Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Apr.25, (1) Oracle computation and relativization
- Apr.27, (2) m-reducibility and simple sets
- May 4, (3) T-reducibility and Post's problem
- May 9, (4) Arithmetical hierarchy and polynomial-time hierarchy
- May 11, (5) Analytical hierarchy and descriptive set theory I
- May 16, (6) Analytical hierarchy and descriptive set theory II

Today's topics

Recap

- The **relativized arithmetical hierarchy** (with oracle $\xi \in \mathbb{N}^{\mathbb{N}}$) for subsets of $\mathbb{N}$ is defined as follows.

$$\begin{array}{rcl} \Sigma_1(\xi) & := & \{\xi\text{-CE sets}\}, \\ \Delta_1(\xi) & := & \{\xi\text{-computable sets}\}, \\ \Sigma_{n+1}(\xi) & := & \{A \mid A \text{ is CE in some } B \in \Sigma_n(\xi)\}, \\ \Delta_{n+1}(\xi) & := & \{A \mid A \text{ is computable in some } B \in \Sigma_n(\xi)\}, \\ \Pi_n(\xi) & := & \{\text{the complement of sets in } \Sigma_n(\xi)\} \end{array}$$

  When ξ is computable, we omit to mention (ξ) or ξ, and they are the usual classes in the arithmetical hierarchy.
- We write $A \leq_{\mathrm{m}} B$ if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for any $x \in \mathbb{N}$, $x \in A \Leftrightarrow f(x) \in B$.
- Let $\mathcal{C}$ be a class of subsets of $\mathbb{N}$.
  A set B is said to be $\mathcal{C}$-**hard** if for every $A \in \mathcal{C}$, $A \leq_{\mathrm{m}} B$.
  A set B is said to be $\mathcal{C}$-**complete** if B is $\mathcal{C}$-hard and $B \in \mathcal{C}$.

Now, the following are typical $\leq_{\mathrm{m}}$-complete sets.

(i) K $= \{e : e \in W_e\}$ is Σ_1-complete.

(ii) MEM $= \{(e, x) : x \in W_e\}$ is Σ_1-complete.

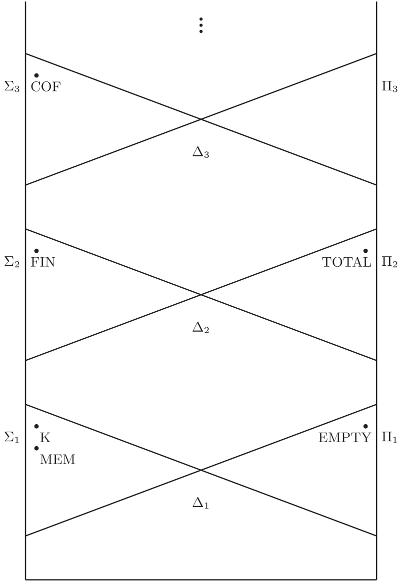
(iii) EMPTY $= \{e : W_e = \varnothing\}$ is Π_1-complete.

(iv) FIN $= \{e : W_e$ is finite$\}$ is Σ_2-complete.

(v) TOTAL $= \{e : \{e\}$ is a total function $\}$ is Π_2-complete.

(vi) COF $= \{e :$ the complement of W_e is finite$\}$ is Σ_3-complete.

(vii) REC $= \{e : W_e$ is recursive$\}$ is Σ_3-complete.



Introduction

- The field that approaches difficult problems in set theory by descriptive methods of sets is called **descriptive set theory**. For instance, the continuum hypothesis is independent from the usual axiomatic set theory, but it is true on classes of well-described sets such as the Borel sets.
- S. C. Kleene and J. Addison made a breakthrough in this field by adopting logical methods such as analytic hierarchy as a means of description.
- We will explain Addison's proof of Kondo's classical theorem, which was a starting point of modern descriptive set theory.

S. C. Kleene

J. Addison

Notation

- The notation here is slightly different from the previous lectures. From now on, we will adopt the standard notation of set theory.
- We identify a natural number n with the set $\{0, 1, \ldots, n-1\}$, and denote the set of natural numbers by $\omega = \{0, 1, \ldots\}$. By $X, Y, \ldots$, we will usually denote subsets of ω.
- Let ${}^{X}Y$ denote the set of functions from X to Y, read as "Y-pre-X".
- Then an element f of ${}^{n}X$ is a function from $\{0, 1, \ldots, n-1\}$ to X, which can be regarded as an n-tuple of elements of X, that is, $(f(0), f(1), \ldots, f(n-1))$.
- Moreover, we define

$$ {}^{\underset{\smile}{\omega}}X := X^{<\omega} = \bigcup_{n \in \omega} {}^{n}X. $$

  Here, ${}^{\underset{\smile}{\omega}}X$ is read as "X-pre-omega-cup".
- For $\xi \in {}^{\omega}X$ or $\xi \in {}^{n}X (n \geq m)$, a sequence $(\xi(0), \xi(1), \ldots, \xi(m-1))$, denoted $\xi \restriction m$ or $\xi[m]$, is called an **initial segment** of ξ (with length m). By $s \subset \xi$, we mean that s is an initial segment of ξ.

- $\mathcal{C} = {}^{\omega}2$ and $\mathcal{N} = {}^{\omega}\omega$ are called the **Cantor space** and the **Baire space**, respectively. They have a natural correspondence with the set of real numbers.

- Between $\mathcal{C}$ and $[0,1]$, there is the following correspondence (continuous surjection) via binary decimal notation:

$$\xi \in {}^{\omega}2 \mapsto \sum_{n \in \omega} \xi(n) \cdot 2^{-(n+1)} \in [0,1].$$

- However, this correspondence is not one-to-one. For example, both $(1,0,0,0,\cdots)$ and $(0,1,1,1,\cdots)$ correspond to $\frac{1}{2}$.

- On the other hand, $\mathcal{N}$ has a one-to-one correspondence with the irrationals in $[0,1]$ by using the notation of continued fractions as follows.

$$\xi \in \mathcal{N} \mapsto \cfrac{1}{1+\xi(0)+\cfrac{1}{1+\xi(1)+\cfrac{1}{1+\xi(2)+\cdots}}}.$$

- Example. the continued fraction ξ with $\xi(2n) = 0, \xi(2n+1) = 2n+1$ expresses $e-2$.

Topology

- In the following, we introduce topological notions of the Baire space, but most of them are directly applicable to the Cantor space.
- For $s \in {}^{<\omega}\omega$, let $[s] = \{\xi \in {}^{\omega}\omega : s \subset \xi\}$. $\{[s] : s \in {}^{<\omega}\omega\}$ is an **open base** of the Baire space.
- A set $G \subset {}^{\omega}\omega$ is **open** if there exists some $A \subset {}^{<\omega}\omega$ such that $G = \bigcup_{s \in A}[s]$.
- The complement of an open set is called **closed**.
- Note that $[s]$ is also a closed set. Because $[s]^c = \bigcup\{[t] : s \not\subset t, t \not\subset s\}$.

- We say that a set $G \subset {}^{\omega}\omega$ is $\Sigma^0_1(\xi)$ if there exists a ξ-CE set A (or equivalently, ξ-computable set A) such that $G = \bigcup_{s \in A}[s]$.
- Note that we here define $\Sigma^0_1(\xi)$ for subsets of ${}^{\omega}\omega$, while in the previous lectures, $\Sigma^0_1(\xi)$ (in the relativized arithmetical hierarchy) for subsets of ω. There is a good reason to use the same notation. The former can be expressed as $G = \{\eta \in {}^{\omega}\omega : \varphi(\eta, \xi)\}$ with a Σ^0_1 formula φ, and the latter as $\{n \in \omega : \varphi(n, \xi)\}$ with a Σ^0_1 formula φ.
- The class $\mathcal{G}$ of open sets coincides with $\bigcup_{\xi} \Sigma^0_1(\xi)$, which is denoted as $\boldsymbol{\Sigma}^0_1$ or $\underset{\sim}{\Sigma}^0_1$.
- A set $F \subset {}^{\omega}\omega$ is $\Pi^0_1(\xi)$ if its complement is $\Sigma^0_1(\xi)$. The class of closed sets $\mathcal{F} = \bigcup_{\xi} \Pi^0_1(\xi)$ is denoted as $\boldsymbol{\Pi}^0_1$ or $\underset{\sim}{\Pi}^0_1$.
- Also, the class of countable unions of closed sets $\mathcal{F}_\sigma = \bigcup_{\xi} \Sigma^0_2(\xi)$ is $\boldsymbol{\Sigma}^0_2$ or $\underset{\sim}{\Sigma}^0_2$.
- Thus, the finite levels of **Borel set**, $\mathcal{G}, \mathcal{F}, \mathcal{F}_\sigma, \ldots$ have been defined in parallel to the arithmetical hierarchy.

Analytical hierarchy

- An **analytic set** is obtained as a projection of a Borel set (or equivalently, just a closed set), the class of such sets $\mathcal{A} = \bigcup_\xi \Sigma^1_1(\xi)$ is denoted as $\mathbf{\Sigma}^1_1$ or $\underset{\sim}{\Sigma}{}^1_1$.
- The class of co-analytic set $\mathcal{CA} = \bigcup_\xi \Pi^1_1(\xi)$ is denoted as $\mathbf{\Pi}^1_1$ or $\underset{\sim}{\Pi}{}^1_1$.
- The class of projections of co-analytic set is $\mathcal{PCA} = \bigcup_\xi \Sigma^1_2(\xi)$ is written as $\mathbf{\Sigma}^1_2$ or $\underset{\sim}{\Sigma}{}^1_2$.
- The finite hierarchy of such **projective sets** corresponds with the analytical hierarchy (with arbitrary oracles).
- Then, the assertions on $\underset{\sim}{\Sigma}{}^1_n$ sets can be regarded as relativization of the assertions on Σ^1_n sets.
- By this method of relativization, Kondo's theorem on the uniformization of the co-analytic sets is obtained as a corollary to Addison's theorem on the uniformization of the Π^1_1 sets.

- As seen in Lecture05-02, there are two types of analytical hierarchies with set quantifiers and function quantifiers. In the following, we mainly deal with function quantifier hierarchies.
- The following two theorems can be proved almost in the same way as the relativized arithmetical hierarchy in Lecture 06-01.

Theorem (Analytical enumeration theorem)

Let $m, n \geq 0$ and $k > 0$. There exists a Σ^1_k subset U of $\mathbb{N}^{n+1} \times (\mathbb{N}^{\mathbb{N}})^m$ such that for any Σ^1_k subset R of $\mathbb{N}^n \times (\mathbb{N}^{\mathbb{N}})^m$ there exists an e such that

$$R(x_1, \cdots, x_n, \xi_1, \cdots, \xi_m) \Leftrightarrow U(e, x_1, \cdots, x_n, \xi_1, \cdots, \xi_m).$$

Theorem (Analytical hierarchy theorem)

For any $n \geq 0$, $\Sigma^1_n \cup \Pi^1_n \subsetneq \Delta^1_{n+1}$.

Proof for the case $n = 0$, i.e., $\Sigma^1_0 \cup \Pi^1_0 \subsetneq \Delta^1_1$

- Let A_n be a Σ^0_n set and but not Π^0_n.
- If we put $B := \bigcup_n \{n\} \times A_n$, then B is no longer arithmetical. That is, $B \notin \Sigma^1_0 \cup \Pi^1_0$.
- On the other hand, since every A_n is Σ^1_1, by the analytical enumeration theorem, there exist a Σ^1_1 formula U such that for each n there exists e_n such that $x \in A_n$ iff $U(e_n, x)$. Now considering $n \mapsto e_n$ as a computable function, we have $(n, x) \in B \Leftrightarrow U(e_n, x)$, which means B is Σ^1_1.
- Also, $B^c = \bigcup_n \{n\} \times A^c_n$, where $\{n\} \times A^c_n$ is Π^0_n and so Σ^1_1. Thus, B^c is also Σ^1_1.
- Therefore, B is a Δ^1_1 set.

- Now, we will define some concepts of ordered sets and trees in second-order arithmetic.
- By identifying a pair (m, n) with its code $\frac{(m+n)(m+n+1)}{2} + m$, we can represent a function with two or more variables by a function with a single variable. Thus, $\xi \in {}^{\omega}\omega$ can also represent a binary relation $\{(m, n) \in \omega^2 : \xi(m, n) \geq 1\}$.
- We say that $\xi (\in {}^{\omega}\omega)$ is a **linear order** (abbreviated as LO) if

$$\{(m, n) : \xi(m, n) \geq 1\} \text{ is a linear ordering on } \mathbb{N}.$$

- Formally, it is expressed as the following Π^0_1 formula.

$$\begin{aligned} \mathrm{LO}(\xi) \quad \Leftrightarrow \quad & \forall m, n(\xi(m, n) + \xi(n, m) \geq 1) \\ & \land \forall m, n(\xi(m, n) \cdot \xi(n, m) \geq 1 \to m = n) \\ & \land \forall m, n, k(\xi(m, n) \cdot \xi(n, k) \geq 1 \to \xi(m, k) \geq 1). \end{aligned}$$

- We denote $\xi(m, n) \geq 1$ by $m \leq_\xi n$ or simply $m \leq n$. Then, $\leq$ and ξ are often used indiscriminately.

- A linear order with no infinite descending sequence is called a **well-order** (abbreviated as WO). Considering that every infinite sequence in WO has an ascending part, we can also express it as follows.

$$\mathrm{WO}(\xi) \Leftrightarrow \mathrm{LO}(\xi) \wedge \forall \eta \exists n \xi(\eta(n), \eta(n+1)) \geq 1.$$

- By using $\leq$, we rewrite it as

$$\mathrm{WO}(\leq) \Leftrightarrow \mathrm{LO}(\leq) \wedge \forall \eta \exists n(\eta(n) \leq \eta(n+1)).$$

- Note that these expressions are Π^1_1.
- A finite sequence $s = (s_0, s_1, \ldots, s_{n-1}) \in {}^{\omega}\omega$ can also be identified with a code.
- Then, for two sequences $s = (s_0, s_1, \cdots, s_{m-1})$ and $t = (t_0, t_1, \cdots, t_{n-1})$, the concatenation $s * t = (s_0, s_1, \cdots, s_{m-1}, t_0, t_1, \cdots, t_{n-1})$ is a binary operation.
- A relation $t \subset s$, defined by $\exists u(t * u = s)$, represents "t is an initial segment of s". Any subset S of ${}^{\omega}\omega$ can be uniquely represented by $\xi \in {}^{\omega}\omega$ s.t. $s \in S \Leftrightarrow \xi(s) \geq 1$.

Trees

Definition

$T \subset {}^{\omega}\omega$ is said to be a **tree** if it is closed under initial segment, i.e.

$$\forall s \in T\ \forall t(t \subset s \to t \in T)$$

- A subset of a tree T is called a subtree of T if it is a tree. A subtree P is called a **path** through T if there is no branching, i.e., $\forall s, t \in P(t \subset s \vee s \subset t)$.
- The set of infinite paths of T is represented by $[T](\subset {}^{\omega}\omega)$.
- A tree with no infinite paths is said to be **well-founded**.

- We consider a partial order $\leq$ on ${}^{\omega}\omega$, defined by $t \leq s \Leftrightarrow s \subseteq t$. Then, in a tree, nodes closer to **root** $\varnothing$ are larger, and and an infinite path $\varnothing = s_0 \subset s_1 \subset s_2 \subset \cdots$ is an infinite descending sequence.
- We also regard an infinite path as a function $f : n \mapsto s_n$. Therefore, the well-foundedness of a tree T can be expressed by the following Π^1_1 formula,

$$\mathrm{WF}(T) \Leftrightarrow \neg\exists f \forall n(f(n) \in T \wedge f(n) \subset f(n+1))$$

- Now, for $k \in \omega$, let $s^\wedge k = s * (k)$ and $k^\wedge s = (k) * s$.
- In a tree T, a node $s^\wedge k \in T$ is called a **child** of s.
- A tree T is said to be **finitely branching** if every $s \in T$ has only a finite number of children.
- For $s \in T$, the subtree rooted at s is written as $T_s = \{t : s * t \in T\}$.
- The following lemma is the most important fact for infinite trees.

Theorem (König's lemma)

Any finitely branching infinite tree T has an infinite path.

Proof.

- Suppose an infinite tree T is finitely branching. We inductively construct an infinite path $\varnothing = s_0 \subset s_1 \subset s_2 \subset \cdots$ through T. Assume it is constructed up to s_i and T_{s_i} is infinite.
- $T_{s_i} = \bigcup_k k^\wedge T_{s_i^\wedge k}$. Since T_{s_i} is finitely branching, $T_{s_i^\wedge k}$ is infinite for some k.
- For such a k, let $s_{i+1} = s_i^\wedge k$. Repeating this operation infinitely many times, an infinite path can be constructed.

Ordinals and well-founded trees

The ordinal number can be seen just as a representative of well-order.
From the set-theoretic convention, $<$ on the ordinals is represented by $\in$. So, ordinal $\sigma + 1$ (the successor of σ) is defined as $\{0, 1, \ldots, \sigma\}$. We also use $\sigma + 1$ to denote $\{0, 1, \ldots, \sigma\}$.

Theorem

A tree T is well-founded $\Longleftrightarrow$ there exists an ordinal number σ and a function $f : T \to \sigma + 1$ such that f is order-preserving $(s \subsetneq t \Leftrightarrow t < s \Leftrightarrow f(t) < f(s))$.

Such an order-preserving function f is denoted as $f : T \xrightarrow{\text{o.p.}} \sigma + 1$ or $T \xrightarrow{f} \sigma + 1$.

Proof.
$(\Leftarrow)$ By contradiction, suppose that a tree T is not well-founded. Then a path ξ exists and

$$\varnothing = \xi(0) > \xi(1) > \xi(2) > \xi(3) > \cdots$$

So, for an order-preserving function f,

$$\sigma \geq f(\xi(0)) > f(\xi(1)) > f(\xi(2)) > f(\xi(3)) > \cdots .$$

Hence, σ is no longer well-ordered, which violates the definition of ordinals.

($\Rightarrow$)

- By way of contradiction. We assume that T has no order-preserving functions, and show inductively that T has a path $\varnothing = s_0 \subset s_1 \subset \cdots$.
- By assumption, $T_\varnothing$ has no order-preserving function.
- We will show that if T_s has no order-preserving function, for some k, $T_{s^\wedge k}$ also has no order-preserving function.
- By contradiction, assume $f_k : T_{s^\wedge k} \xrightarrow{\text{o.p.}} \sigma_k$ for all k. Then let $\sigma := \sup_k(\sigma_k + 1)$ and define

$$f(t) := \begin{cases} \sigma & \text{if } t = \varnothing \\ f_k(t') & \text{if } t = k^\wedge t' \end{cases}$$

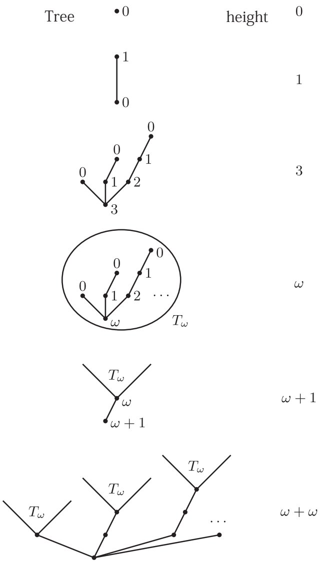
- Thus, $f : T_s \xrightarrow{\text{o.p.}} \sigma$, contrary to the assumption. $\square$

Definition

- The **height** of T is the smallest ordinal number σ such that there exists $f : T \xrightarrow{\text{o.p.}} \sigma + 1$, represented by $||T||$.
- If T is a recursive well-founded tree, $||T||$ is said to be **computable**.
- In addition, set $||T|| = -1$ when T is empty, and set $||T|| = \infty$ when T is not well-founded.

Example

Right-hand-ride is a typical well-founded tree T. Each vertex has an order-preserving ordinal, and its height $||T||$ is shown on the right side.



Lemma

For any countable ordinal σ, there is a tree T such that $||T|| = \sigma$.

Proof. For countable ordinals σ,

$$T_\sigma := \{(\sigma_0, \sigma_1, \cdots, \sigma_k) : \sigma \geq \sigma_0 > \sigma_1 > \cdots > \sigma_k, k < \omega\}.$$

Since σ is countable, identifying it with ω, T_σ can be regarded as a subset of ${}^{\omega}\omega$. Finally, it is easy to show $||T_\sigma|| = \sigma$ by transfinite induction. □

If f is an order-preserving function of a tree T, then $f_s(t) = f(s * t)$ is an order-preserving function of subtree $T_s = \{t : s * t \in T\}$, and $f(s) = f_s(\varepsilon) \geq ||T_s||$.

Theorem

For any T, $||T|| = \sup_{s \neq \varepsilon}(||T_s|| + 1)$.

Proof.

- If T is not well-founded, then both sides are $+\infty$. Therefore, we assume T is well-founded, and suppose $\sigma = ||T||$ and $f : T \xrightarrow{\text{o.p.}} \sigma + 1$.
- If $s \neq \varepsilon$, then $||T_s|| \leq f(s) < f(\varepsilon) = ||T||$. So, $\sup(||T_s|| + 1) \leq ||T||$.
- Suppose $\sigma = \sup(||T_s|| + 1) < ||T||$.
- We define a function $h : T \to \sigma + 1$ as

$$h(s) = \begin{cases} ||T_s|| & \text{if } s \neq \varepsilon \\ \sigma & \text{if } s = \varepsilon. \end{cases}$$

- Then h is order-preserving, and so $||T|| \leq \sigma$, which is a contradiction.
- Hence, $||T|| = \sigma$.

□

Further Reading

- Kozen, D. C. (2006). *Theory of computation* (Vol. 170). Heidelberg: Springer.
- Soare, R. I. (2016). *Turing computability. Theory and Applications of Computability*. Springer.

Thank you for your attention!