Logic and Computation II

Part 6. Recursion-theoretic hierarchies

Kazuyuki Tanaka

BIMSA

April 27, 2023

Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Apr.25, (1) Oracle computation and relativization
- Apr.27, (2) m-reducibility and simple sets
- May 4, (3) T-reducibility and Post's problem
- May 9, (4) Arithmetical hierarchy and polynomial-time hierarchy
- May 11, (5) Analytical hierarchy and descriptive set theory I
- May 16, (6) Analytical hierarchy and descriptive set theory II

Today's topics

Recap

- Fix a function $\xi : \mathbb{N} \to \mathbb{N}$. Then, a function $f : \mathbb{N}^n \to \mathbb{N}$ is said to be **computable** in **oracle** ξ if there exists an algorithm that computes f using ξ as a database.
- The three classes of functions (primitive recursive / recursive / partial recursive) are extended as primitive recursive in ξ / recursive in ξ / partial recursive in ξ, by adding ξ to the initial functions in each definition.
- Almost all the theorems of recursion theory can be extended to statements with oracle ξ, which are called **relativizations** of the original theorems.
- Relativized Kleene normal form theorem: A partial recursive function in ξ can be expressed as $U(\mu y T^\xi(e, x_1, \cdots, x_n, y))$, also denoted $\{e\}^\xi(x_1, \cdots, x_n)$.
- A partial recursive functional $F : \mathbb{N}^n \times (\mathbb{N}^\mathbb{N})^k \to \mathbb{N}$ is represented as

$$F(x_1, \cdots, x_n, \xi_1, \cdots, \xi_k) = U(\mu y T(e, x_1, \cdots, x_n, y, \xi_1 \restriction y, \cdots, \xi_k \restriction y)).$$

Definition (Relativized arithmetical hierarchy)

Given a $\xi : \mathbb{N} \to \mathbb{N}$ and $k \geq 0$, the following set A is said to be $\Sigma_{2k+1}(\xi)$ (with index e).

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 \forall y_2 \cdots \exists y_{2k-1} \forall y_{2k} \{e\}^{\xi}(x_1, \ldots, x_n, y_1, \ldots, y_{2k}) \downarrow .$$

The following set A is a $\Sigma_{2k+2}(\xi)$ set (with index e).

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 \forall y_2 \cdots \forall y_{2k} \exists y_{2k+1} \{e\}^{\xi}(x_1, \ldots, x_n, y_1, \ldots, y_{2k}) \uparrow .$$

$\Pi_k(\xi)$ is the complement of $\Sigma_k(\xi)$. $\Delta_k(\xi)$ is $\Sigma_k(\xi)$ and $\Pi_k(\xi)$.

Theorem (Relativized arithmetical hierarchy theorem)

For every $k \geq 1$, $\Sigma_k(\xi) \cup \Pi_k(\xi) \subsetneq \Delta_{k+1}(\xi)$.

Theorem (Post)

A is $\Delta_{k+1}(\xi)$ if and only if there exists some $\Sigma_k(\xi)$ set B such that A is computable in χ_B $(A \leq_T B)$. In particular, A is Δ_2 if and only if $A \leq_T \mathrm{K}$.

introduction

- The early concern in recursion theory or computability theory was to understand the structure of the m-degrees and T-degrees of CE sets.
- The **m-degree** of a set A is the equivalence class of A in the many-to-one reducibility $\leq_{\mathrm{m}}$. The **T-degree of** A is the equivalence class of A in the Turing reducibility $\leq_{\mathrm{T}}$.
- Obviously, there are at least two CE T-degrees. That is, the degree of the computable sets (or the degree of $\varnothing$) and the degree of the complete CE sets (or the degree of the halting problem).
- Since any m-degree is a subset of a T-degree, there are at least two CE m-degrees (except for $\varnothing$ and $\mathbb{N}$).

- Furthermore, Post showed that there are more than two m-degrees of CE sets, and raised the corresponding question about T-degrees (1944).

- **Post's problem** motivated the deep research on degree structures, and was independently solved by Friedberg[a] and Muchnik[b].

- The technique used in their proof is called the **finite injury priority method**, and subsequently many improvements and developments have been made, such as infinite injury priority method and tree injury priority method.

Emil Post

R. M. Friedberg

A. A. Muchnik

[a] R. M. Friedberg, Two recursively enumerable sets of incomparable degrees of unsolvability, Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 236-238

[b] A. A. Muchnik, Negative answer to the problem of reducibility of the theory of algorithms(Russian), Dokl. Akad. Nauk SSSR 108 (1956), 194-197.

- First, let us review the basic concepts and results in part 1 of last semester.

Recall Lecture01-06

- A sequence (or set) of partial computable functions $\varphi_0, \varphi_1, \varphi_2, \dots$ (with repetition) is called a CE numbering, if $\varphi(e, x) := \varphi_e(x)$ is a partial computable function.
- A sequence of CE sets $A_0, A_1, A_2, \dots$, is called a CE numbering if $\{\langle e, x\rangle : x \in A_e\}$ is CE.
- The CE numbering of partial computable functions $\varphi_0, \varphi_1, \varphi_2, \dots$ is called a Gödel numbering if for any CE numbering $\psi_0, \psi_1, \psi_2, \dots$, there exists a computable function σ such that for any e,

$$\psi_e(x) \sim \varphi_{\sigma(e)}(x).$$ (both undefined, or both defined and the same value)

- A typical Gödel numbering is $\{\{e\} : e \in \mathbb{N}\}$, where $\{e\}$ is Kleene's bracket notation.
- For a Gödel numbering $\varphi_0, \varphi_1, \ldots$, let $W_e = \{x \mid \varphi_e(x)\downarrow\}$, where $\varphi_e(x)\downarrow$ means that φ_e is defined at x. Then, $W_0, W_1, \ldots$ is a CE numbering.
- A typical incomputable CE set is the halting problem K defined as follows.

$$\mathrm{K} \quad := \quad \{x \mid \varphi_x(x)\downarrow\} \quad = \quad \{x \mid x \in W_x\}.$$

- For $A, B \subset \mathbb{N}$, if there exists a computable function f, for any x,

$$x \in A \quad \Leftrightarrow \quad f(x) \in B$$

  then we write $A \leq_{\mathrm{m}} B$.
- $\varnothing$ and $\mathbb{N}$ are minimal with respect to $\leq_{\mathrm{m}}$. K is an m-complete CE set. Also, there is the degree of computable sets (except for $\varnothing$ and $\mathbb{N}$).

Theorem (Lecture01-06)

For any $A \subset \mathbb{N}$, the following statements are equivalent.
(1) $A \leq_{\mathrm{m}} \mathrm{K}$.
(2) $A \leq_1 \mathrm{K}$.
(3) A is CE.

Definition (Lecture01-06)

We say that a set A is **m-complete** (with respect to CE) if A is CE and $B \leq_{\mathrm{m}} A$ for any CE set B.

- If A is computable in oracle B (or recursive in χ_B), we write $A \leq_{\mathrm{T}} B$.
- If $A \leq_{\mathrm{m}} B$ then $A \leq_{\mathrm{T}} B$.
- $\mathrm{K}^c \leq_{\mathrm{T}} \mathrm{K}$ is obvious, where K^c is the complement of K. But not $\mathrm{K}^c \leq_{\mathrm{m}} \mathrm{K}$.
- If $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} A$, we write $A \equiv_{\mathrm{m}} B$. If $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$, then $A \equiv_{\mathrm{T}} B$.
- The m-degree of A is $\{B : B \equiv_m A\}$. The T-degree of A is $\{B : B \equiv_T A\}$

Homework

Show that $\mathrm{K}^c \leq_{\mathrm{m}} \mathrm{K}$ does not hold.

- The minimum degree with respect to $\leq_{\mathrm{m}}$ (except for $\varnothing$ and $\mathbb{N}$) is the equivalence class consisting of all computable sets.
- The maximum CE m-degree is the class of m-complete CE sets.
- Post showed that there exists a CE m-degree between these two.

Theorem (Post theorem, 1944)

There exists a CE set that is neither computable nor m-complete.

- Following the above theorem, Post also sought an intermediate T-degree, which is known as Post's problem. To challenge it, various notions of CE sets (such as immune sets, simple sets, and productive sets) were introduced.

Definition

An infinite set $B \subset \mathbb{N}$ that does not contain an infinite CE subset is called an **immune set**. A CE set $A \subset \mathbb{N}$ whose complement is an immune set is called a **simple set**.

- A simple set is a CE set that has a nonempty intersection with any infinite CE set and whose complement is an infinite set.
- Simple sets are not computable. This is because if it were computable, then its complement would be an infinite CE set. As we will see later, this set is closely related to the incompleteness theorem.
- First, we must show the existence of simple sets, which is easily derived from the following lemma.

Lemma (Dekker)

Let $f : \mathbb{N} \to \mathbb{N}$ be a computable injection. Then

$$\mathrm{Range}(f) \equiv_{\mathrm{T}} \{n : \exists m > n \;\; (f(m) < f(n))\}.$$

The set on the right-hand side is called the **deficiency set** of f, denoted by $\mathrm{Dfc}(f)$.

proof

- Let $f : \mathbb{N} \to \mathbb{N}$ be a computable injection and $A = \mathrm{Range}(f)$ and $B = \mathrm{Dfc}(f)$.
- To show $B \leq_{\mathrm{T}} A$.
- $n \in B$ is equivalent to $\exists k < f(n)(k \in A \land f^{-1}(k) > n)$, the latter of which is computable in A. Note that if $k \in A$ is known, it is easy to compute the value $f^{-1}(k) = \mu x(f(x) = k)$.
- To show $A \leq_{\mathrm{T}} B$.
- The complement B^c of B is always infinite. Because for any x, if we take y_x such that $f(y_x) = \min\{f(y) : y \geq x\}$, then $y_x \in B^c \land y_x \geq x$.
- Obviously $n \in A$ is equivalent to $\exists l \leq k(f(l) = n))$ for a sufficiently large k. So it is also equivalent to a Σ_1 formula $\exists k > n(k \in B^c \land f(k) > n \land \exists l \leq k(f(l) = n))$.
- Also, if $k \in B^c \land f(k) > n$, then $\forall l > k(f(l) > f(k) > n)$, and so $n \in A$ is equivalent to a Π_1 formula $\forall k > n(k \in B^c \land f(k) > n \to \exists l \leq k(f(l) = n))$.
- Therefore, A is computable in B. $\square$

In part 1, we prove that any nonempty CE set A is represented as the range of computable injection f. Then, if A is not computable, then Dfc(f) is a simple set. For example, setting $A = \mathrm{K}$ gives a simple set.

Lemma (1)

$f : \mathbb{N} \to \mathbb{N}$ is a computable injection, and if $\mathrm{Range}(f)$ is not computable, then $\mathrm{Dfc}(f)$ is a simple set.

Proof

Since $B = \mathrm{Dfc}(f)$ is also not computable, it is clear that its complement is not finite.
By way of contradiction, we assume that there exists an infinite CE set $C \subset B^c$.
Then by the second half of the proof for the lemma in page 12, $n \in \mathrm{Range}(f)$ is equivalent to

$$\exists k > n(k \in C \land f(k) > n \land \exists l \leq k(f(l) = n))$$

and

$$\forall k > n(k \in C \land f(k) > n \to \exists l \leq k(f(l) = n)),$$

which is computable and contradicts the assumption. □

Incompleteness theorems and simple sets (1/2)

- For any CE set C, in a proper arithmetic system T, there exists Σ_1 formula $\varphi(x)$

$$n \in C \Leftrightarrow T \vdash \varphi(\overline{n}).$$

  Now suppose C is a simple set. Since $\{n : T \vdash \neg\varphi(\overline{n})\}$ is CE, if it is an infinite set, it has non-empty intersection with C, which implies the inconsistency of T.
- Thus, if T is a consistent system, $\{n : T \vdash \neg\varphi(\overline{n})\}$ is finite.
- On the other hand, since C^c is an infinite set, there are infinitely many n such that neither $\varphi(\overline{n})$ nor $\neg\varphi(\overline{n})$ can be proved in T.

Incompleteness theorems and simple sets (2/2)

- Various concrete examples of C or $\varphi(x)$ have been studied in relation to the incompleteness theorem. One of them is the set of non-random numbers.
- For $n \in \mathbb{N}$, $\mu e(\{e\}(0) = n)$ can be regareded as a minimal program that outputs n, and such e is called the **Kolmogorov complexity** of n, represented by $\mathrm{K}(n)$.
- When $\mathrm{K}(n) \geq n$, n is called **random**.
- Then the set $\{n : \mathrm{K}(n) < n\}$ of non-random numbers is a simple set.
- It turns out that there are only finitely many numbers that can be proven to be random in an appropriate system of arithmetic.

Homework

Show that $\{n : \mathrm{K}(n) < n\}$ is a simple set.

Definition

A set $A \subset \mathbb{N}$ that satisfies the following condition is called **productive**:

- There exists a computable function f such that $f(x) \in A - W_x$ for each $W_x \subset A$.

Such an f is called a **productive function** for A.

Productive sets are not CE sets.

Example

The complement K^c of K is a productive set whose productive function is the identity map $\lambda x.x$, where $(x \mapsto f(x)$ is represented as $\lambda x.f(x))$.
To show this, suppose $W_x \subset \mathrm{K}^c$. By the definition of K, $x \in W_x \Leftrightarrow x \in \mathrm{K}$. Then either $x \in W_x \wedge x \in K$ or $x \notin W_x \wedge x \notin K$, where the former contradicts with $W_x \subset K^c$. So, only the latter case holds, that is, $x \in \mathrm{K}^c - W_x$.

Lemma (2)

A productive set contains an infinite CE subset. Hence the complement of a simple set is not a productive set.

Proof Let C be a productive set with a productive function f. We will construct an infinite CE subset of C by applying f repeatedly from $\varnothing$.
First, let i_0 be the index of the empty set. That is,

$$W_{i_0} = \varnothing \subset C.$$

Suppose now that $W_{i_n} \subset C$ has been constructed. Then, since $f(i_n) \in C - W_{i_n}$, by putting

$$W_{i_{n+1}} \quad := \quad W_{i_n} \cup \{f(i_n)\},$$

we have $W_{i_{n+1}} \subset C$.
Here, since i_{n+1} is computable in i_n, the set $\{f(i_0), f(i_1), f(i_2), \dots\}$ is an infinite CE subset of C. The second half follows from the definition of simple sets. □

A CE set A such that $B \leq_{\mathrm{m}} A$ for any CE set B is called an m-complete CE set. In particular, K is an m-complete CE set.

Lemma (3)

If A is an m-complete CE set, then A^c is a productive set.

Proof Let A be an m-complete CE set. Then there exists a computable function f such that for any x

$$x \in \mathrm{K} \Leftrightarrow f(x) \in A.$$

Thus $\mathrm{K}^c = f^{-1}(A^c)$.
Now let $\tau(e)$ be the index of $\lambda x.\varphi_e(f(x))$. That is,

$$W_{\tau(e)} = \{x \mid \varphi_e(f(x))\downarrow\}$$

Then, for $W_e \subset A^c$,

$$W_{\tau(e)} = \{x \mid f(x) \in W_e\} = f^{-1}(W_e) \subset f^{-1}(A^c) = \mathrm{K}^c.$$

From the example in page 17, the identity map $\lambda x.x$ is a productive function on K^c, so

$$\tau(e) \in \mathrm{K}^c - W_{\tau(e)} = f^{-1}(A^c) - f^{-1}(W_e) = f^{-1}(A^c - W_e).$$

That is, $f(\tau(e)) \in A^c - W_e$. Thus $f \circ \tau$ is a productive function for A^c. □

Now we are ready to show the post theorem.

Theorem (Post theorem, 1944)

There exists an incomputable CE set that is not m-complete.

Proof By Lemma (1) there exists a simple set A. By Lemma (2) A^c is not productive. By Lemma (3) A is not m-complete. But from the definition of simple sets, A is CE . □

Homework

If $A \leq_{\mathrm{m}} B$ and A is productive, show that B is also productive.

Further Reading

- Kozen, D. C. (2006). *Theory of computation* (Vol. 170). Heidelberg: Springer.
- Soare, R. I. (2016). *Turing computability. Theory and Applications of Computability*. Springer.

Thank you for your attention!