Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Apr.25, (1) Oracle computation and relativization
- Apr.27, (2) m-reducibility and simple sets
- May 4, (3) T-reducibility and Post's problem
- May 9, (4) Arithmetical hierarchy and polynomial-time hierarchy
- May 11, (5) Analytical hierarchy and descriptive set theory I
- May 16, (6) Analytical hierarchy and descriptive set theory II

Recap

- $A \leq_{\rm m} B$, if there exists a computable function f s.t. $x \in A \Leftrightarrow f(x) \in B$ for any x.
- $A \leq_{\rm T} B$, if A is computable in oracle B (i.e., recursive in χ_B).
- A CE set A is **(T-)complete** / **m-complete** if $B \leq_{\rm T} A$ / $B \leq_{\rm m} A$ for any CE set B.

Theorem (Post's theorem, 1944)

There exists a CE set that is neither computable nor m-complete.

- A CE set A is a **simple** set if A^c is infinite and A has a common element with each infinite CE set. A simple set satisfies Post's theorem.

Theorem (Post's problem, finite injury priority argument due to Friedberg, Muchnik)

There exists a CE set that is neither computable nor T-complete.

- A set A is a **low** set if $A' := {\rm K}^A \leq_{\rm T} {\rm K}$. A simple low set is a solution to Post's problem.

Theorem (Baker, Gill, Solovay (1975))

(1) There exists a computable oracle A such that $\mathrm{P}^A = \mathrm{NP}^A$.
(2) There exists a computable oracle A such that $\mathrm{P}^A \neq \mathrm{NP}^A$.

Proof To show (1)

- Let A be a PSPACE complete problem such as TQBF (Lecture02-06). First, obviously $\mathrm{P}^A \subset \mathrm{NP}^A \subset \mathrm{PSPACE}^A$.
- Since A is PSPACE, one can compute PSPACE^A in PSPACE without using A as an oracle. That is, $\mathrm{PSPACE}^A \subset \mathrm{PSPACE}$.
- Finally, due to the PSPACE completeness of A, $\mathrm{PSPACE} \subset \mathrm{P}^A$.
- Therefore, $\mathrm{P}^A = \mathrm{NP}^A = \mathrm{PSPACE}^A$.

To show (2) [This is not a priority argument. We may set $s = e$.]

- For any $A \subset \{0,1\}^*$, $B = \{0^{|x|} : x \in A\}$ is in NP^A.
- So, we only need to construct a computable $A = \bigcup_s A_s$ such that $B \notin \mathrm{P}^A$.
- Let M_e enumerate deterministic machines (or sets accepted by such machines) running in polynomial p_e time.
- We want to show $R_e : M_e^A \neq B$ for all e. That is, for each e, we guarantee the existence of n such that
$$M_e^A(0^n) \neq B(0^n).$$
- Assume that A_s's are constructed up to step $s = e$. Then, take a n greater than any number used in the previous constructions and $2^n > p_e(n)$. Consider whether or not a word with length n should be put into A_{s+1}.
- When $M_e^{A_s}(0^n) = 1$, set $A_{s+1} = A_s$. Since a word with length n will never be added to A, we have $B(0^n) = 0$.
- Next assume $M_e^{A_s}(0^n) = 0$. Since this computation queries the oracle A_s at most $p_e(n)$ times, by $2^n > p_e(n)$ there is a word x of length n that is irrelevant to the oracle query. So setting $A_{s+1} = A_s \cup \{x\}$, we have $M_e^{A_{s+1}}(0^n) = 0$, but $B(0^n) = 1$. □

Today's topics

1. Recap
2. Relativized arithmetical hierarchy
3. Complete problems in the arithmetic hierarchy
4. Polynomial time hierarchy

- Today, we consider examples of sets belonging to various classes in arithmetic hierarchies and their relativizations.
- Recall:
  - $A = W_e^\xi$ is called ξ-CE if it is the domain of a partial recurisive function $\{e\}^\xi$ with oracle ξ. In particular when $\xi = \chi_B$, we say A is CE in B.
  - A set A is computable in B if A is recursive in χ_B, written as $A \leq_T B$ (A is Turing reducible to B).
- Then a **relativized arithmetical hierarchy** for subsets of $\mathbb{N}^k$ is defined as follows.

$$
\begin{aligned}
\Sigma_1(\xi) &:= \{\xi\text{-CE sets}\}, \\
\Delta_1(\xi) &:= \{\xi\text{-computable sets}\}, \\
\Sigma_{n+1}(\xi) &:= \{A \mid A \text{ is CE in some } B \in \Sigma_n(\xi)\}, \\
\Delta_{n+1}(\xi) &:= \{A \mid A \text{ is computable in some } B \in \Sigma_n(\xi)\}, \\
&:= \{A \mid A \leq_T B \text{ for some } B \in \Sigma_n(\xi)\}, \\
\Pi_n(\xi) &:= \{\text{the complement of sets in } \Sigma_n(\xi)\}
\end{aligned}
$$

When ξ is a computable function, we omit to mention (ξ) or ξ, and classes $\Sigma_n, \Pi_n, \Delta_n$ are usual **arithmetical hierarchy** .

- Since there is a computable bijection between $\mathbb{N}^k$ and $\mathbb{N}$, we will mainly discuss sets and functions on $\mathbb{N}$ below.
- We write $A \leq_{\mathrm{m}} B$ if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for any $x \in \mathbb{N}$, $x \in A \Leftrightarrow f(x) \in B$.
- A set B is called **m-hard** if for every CE set A, $A \leq_{\mathrm{m}} B$; moreover, if B itself is also CE , then B is called **m-complete**.
- In the following, such sets will be generalized to Σ_n etc.
- Since we only treat with m-reducibility, by $\mathcal{C}$-hardness (completeness) we mean m-hardness (completeness) with respect to the sets in $\mathcal{C}$.
- More strictly, let $\mathcal{C}$ be a class of sets.
  A set B is said to be $\mathcal{C}$-**hard** if for every $A \in \mathcal{C}$, $A \leq_{\mathrm{m}} B$.
  A set B is said to be $\mathcal{C}$-**complete** if B is $\mathcal{C}$-hard and $B \in \mathcal{C}$.
- Clearly, if $A \leq_{\mathrm{m}} B$ and $B \in \Sigma_n$ (Π_n, Δ_n), then so is A.
- A Σ_n-complete set is not Π_n, since arithmetical hierarchy is strict.

Now, the following are typical m-complete sets.

(i) $\mathsf{K} = \{e : e \in W_e\}$ is Σ_1-complete.

(ii) $\mathsf{MEM} = \{(e, x) : x \in W_e\}$ is Σ_1-complete.

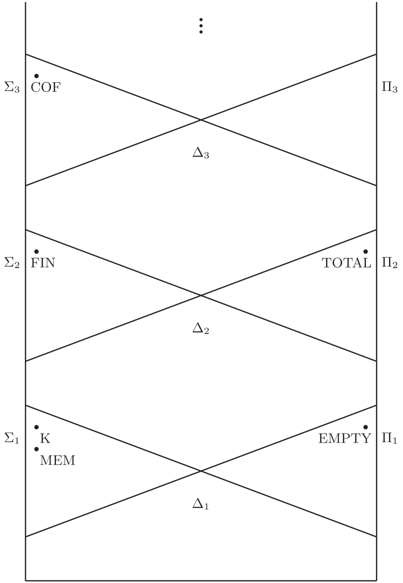
(iii) $\mathsf{EMPTY} = \{e : W_e = \varnothing\}$ is Π_1-complete.

(iv) $\mathsf{FIN} = \{e : W_e \text{ is finite}\}$ is Σ_2-complete.

(v) $\mathsf{TOTAL} = \{e : \{e\} \text{ is a total function }\}$ is Π_2-complete.

(vi) $\mathsf{COF} = \{e : \text{the complement of } W_e \text{ is finite}\}$ is Σ_3-complete.

(vii) $\mathsf{REC} = \{e : W_e \text{ is recursive}\}$ is Σ_3-complete.



(i) K $= \{e : e \in W_e\}$ is Σ_1-complete.

$(\because)$ A is CE $\Leftrightarrow$ $A \leq_{\mathrm{m}} \mathrm{K}$ (Lecture01-06), and K is Σ_1.

(ii) MEM $= \{(e, x) : x \in W_e\}$ is Σ_1-complete.

$(\because)$ K $\leq_{\mathrm{m}}$ MEM, and MEM is Σ_1.

(iii) EMPTY $= \{e : W_e = \varnothing\}$ is Π_1-complete.

($\because$) It is sufficient to show NONEMP$= \{e : \exists x\ x \in W_e\}$, the complement of EMPTY, is Σ_1-complete.

- It is clear that NONEMP is Σ_1. Any CE set A can be written as $x \in A \leftrightarrow \exists y T(x, y)$ for some primitive recursive relation $T(x, y)$.
- Then by the parameter theorem, there exists a computable function f such that $T(x, y) \leftrightarrow y \in W_{f(x)}$.
- So $x \in A \leftrightarrow f(x) \in$ NONEMP.

Recall: Parameter theorem

(Lecture01-05)
There exists a primitive recursive function S^m_n: $\mathbb{N}^{m+1} \to \mathbb{N}$ s.t.

$$\{e\}^{m+n}(x_1, \cdots, x_n, y_1, \cdots, y_m) \sim \{S^m_n(e, y_1, \cdots, y_m)\}^n(x_1, \cdots, x_n).$$

(iv) FIN $= \{e : W_e$ is finite$\}$ is Σ_2-complete.

($\because$) "W_e is finite" can be expressed as $\exists y \forall x > y\ x \notin W_e$, which is Σ_2.
Any Σ_2 set A can be expressed as $\exists y \forall z R(x, y, z)$ with a primitive recursive relation R.
Then by using its complement A^c, we define partial computable function ψ as follows:

$$\psi(x, w) = \begin{cases} 0 & \text{if } \forall y \leq w \exists z \neg R(x, y, z) \\ \uparrow & \text{otherwise} \end{cases}$$

By the parameter theorem, there exists a computable function f s.t. $\{f(x)\}(w) \sim \psi(x, w)$.
Then,

$$x \in A \Rightarrow W_{f(x)} \text{ is finite} \Rightarrow f(x) \in \mathrm{FIN}(\subset \mathrm{TOTAL}^c)$$
$$x \in A^c \Rightarrow \forall w \{f(x)\}(w)\downarrow \Rightarrow f(x) \in \mathrm{TOTAL} \subset \mathrm{FIN}^c$$

That is, $A \leq_{\mathrm{m}} \mathrm{FIN}$. So FIN is Σ_2-complete.

(v) TOTAL $= \{e : \{e\}$ is a total function $\}$ is Π_2-complete.

From the discussion above, $B \leq_{\mathrm{m}}$ TOTAL for any Π_2 set B.
"$\{e\}$ is a total function" itself can be expressed by Π_2 from $\forall x \exists y \{e\}(x) = y$.

Relativization is useful when discussing levels above Σ_3 or Π_3.
For example, the finiteness problem relative to oracle A is

$$\mathrm{FIN}^A = \{e \mid W_e^{\chi_A} \text{ is finite}\}.$$

Lemma

$\mathrm{FIN}^{\mathrm{K}}$ is Σ_3-complete. More generally, if A is Σ_n-complete, then FIN^A is Σ_{n+2}-complete.

Proof.
Let A be a Σ_n-complete set. The finiteness of $W_e^{\chi_A}$ can be expressed as

$$\exists y \forall x > y \; x \notin W_e^{\chi_A}$$

where χ_A is Δ_{n+1} and $x \notin W_e^{\chi_A}$ is Π_{n+1}. So FIN^A is Σ_{n+2}.
To prove the completeness of Σ_{n+2}, since A is Σ_n-complete, the set Σ_{n+2} can be expressed as $\Sigma_2(A)$. All that remains is to relativize the proof that FIN is Σ_2-complete with oracle A. □

(vi) COF $= \{e :$ the complement of W_e is finite$\}$ is Σ_3-complete.

$(\because)$ "The complement of W_e is finite" can be expressed as $\exists y \forall x > y\ x \in W_e$, which is $\mathrm{FIN}^{\mathrm{MEM}}$. By the above lemma, COF is Σ_3-complete.

(vi) REC $= \{e : W_e$ is recursive (decidable)$\}$ is Σ_3-complete.

Proof.

- W_e is recursive iff its complement can be also expressed as W_d for some d, and thus REC is Σ_3.
- To show Σ_3-completeness, let A be any Σ_3 set. Then there exists a Π_2 set P that it can be expressed as
$$x \in A \Leftrightarrow \exists y P(x, y).$$
By the complement of FIN, there is a computable function g such that
$$P(x, y) \Leftrightarrow \text{"}W_{g(x,y)} \text{ is an infinite set"}.$$

- Let K^s and $W^s_{g(x,y)}$ denote recursive approximation sequences of CE sets K and $W_{g(x,y)}$, respectively. Then finite sets V^s_x and numbers $a^s_{x,y}$ are inductively defined as follows.
- First, set $V^0_x = \varnothing$ and $a^0_{x,y} = y$ for all y.
- Now, assume V^s_x and $a^s_{x,y}$ have been constructed up to s, and the following holds

$$\mathbb{N} - V^s_x = \{a^s_{x,0} < a^s_{x,1} < \cdots < a^s_{x,y} < \dots\}.$$

  Next, let $Q(x, y, s)$ be the following recursive relation

$$W^{s+1}_{g(x,y)} \neq W^s_{g(x,y)} \vee y \in K^{s+1} - K^s.$$

  Then put $V^{s+1}_x = V^s_x \cup \{a^s_{x,y} : y \leq s \wedge Q(x, y, s)\}$.
- Finally, list the elements of $\mathbb{N} - V^{s+1}_x$ from smallest to largest, and let $a^{s+1}_{x,y}$ be the y-th element.
- Since this construction is computable, $V_x = \cup_s V^s_x$ is a CE set, and moreover there is a computable function f such that $W_{f(x)} = V_x$.

- Suppose $x \in A$.
  - Then we can take a y such that $P(x,y)$. Since $W_{g(x,y)}$ is an infinite set, there are infinitely many s such that $W^{s+1}_{g(x,y)} \neq W^s_{g(x,y)}$.
  - Thus, there are infinite many $s \geq y$ such that $Q(x,y,s)$ holds, and for such s, the y-th element $a^s_{x,y}$ of $\mathbb{N} - V^s_x$ is removed from $\mathbb{N} - V^s_x$. So at most y elements remain in $\mathbb{N} - V_x$. Therefore, $W^c_{f(x)} = \mathbb{N} - V_x$ is finite and $f(x) \in \mathrm{COF} \subset \mathrm{REC}$.
- Next, suppose $x \notin A$.
  - We will prove $K \leq_{\mathrm{T}} W_{f(x)}$.
  - Take y arbitrarily. By $x \notin A$, $W_{g(x,y)}$ is a finite set.
  - Therefore, for sufficiently large $s \geq y$, $Q(x,y,s)$ holds only if $y \in K^{s+1} - K^s$. Since there is at most one such s, $Q(x,y,s)$ holds finitely many times. Therefore, for sufficiently large s, $a^s_{x,y}$ is constant and we denote it by $a_{x,y}$.
  - $W_{f(x)} = V_x = \{a_{x,0} < a_{x,1} < \cdots < a_{x,y} < \dots\}$.
  - $s(y) := \mu s[a^s_{x,y} = a_{x,y}]$ is computable in $W_{f(x)}$.
  - For every $s \geq s(y)$, since $a^s_{x,y} = a_{x,y}$, $y \in K^{s+1} - K^s$ does not hold. So

$$y \in K \Leftrightarrow y \in K^{s(y)}.$$

  - Since the right hand side is computable in $W_{f(x)}$, so is K. Therefore, $f(x) \notin$REC.
- Therefore, $x \in A \Leftrightarrow f(x) \in$ REC.

Finally, we discuss the polynomial-time version of arithmetical hierarchy. We defined P^A and NP^A for the set $A \subset \Omega^*$. For a class $\mathcal{C}$ of sets,

$$\mathsf{P}(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \mathsf{P}^A, \quad \mathsf{NP}(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \mathsf{NP}^A.$$

Definition (Polynomial time hierarchy)

The **polynomial-time hierarchy** (PH) is defined inductively defined as follows

- $\Sigma_0^{\mathrm{P}} = \Pi_0^{\mathrm{P}} = \mathsf{P}$,
- $\Sigma_{n+1}^{\mathrm{P}} = \mathsf{NP}(\Sigma_n^{\mathrm{P}})$,
- $\Pi_{n+1}^{\mathrm{P}} = \text{co-}\Sigma_{n+1}^{\mathrm{P}}$,
- $\Delta_{n+1}^{\mathrm{P}} = \mathsf{P}(\Sigma_n^{\mathrm{P}})$
- $\mathsf{PH} = \bigcup_n \Sigma_n^{\mathrm{P}}$

Then it is easy to see that:

Lemma

PH $\subset$ PSPACE

Proof. NP(PSPACE) $\subset$ PSPACE(PSPACE) $\subset$ PSPACE. □

Lemma

If PH $=$ PSPACE, then $\Sigma_n^{\mathrm{P}} = \Sigma_{n+1}^{\mathrm{P}}$ for some n.

Proof. If TQBF $\in \Sigma_n^{\mathrm{P}}$ then PSPACE $\subset \Delta_{n+1}^{\mathrm{P}}$. □

Homework

Given A as an NP-complete set, show the following.

(1) $\Sigma_1^{\mathrm{P}} = \{B : B \leq_{\mathrm{m}}^{\mathrm{P}} A\}$.

(2) $\Delta_2^{\mathrm{P}} = \{B : B \leq_{\mathrm{T}}^{\mathrm{P}} A\}$.

(3) $\Sigma_{n+1}^{\mathrm{P}} = \Sigma_n^{\mathrm{P}}(A)$.

Further Reading

- Kozen, D. C. (2006). *Theory of computation* (Vol. 170). Heidelberg: Springer.
- Soare, R. I. (2016). *Turing computability. Theory and Applications of Computability*. Springer.

Thank you for your attention!