, $h : {}^{\omega}2 \to \{s_i\}$ is an order-preserving from ${}^{\omega}2$ to T. We also define an order-preserving injection $g : {}^{\omega}\omega \to {}^{\omega}2$ as, for instance,

$$(l, m, n) \mapsto (\overbrace{1, 1, \ldots, 1}^{l}, 0, \overbrace{1, 1, \ldots, 1}^{m}, 0, \overbrace{1, 1, \ldots, 1}^{n}).$$

Then, $h \circ g : {}^{\omega}\omega \xrightarrow{g} {}^{\omega}2 \xrightarrow{h} T$ is also order-preserving, and so $f = h \circ g$ is an order-preserving function f from S to T..

Next suppose $||T|| < \infty$. We will prove by induction on $||S||$.

- When $||S|| = 0$, S is a singleton set, and then any function from S to T is order-preserving.
- Assume $||S|| > 0$. Then, for each $(n) \in S$, $||S_n|| < ||S|| \leq ||T|| = \sup_{t \neq \varepsilon}(||T_t|| + 1)$.
  - Then, for each $(n) \in S$, take t_n such that $||S_n|| \leq ||T_{t_n}||$.
  - By the induction hypothesis, there exists a $f_n : S_n \to T_{t_n}$.
  - Then, we define a function f as follows

$$f(s) = \begin{cases} \varepsilon & \text{if } s = \varepsilon \\ t_n * f_n(t) & \text{if } s = n^\wedge t \in S. \end{cases}$$

  - Thus f is an order-preserving function from S to T. □

Corollary

"$||S|| \leq ||T||$" is Σ^1_1.

Proof. We can easily see that
$A = \{(\xi, \eta) : \xi, \eta$ are codes of S, T respectively, and $||S|| \leq ||T||\}$ is Σ^1_1, since

$$(\xi,\eta) \in A \quad \Leftrightarrow \quad \exists f \forall s, t(\xi(s)\cdot\xi(t) \geq 1 \wedge s \subset t \\ \to \eta(f(s))\cdot\eta(f(t)) \geq 1 \wedge f(s) \subset f(t)).$$

Note that variables s, t are treated as number variables, and f as a function variable. □

Corollary

If T is a well-founded tree, then $\{S : ||S|| \leq ||T||\}$ is Δ^1_1.

Proof. Assume T is a well-founded tree. Then

$$||S|| \not\leq ||T|| \Leftrightarrow ||T|| < ||S|| \Leftrightarrow \underbrace{\exists n\ ||T|| \leq ||S_n||}_{\Sigma^1_1}$$

So "$||S|| \leq ||T||$" is Δ^1_1. Note that if we allow $||T|| = \infty$, the rightmost formula does not imply the middle formula. □

- A tree $T \subset {}^{\omega}({}^{2}\omega)$ consisting of a finite sequence of ordered pairs of natural numbers is called a **tree of pairs**.
- It can also be viewed as a set of pairs of sequences s, t of the same length.
- Define the set of paths in a tree T of pairs

$$[T] := \{(\xi, \eta) \in {}^{2}({}^{\omega}\omega) : \forall m(\xi \restriction m, \eta \restriction m) \in T\}.$$

Theorem (Recall: Normal form theorem for analytical formulas, Lecture04-06)

For each $i \geq 1$, for any Σ^1_i formula (or Π^1_i formula), there exists an equivalent $\mathrm{fnc}\text{-}\Sigma^1_i$ formula (or $\mathrm{fnc}\text{-}\Pi^1_i$ formula) whose arithmetical part is Σ^0_1 or Π^0_1.

- So, let $\varphi(\xi)$ be any Σ^1_1 formula of the normal form $\exists\eta\forall k R(e, \xi \restriction k, \eta \restriction k)$ (R is primitive recursive). We put

$$T_e = \{(s, t) : \mathrm{leng}(s) = \mathrm{leng}(t) \wedge R(e, s, t)\}$$

where $\mathrm{leng}(x)$ denotes the length of x, and T_e can be viewed as the e-th tree of pairs.
- Finally, $\varphi(\xi)$ can be expressed as

$$\varphi(\xi) \leftrightarrow \exists\eta\ (\xi, \eta) \in [T_e].$$

Corollary (1)

For any Σ^1_1 formula $\varphi(\xi)$ there exists a primitive recursive pair tree T_e such that

$$\varphi(\xi) \Leftrightarrow T_e^\xi \notin \mathrm{WF},$$

where $T_e^\xi := \{t \in {}^\omega\omega : (\xi \restriction \mathrm{leng}(t), t) \in T_e\}$.

Corollary (2)

WF is Π^1_1, but not $\utilde{\Sigma}^1_1$.

Proof. The first half is clear from Corollary (1). To show the second half, assume for the contrary that WF be $\utilde{\Sigma}^1_1$. Then there exists a Σ^1_1 formula $\varphi(\xi, \eta)$ and an oracle γ such that $\varphi(\xi, e^\wedge\gamma) \Leftrightarrow T_e^\xi \in \mathrm{WF}$ for all e and ξ.
Again by Corollary (1), there exists some d such that $\varphi(\xi, \xi) \Leftrightarrow T_d^\xi \notin \mathrm{WF}$. Since $T_d^\xi \notin \mathrm{WF} \Leftrightarrow \neg\varphi(\xi, d^\wedge\gamma)$, letting $\xi = d^\wedge\gamma$, we reaches a contradiction. □

For an ordinal σ, let $\mathrm{WF}_\sigma \equiv \{T : ||T|| \leq \sigma\}$.

Theorem (Σ^1_1 boundedness theorem)

If A is a Σ^1_1 subset of WF, then there exists a $\sigma(< \omega_1^{\mathrm{CK}})$ such that $A \subset \mathrm{WF}_\sigma$.

The ordinal ω_1^{CK}(read as "omega-1 Church Kleene") is the upper bound of recursive ordinal numbers, i.e., the smallest non-recursive (countable) ordinals.

Proof. Let A be a Σ^1_1 subset of WF. By way of contradiction, assume $\forall \sigma < \omega_1^{\mathrm{CK}} \exists S \in A(||S|| \geq \sigma)$. Then,

$$T \in \mathrm{WF} \Leftrightarrow \exists S[S \in A \wedge ||S|| \geq ||T||].$$

So WF is Σ^1_1, which contradicts Corollary (2). □

Kleene's $\mathcal{O}$

- So far, we have treated countable ordinals ($< \omega_1^{\mathrm{CK}}$) in terms of recursive well-founded trees. There is another way to express them as natural numbers. Here, we will introduce Kleene's notation system $\mathcal{O}$.
- Kleene's system consists of set $\mathcal{O}$ of natural numbers and a binary relation $<_{\mathcal{O}}$ on it. Then, $(\mathcal{O}, <_{\mathcal{O}})$ is the minimal transitive set satisfying
  (1) $1 \in \mathcal{O}$
  (2) If $y \in \mathcal{O}$, then $y <_{\mathcal{O}} 2^y$
  (3) If a recursive function $\{y\}(n)$ is monotonically increasing with respect to $<_{\mathcal{O}}$, then for all n, $\{y\}(n) <_{\mathcal{O}} 3 \cdot 5^y$.
- Then $\mathcal{O}$ and $<_{\mathcal{O}}$ are m-complete Π^1_1 sets.
- The ordinal number $|a|$ represented by $a \in \mathcal{O}$ is determined inductively as follows:

$$|1| = 0, \ |2^y| = |y| + 1, \ |3 \cdot 5^y| = \lim_n |\{y\}(n)|.$$

- Then, for a recursive well-founded tree T, there exists $y \in \mathcal{O}$ such that $||T|| = |y|$.

- Kleene used the notation system $\mathcal{O}$ to extend the arithmetic hierarchy on the set of natural numbers to the transfinite hierarchy. For each $y \in \mathcal{O}$, we define the set H_y of natural numbers as follows:

$$H_1 := \varnothing,\ H_{2^y} := H'_y \text{ (jump)},\ H_{3\cdot 5^y} := \{(x,n) : x \in H_{\{y\}(n)}\}.$$

- A set of natural numbers that can be computed with H_y $(y \in \mathcal{O})$ as an oracle is called **hyperarithmetic**.
- Then it can be shown that Hyp, the class of hyperarithmetic sets, coinsides with the class of Δ^1_1 sets (reference [Rog]).
- The hyperarithmetic hierarchy Σ_α in the Baire space ${}^\omega\omega$ is defined by relativizing the hierarchy $\{H_y\}$ as follows: $R(\subset {}^\omega\omega)$ is $\Sigma_{|y|}$ if there exists some $e \in \omega$ such that

$$\xi \in R \Leftrightarrow e \in H^\xi_y$$

- Even in the Baire space, the class of hyperarithmetic sets coincides with the class of Δ^1_1 sets (see **Souslin-Kleene theorem**, [Rog], page 454). .

Thank you for your attention!