Logic and Computation II

Part 6. Recursion-theoretic hierarchies

Kazuyuki Tanaka

BIMSA

May 5, 2023

北京雁栖湖应用数学研究院
YANQI LAKE BEIJING INSTITUTE OF MATHEMATICAL SCIENCES AND APPLICATIONS

Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Apr.25, (1) Oracle computation and relativization
- Apr.27, (2) m-reducibility and simple sets
- May 4, (3) T-reducibility and Post's problem
- May 9, (4) Arithmetical hierarchy and polynomial-time hierarchy
- May 11, (5) Analytical hierarchy and descriptive set theory I
- May 16, (6) Analytical hierarchy and descriptive set theory II

Today's topics

1. Recap
2. Introduction
3. Low sets
4. Polynomial-time reducibility

Recap

- $A \leq_{\mathrm{m}} B$, if there exists a computable function f such that for any x,

$$x \in A \quad \Leftrightarrow \quad f(x) \in B.$$

- $A \leq_{\mathrm{T}} B$, if A is computable in oracle B (i.e., recursive in χ_B).
- A set A is said to be **(T-)complete**/**m-complete** (with respect to CE) if A is CE and $B \leq_{\mathrm{T}} A$ / $B \leq_{\mathrm{m}} A$ for any CE set B.

Theorem (Post's theorem, 1944)

There exists a CE set that is neither computable nor m-complete.

- **Post's problem**: Is there a CE set that is neither computable nor (T-)complete.
- To challenge this problem, various notions of CE sets (such as immune sets, simple sets, and productive sets) were introduced. A simple set satisfies Post's theorem.

Introduction

- Post's problem was independently solved by Friedberg (1957) and Muchnik (1956). Their proof technique is now called the **finite injury priority argument**.
- Although this proof method is already common in the study of computability, it is still difficult for a novice to grasp the argument. So, it may be a good idea to start with a quick look at its outline, and then gradually deepen your understanding by reading the proof repeatedly.
- Now, if $A \leq_{\mathrm{T}} B$ but not $B \leq_{\mathrm{T}} A$, we write $A <_{\mathrm{T}} B$. Then, Post's problem can be expressed as follows.

Theorem (Friedberg, Mucinik)

There exists a set A such that $\varnothing <_{\mathrm{T}} A <_{\mathrm{T}} \mathrm{K}$.

Low sets

- In the last lecture, we proved Post's theorem by showing the existence of a simple set, which is incomputable CE set that is not m-complete.
- Today we introduce the notion of **low sets** to extend from "non-m-complete" to "non-T-complete".
- Fix a set $A \subset \mathbb{N}$, and let $\{\varphi_e^A\}$ be a Gödel numbering of partial recursive functions $\varphi_0, \varphi_1, \ldots$ in A. Suppose W_x^A and K^A are also defined naturally as follows:

$$W_x^A := \{z \mid \varphi_x^A(z)\downarrow\},$$

$$\mathrm{K}^A \quad := \quad \{x \mid \varphi_x^A(x)\downarrow\} \quad = \quad \{x \mid x \in W_x^A\}.$$

- We can prove that K^A is not computable in A, etc., in the same way as $A = \varnothing$.
- K^A is also written as A' and called A-**jump**.

Definition

A set A such that $A' \leq_{\mathrm{T}} \mathrm{K}$ is called **low**.

Lemma

$A <_{\mathrm{T}} \mathrm{K}$ if A is a low set.

Proof If A is a low set, $A <_{\mathrm{T}} A' \leq_{\mathrm{T}} \mathrm{K}$, and so $A <_{\mathrm{T}} \mathrm{K}$. □

Thus, to solve Post's problem, it is sufficient to prove the following:

Lemma (main lemma for Post's problem)

There exists a simple low set.

- We introduce some notations related to oracle computations.
- By "$\varphi^A_{e,s}(x) = y$", we denote the computation of $\varphi^A_e(x) = y$ will be completed within s steps, and if it exceeds s steps, we denote it as $\varphi^A_{e,s}(x)\uparrow$.
- For a given s, it is decidable whether or not the computation terminates within s steps. Thus, "$\varphi^A_{e,s}(x) = y$" is a function computable in A (in fact, primitive recursive in A). Also, $\uparrow$ can be regarded as a finite value.
- It doesn't matter how you measure the number of steps. What we essentially need is
$$\varphi^A_e(x) = y\ (<\infty) \Leftrightarrow \exists \sigma \subset A\ \exists s\ \forall \tau \supseteq \sigma\ \forall t \geq s\ \varphi^\tau_{e,t}(x) = y.$$
- Here $\sigma \subset A$ means σ is an initial segment of χ_A. Let $W^A_{e,s} := \mathrm{dom}\varphi^A_{e,s}$.

Proof

- In the finite injury priority argument, a desired CE set A is constructed as the infinite sum $\bigcup_s A_s$ of finite sets A_s, where $A_0 = \varnothing$ and A_s is "the (finite) set of numbers that are verified to be members of A within s step". Once an element is determined to be a member of A, it is never removed. Thus $A_s \subset A_{s+1}$ for each s.
- To ensure that A is low and simple, we construct A_s to satisfy several requirements.
- A **positive requirement** is satisfied by adding some elements to a desired set A and a **negative requirement** is by excluding some elements from A.
- Satisfying one requirement may **injure** another requirement that is already satisfied. So, **priorities** are set to all requirements, so that a requirement will be injured by only a finite number of requirements (with higher-priority).

- A is low and simple if all of the following are satisfied.
  (i) A is CE,
  (ii) A^c is infinite,
  (iii) A has a common element with each infinite CE set, and
  (iv) $\mathrm{K}^A \leq_{\mathrm{T}} \mathrm{K}$.
- In the above, condition (i) naturally holds from the inductive construction of A. Condition (ii) is also easily satisfied.
- The essential ones are the positive condition (iii) and the negative condition (iv). Rewriting these into *requirements* for each e, we have

$$\begin{aligned} P_e &: |W_e| = \infty \Rightarrow A \cap W_e \neq \varnothing \\ N_e &: \exists^\infty s\ \varphi_{e,s}^{A_s}(e)\downarrow \Rightarrow \varphi_e^A(e)\downarrow . \end{aligned}$$

Here, $\exists^\infty$ means "exists infinitely many".

- It is clear that (iii) holds if P_e holds for each e.
- Next, we show that (iv) holds if N_e holds for each e. First, assume that $s \mapsto A_s$ is computable.
  If N_e holds, then

$$\exists^\infty s\ \varphi_{e,s}^{A_s}(e)\downarrow \Rightarrow\ \varphi_e^A(e)\downarrow \Rightarrow\ \exists t \forall s > t\ \varphi_{e,s}^{A_s}(e)\downarrow$$

$$\Rightarrow\ \forall t \exists s > t\ \varphi_{e,s}^{A_s}(e)\downarrow \equiv\ \exists^\infty s\ \varphi_{e,s}^{A_s}(e)\downarrow.$$

- Thus, $\mathrm{K}^A = \{e : \varphi_e^A(e)\downarrow\}$ is a Δ_2 set.

Corollary (Lecture 06-01)

A is Δ_2 if and only if $A \leq_T \mathrm{K}$.

- By the above fact, we have $\mathrm{K}^A \leq_{\mathrm{T}} \mathrm{K}$.

- Now we explain why N_e is a negative requirement.
- We define the following computable function r as a tool to control N_e:

$$r(e, s) = u(A_s, e, e, s).$$

  Here, the right-hand side is called the **use function**, which is 1 + the maximum number used in the computation of $\varphi_{e,s}^{A_s}(e)$, and 0 if the computation never halts.
- If $s \mapsto A_s$ is assumed to be computable, then r is also computable, which is called the **restraint function**.
- That is, given A_s, if $\varphi_{e,s}^{A_s}(e)\downarrow$, then by excluding (not adding) elements x less than $r(e, s)$ from A, we have $A \upharpoonright r = A_s \upharpoonright r$, so $\varphi_e^A(e)\downarrow$, and N_e works as a negative requirement.

- Among all P_e and N_e, set the priority as

$$P_0 > N_0 > P_1 > N_1 > P_2 > N_2 > \dots$$

- Note that for any requirement there are only a finitely many requirements with higher priorities. Numbers below $r(e,s)$ are added to A only for P_i with $i < e$.
- Now, we show the construction of A.
  - Step $s = 0$: Set $A_0 = \varnothing$.
  - Step $s+1$: Assume that A_s is obtained.
    If there is an $i \leq s$ which satisfies (i) $W_{i,s} \cap A_s = \varnothing$, and
    (ii) $\exists x \in W_{i,s}(x > 2i \wedge \forall e \leq i \; r(e,s) < x)$,
    then choose the smallest x that satisfies (ii) and set $A_{s+1} = A_s \cup \{x\}$.
    Then the requirement P_i is satisfied, and after $s+1$ it will never receive attention.

    If there is no such $i \leq s$, put $A_{s+1} = A_s$.

- When $A_{s+1} = A_s \cup \{x\}$, for e such that $x \leq r(e,s)$, N_e is **injured** by x at $s+1$. Then, we have

Claim 1

For every e, N_e is injured at most finitely many times.

$(\because)$ N_e can be injured only by P_i for $i < e$.

Claim 2

For all e, $r(e) = \lim_s r(e,s)$ exists and hence N_e holds.

$(\because)$ Fix any e. From Claim 1, there exists a step s_e such that N_e is not injured after s_e. But if $\varphi_{e,s}^{A_s}(e)\downarrow$ for $s > s_e$, then for $t \geq s$, $r(e,t) = r(e,s)$ and so $r(e) = \lim_s r(e,s)$ exists. Hence $A_s \upharpoonright r = A \upharpoonright r$ and $\varphi_e^A(e)\downarrow$, which implies N_e holds.

Claim 3

P_i holds for all i.

$(\because)$ Suppose that W_i is an infinite set. From Claim 2, we take such an s that

$$\forall t \geq s \; \forall e \leq i \; r(e,t) = r(e).$$

We may assume that no P_j with $j < i$ receives attention after $s'(\geq s)$, In addition, take $t > s'$ such that

$$\exists x \in W_{i,t}(x > 2i \wedge \forall e \leq i \; r(e) < x).$$

Then we already have $W_{i,t} \cap A_t \neq \varnothing$ or P_i receives attention at $t+1$. In either case, $W_{i,t} \cap A_{t+1} \neq \varnothing$,and so P_i holds.

From the above, $A = \bigcup_{s \in \mathbb{N}} A_s$ is a simple low set. Also, A^c is infinite, since from condition (ii) that $x > 2i$, we have $| \{x \in A : x \leq 2i\} | \leq i$. □

Friedberg and Mucinik actually proved the following assertion.

Theorem (Friedberg, Muchnik)

There exist CE sets A, B such that $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$.

It is clear that A, B in this theorem are neither computable nor complete. By the finite injury priority argument, these sets are constructed as $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$ with the following requirements:

$$\begin{aligned} R_{2e} &: \quad A \neq W_e^B \\ R_{2e+1} &: \quad B \neq W_e^A \end{aligned}$$

Among many generalizations of the above theorem, the following theorem is particularly important.

Theorem (G. E. Sacks*)

Let C be an incomplete CE set.
(1) There is a simple set A such that $C \not\leq_{\mathrm{T}} A$.
(2) There exists low CE sets A, B s.t. $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$ with $C = A \cup B$ and $A \cap B = \varnothing$.

*For more detalis, refer to Soare (2016).

Polynomial-time reducibility

- Finally, we discuss the polynomial-time versions of m-reduction and T-reduction.
- A is **polynomial (time) reducible** to B ($A \leq_{\mathrm{P}} B$) if there exists a polynomial time computable function f and $x \in A \Leftrightarrow f(x) \in B$. This is a kind of m-reduciblity, which also written as $A \leq_{\mathrm{m}}^{\mathrm{P}} B$.
- On the other hand, A is **polynomial-time Turing reducible** to B ($A \leq_{\mathrm{T}}^{\mathrm{P}} B$ or $A \in \mathrm{P}^B$) if there exists a polynomial q and a deterministic Turing machine M^B with oracle B that can decide whether $x \in A$ within $O(q(|x|))$ time.
- We will not consider how to measure the time required for querying the oracle ($n \in B$). We only treat it very naively as shown in the proof of the next theorem.
- Furthermore, making M^B nondeterministic, we also defines $A \in \mathsf{NP}^B$.

It is clear that if $A \leq_{\mathrm{m}}^{\mathrm{P}} B$ then $A \leq_{\mathrm{T}}^{\mathrm{P}} B$. The reverse does not hold over a large class such as EXP(TIME) (Ladner, Lynch, and Selman [1975]).

Theorem (Baker, Gill, Solovay (1975))

(1) There exists a computable oracle A such that $\mathrm{P}^A = \mathrm{NP}^A$.
(2) There exists a computable oracle A such that $\mathrm{P}^A \neq \mathrm{NP}^A$.

Proof To show (1)

- Let A be a PSPACE complete problem such as TQBF (Lecture02-06). First, obviously $\mathrm{P}^A \subset \mathrm{NP}^A \subset \mathrm{PSPACE}^A$.
- Since A is PSPACE, one can compute PSPACE^A in PSPACE without using A as an oracle. That is, $\mathrm{PSPACE}^A \subset \mathrm{PSPACE}$.
- Finally, due to the PSPACE completeness of A, $\mathrm{PSPACE} \subset \mathrm{P}^A$.
- Therefore, $\mathrm{P}^A = \mathrm{NP}^A = \mathrm{PSPACE}^A$.

To show (2)

- For any $A \subset \{0,1\}^*$, $B = \{0^{|x|} : x \in A\}$ is in NP^A.
- So, we only need to construct a computable $A = \bigcup_s A_s$ such that $B \notin \mathrm{P}^A$.
- Let M_e enumerate deterministic machines (or sets accepted by such machines) running in polynomial p_e time.
- We want to prove $R_e : M_e^A \neq B$ for all e. That is, for each e, we guarantee the existence of n such that
$$M_e^A(0^n) \neq B(0^n).$$
- Assume that A_s is constructed at step $s = e$. Then, take n greater than any number used in the previous constructions and $2^n > p_e(n)$.
- When $M_e^{A_s}(0^n) = 1$, set $A_{s+1} = A_s$. Since a word with length n will never be added to A, we have $B(0^n) = 0$.
- Next assume $M_e^{A_s}(0^n) = 0$. Since this computation queries the oracle A_s at most $p_e(n)$ times, by the assumption $2^n > p_e(n)$ there is a word x of length n that is irrelevant to the oracle query. So if we set $A_{s+1} = A_s \cup \{x\}$, $M_e^{A_{s+1}}(0^n) = 0$, but $B(0^n) = 1$. □

Further Reading

- Kozen, D. C. (2006). *Theory of computation* (Vol. 170). Heidelberg: Springer.
- Soare, R. I. (2016). *Turing computability. Theory and Applications of Computability*. Springer.

Thank you for your attention!