Logic and Computation II

Part 6. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 6. Schedule

- Apr.25, (1) Oracle computation and relativization
- Apr.27, (2) m-reducibility and simple sets
- May 4, (3) T-reducibility and Post's problem
- May 9, (4) Arithmetical hierarchy and polynomial-time hierarchy
- May 11, (5) Analytical hierarchy and descriptive set theory I
- May 16, (6) Analytical hierarchy and descriptive set theory II

Today's topics

1. Introduction
2. Oracle computation
3. Relativization
4. Relativized arithmetical hierarchy
5. Post's theorem

Introduction

- Fix a function $\xi : \mathbb{N} \to \mathbb{N}$. Then, a function $f : \mathbb{N}^n \to \mathbb{N}$ is said to be **computable** in ξ if there exists an algorithm that computes f using ξ as a database.
- Consider a Turing machine as a computational model. Besides the usual input tape and working tapes, it is equipped with an infinite tape storing ξ as data, from which necessary information (values of $\xi(n)$) can be retrieved.
- Such a machine is called an **oracle Turing machine**. A function that can be computed by **oracle** ξ is called ξ-**computable** or **computable in** ξ.
- The three classes of functions defined in part 1 in last semester (primitive recursive functions, recursive functions, and partial recursive functions) are extended as primitive recursive functions in ξ, recursive functions in ξ, and partial recursive functions in ξ, by adding ξ to the initial functions in each definition.

Primitive recursive in ξ

Definition

Given a function $\xi : \mathbb{N} \to \mathbb{N}$, the functions primitive recursive in ξ are defined as below.

1. Constant 0, **successor function** $\mathrm{S}(x) = x + 1$, **projection** $\mathrm{P}^n_i(x_1, x_2, \ldots, x_n) = x_i$ $(1 \leq i \leq n)$ and ξ are primitive recursive in ξ.
2. **Composition**.
   If $g_i : \mathbb{N}^n \to \mathbb{N}$, $h : \mathbb{N}^m \to \mathbb{N}\,(1 \leq i \leq m)$ are primitive recursive in ξ, so is $f = h(g_1, \ldots, g_m) : \mathbb{N}^n \to \mathbb{N}$ defined as below:
   $$f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$
3. **Primitive recursion**.
   If $g : \mathbb{N}^n \to \mathbb{N}$, $h : \mathbb{N}^{n+2} \to \mathbb{N}$ are primitive recursive in ξ, so is $f : \mathbb{N}^{n+1} \to \mathbb{N}$ defined as below:
   $$f(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n),$$
   $$f(x_1, \ldots, x_n, y + 1) = h(x_1, \ldots, x_n, y, f(x_1, \ldots, x_n, y)).$$

Recursive in ξ

Definition

The functions recursive in ξ are defined as below.

1. Constant 0,
   Successor function $\mathrm{S}(x) = x + 1$,
   Projection $\mathrm{P}_i^n(x_1, x_2, \cdots, x_n) = x_i$ $(1 \leq i \leq n)$ and ξ are recursive in ξ.
2. **Composition**. Analogous to primitive recursive in ξ.
3. **Primitive recursion**. Analogous to primitive recursive in ξ.
4. **minimalization** (minimization).
   Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be recursive in ξ satisfying that $\forall x_1 \cdots \forall x_n \exists y\ g(x_1, \cdots, x_n, y) = 0$. Then, the function $f : \mathbb{N}^n \to \mathbb{N}$ defined by

$$f(x_1, \cdots, x_n) = \mu y(g(x_1, \cdots, x_n, y) = 0)$$

   is recursive in ξ, where $\mu y(g(x_1, \cdots, x_n, y) = 0)$ denotes the smallest y such that $g(x_1, \cdots, x_n, y) = 0$.

Partial recursive in ξ (part 1/3)

Definition

The function partial recursive in ξ are defined as follows.

1. Constant 0, **Successor function** $\mathrm{S}(x) = x + 1$, **Projection** $\mathrm{P}_i^n(x_1, x_2, \cdots, x_n) = x_i$ $(1 \leq i \leq n)$ and ξ are partial recursive in ξ.
2. **Composition**. If $g_i : \mathbb{N}^n \to \mathbb{N}, h : \mathbb{N}^m \to \mathbb{N}(1 \leq i \leq m)$ are partial recursive in ξ, the composed function $f = h(g_1, \cdots, g_m) : \mathbb{N}^n \to \mathbb{N}$ defined by

$$f(x_1, \cdots, x_n) \sim h(g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n))$$

   is partial recursive in ξ, where $h(g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n)) = z$ means that each $g_i(x_1, \cdots, x_n) = y_i$ is defined and $h(y_1, \cdots, y_m) = z$.

Note: By $f(x_1, \cdots, x_n) \sim g(x_1, \cdots, x_n)$, we mean that either both functions are undefined or defined with the same value.

Partial recursive in ξ (part 2/3)

Definition

3. **Primitive recursion.**
If $g : \mathbb{N}^n \to \mathbb{N}, h : \mathbb{N}^{n+2} \to \mathbb{N}$ are partial recursive in ξ, the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$\begin{aligned} f(x_1, \cdots, x_n, 0) &\sim g(x_1, \cdots, x_n) \\ f(x_1, \cdots, x_n, y+1) &\sim h(x_1, \cdots, x_n, y, f(x_1, \cdots, x_n, y)) \end{aligned}$$

is partial recursive in ξ.

Partial recursive in ξ (part 3/3)

Definition

4. **Minimization**.

- Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be partial recursive in ξ.
- If "$g(x_1, \cdots, x_n, c) = 0$, and for each $z < c$, $g(x_1, \cdots, x_n, z)$ is defined with non-zero values", then we put $\mu y(g(x_1, \cdots, x_n, y) = 0) = c$;
  if there is no such c, then $\mu y(g(x_1, \cdots, x_n, y) = 0)$ is undefined.
- Then $f : \mathbb{N}^n \to \mathbb{N}$ satisfying

$$f(x_1, \cdots, x_n) \sim \mu y(g(x_1, \cdots, x_n, y) = 0)$$

  is partial recursive in ξ.

Definition

An n-ary relation $R \subset \mathbb{N}^n$ is called **(primitive) recursive in** ξ, if its characteristic function $\chi_R : \mathbb{N}^n \to \{0,1\}$ is (primitive) recursive in ξ;

$$\chi_R(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if } R(x_1,\ldots,x_n) \\ 0 & \text{otherwise} \end{cases}$$

- All the theorems of recursion theory mentioned in part 1 of the last semester can be extended to statements with oracles, which are called **relativizations** of the original theorems. We will show some examples of relativization in the following slides.
- The (partial) recursive functions in ξ also match the (partial) computable functions in ξ, and the domain of a partial recursive function in ξ is called compututably enumerable in ξ (ξ-CE).

Theorem (Relativized Kleene normal form theorem)

There are a primitive recursive function $U(y)$ and a primitive recursive relation in ξ $T^\xi(e, x_1, \cdots, x_n, y)$ such that if $f(x_1, \cdots, x_n)$ is partial recursive in ξ , then there exists e such that

$$f(x_1, \cdots, x_n) \sim U(\mu y T^\xi(e, x_1, \cdots, x_n, y)),$$

where $\mu y T^\xi(e, x_1, \cdots, x_n, y)$ takes the smallest value y satisfying $T^\xi(e, x_1, \cdots, x_n, y)$; if there is no such y, it is undefined.

Proof.

- We define a relation $T^\xi(e, x_1, \cdots, x_n, y)$ as follows:
  $T^\xi(e, x_1, \cdots, x_n, y) \Leftrightarrow$ "y is the Gödel number (code) of the whole computation process γ of TM of index e with input $(x_1, \cdots, x_n)$ and oracle ξ"
- The whole computation process γ is a sequence of configurations $\alpha_0 \triangleright \alpha_1 \triangleright \cdots \triangleright \alpha_n$ with an initial α_0 and an accepting α_n, which can regarded as a word over $\Omega \cup Q \cup \{\triangleright\}$.
- In general, it is not decidable whether a whole computation process γ exists or not. But for a given γ, we can easily check that for each $i < n$, $\alpha_i \triangleright \alpha_{i+1}$ is a valid transition of a TM, as well as α_0 and α_n are an initial and accepting configurations.

Some remarks on the proof

- A primitive recursive function $U(y)$ that extracts the output from the code of the computational process does not depend on ξ.
- We call $U(\mu y T^{\xi}(e, x_1, \cdots, x_n, y))$ a **partial recursive function in** ξ of **index** e, denoted as $\{e\}^{\xi}(x_1, \cdots, x_n)$.
- If ξ in $\{e\}^{\xi}(x_1, \cdots, x_n)$ is regarded as an argument, it can be rewritten as $\{e\}(x_1, \cdots, x_n, \xi)$.
- Notice that to evaluate $\{e\}(x_1, \cdots, x_n, \xi)$, at most the initial segment $\xi \upharpoonright y$ is used in the calculation, where y is the code of the whole calculation process γ. Furthermore, if the finite sequence $\xi \upharpoonright y$ is identified with its code, $\{e\}(x_1, \cdots, x_n, \xi \upharpoonright y)$ becomes an ordinary partial recursive function.

$\square$

Definition

Let $U(y)$ and T be primitive recursive functions defined in and after the relativized Kleene normal form theorem. The following function $F : \mathbb{N}^n \times (\mathbb{N}^\mathbb{N})^k \to \mathbb{N}$ is called a **partial recursive functional** with **index** e,

$$F(x_1, \cdots, x_n, \xi_1, \cdots, \xi_k) = U(\mu y T(e, x_1, \cdots, x_n, y, \xi_1 \restriction y, \cdots, \xi_k \restriction y)).$$

- Here $\mathbb{N}^\mathbb{N}$ is the set of total functions from $\mathbb{N}$ to $\mathbb{N}$. The domain D of a partial recursive functional $F : \mathbb{N}^n \times (\mathbb{N}^\mathbb{N})^k \to \mathbb{N}$ is

$$(x_1, \cdots, x_n, \xi_1, \cdots, \xi_k) \in D \Leftrightarrow \exists y T(e, x_1, \cdots, x_n, y, \xi_1 \restriction y, \cdots, \xi_k \restriction y),$$

  which is called a CE set (in a broad sense) or Σ^0_1 set.
- Such general classes will be treated in the following lectures.

Theorem (Relativized enumeration theorem)

$\{e\}^\xi(x_1, \cdots, x_n)$ is partial recursive in ξ on $e, x_1, \cdots, x_n$, and it is also a partial recursive functional on $e, x_1, \cdots, x_n, \xi$.

Theorem (Relativized parameter theorem)

For any $m, n \geq 1$, there exists a primitive recursive function $S^m_n : \mathbb{N}^{m+1} \to \mathbb{N}$ such that

$$\{e\}^\xi(x_1, \cdots, x_n, y_1, \cdots, y_m) \sim \{S^m_n(e, y_1, \cdots, y_m)\}^\xi(x_1, \cdots, x_n).$$

Theorem (Relativized recursion theorem)

Let $f(x_1, \cdots, x_n, y)$ be partial recursive in ξ. There exists e such that

$$\{e\}^\xi(x_1, \cdots, x_n) \sim f(x_1, \cdots, x_n, e).$$

Relativized arithmetical hierarchy

- Before dealing with the relativization of hierarchies, recall some basic definitions.
- We inductively define hierarchical classes of formulas Σ_n and Π_n in Lecture04-02. The sets of natural numbers defined by the Σ_1 formulas coincides with the CE sets as proved in Lecture04-03.
- When discussing the form of formulas, the same kind of quantifiers are joined together as follows.

$$\exists x_1 \cdots \exists x_n \varphi(x_1, \cdots, x_n) \Leftrightarrow \exists x \varphi(c(x, 0), \cdots, c(x, n-1))$$

where $c(x, i)$ is a primitive recursive function that extracts the i-th element x_i in the sequence with code x.

Definition (Relativized arithmetic hierarchy)

Given a $\xi : \mathbb{N} \to \mathbb{N}$ and $k \geq 0$, the following set A is said to be $\Sigma_{2k+1}(\xi)$ (with index e).

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 \forall y_2 \cdots \exists y_{2k-1} \forall y_{2k} \{e\}^{\xi}(x_1, \ldots, x_n, y_1, \ldots, y_{2k}) \downarrow .$$

The following set A is a $\Sigma_{2k+2}(\xi)$ set (with index e).

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 \forall y_2 \cdots \forall y_{2k} \exists y_{2k+1} \{e\}^{\xi}(x_1, \ldots, x_n, y_1, \ldots, y_{2k}) \uparrow .$$

$\Pi_k(\xi)$ is the complement of $\Sigma_k(\xi)$. $\Delta_k(\xi)$ is $\Sigma_k(\xi)$ and $\Pi_k(\xi)$.

- Here, $\downarrow$ / $\uparrow$ means that the function is defined / undefined.
- We fix the arity n of set $A \subset \mathbb{N}^n$ arbitrarily so that $\Sigma_k(\xi)$ and $\Pi_k(\xi)$ sets are treated complementary. In fact, it is enough to consider the case $n = 1$ using the sequence code $c(x, i)$.

Homework

Show that if R, S are $\Sigma_3(\xi)$ sets, so is $R \cap S$. Show that if R is defined by a $\Sigma_3(\xi)$-formula φ, so is the set defined by $\forall y < z \varphi$.

Theorem (Relativized arithmetical enumeration theorem)

For each $k \geq 1$, there exists $\Sigma_k(\xi)$ (or $\Pi_k(\xi)$) subset U of $\mathbb{N}^{n+1}$ with the following property (U is called a universal set). For any $\Sigma_k(\xi)$ (or $\Pi_k(\xi)$) subset R of $\mathbb{N}^n$, there exists some e such that

$$R(x_1, \cdots, x_n) \Leftrightarrow U(e, x_1, \cdots, x_n).$$

Proof.

- In the case of $\Sigma_1(\xi)$, it follows from the relativized enumeration theorem. For the $\Pi_1(\xi)$ set, take the complement of universal set U for $\Sigma_1(\xi)$.
- For $k > 1$, a $\Sigma_k(\xi)$ formula is obtained from a $\Sigma_1(\xi)$ or $\Pi_1(\xi)$ formula by adding appropriate arithmetical quantifiers in the front. Since there is a universal set (or formula) for $\Sigma_1(\xi)$ or $\Pi_1(\xi)$, the formula obtained from it by adding appropriate arithmetical quantifiers is universal for $\Sigma_k(\xi)$. Similarly for $\Pi_k(\xi)$. □

Theorem (Relativized arithmetical hierarchy theorem)

For every $k \geq 1$,

$$\Sigma_k(\xi) \cup \Pi_k(\xi) \subsetneq \Delta_{k+1}(\xi).$$

Proof.

- keys: relativized arithmetical enumeration theorem and diagonalization argument.
- By the relativized arithmetical enumeration theorem, there exists a universal $\Sigma_k(\xi)$ subset U of $\mathbb{N}^2$. Then consider the $\Pi_k(\xi)$ subset $V(e)$ of $\mathbb{N}^1$ defined by $\neg U(e,e)$.
- If $V(e)$ is $\Sigma_k(\xi)$, then there exists some e_0 such that $V(e) \Leftrightarrow U(e_0,e)$. By substituting $e = e_0$, we have $\neg U(e_0,e_0) \Leftrightarrow V(e_0) \Leftrightarrow U(e_0,e_0)$, which is a contradiction.
- Therefore, $V(e)$ is not $\Sigma_k(\xi)$.
- Furthermore, by setting $W(e) \Leftrightarrow \neg V(e)$, $W(e)$ is not $\Pi_k(\xi)$, but a $\Sigma_k(\xi)$ set.
- So, if we set $Z(e,d) \Leftrightarrow (V(e) \wedge d = 0) \vee (W(e) \wedge d > 0)$, then $Z(e,d)$ is clearly a $\Delta_{k+1}(\xi)$ subset of $\mathbb{N}^2$, which is neither $\Sigma_k(\xi)$ nor $\Pi_k(\xi)$. □

Comments on $k = 0$

- Note that we have not defined $\Sigma_0(\xi), \Pi_0(\xi)$. To define $\Sigma_0(\xi), \Pi_0(\xi)$ in the formal arithmetical hierarchy, ξ must also be a formal object such as a formula.
- However, $\Sigma_0(\xi), \Pi_0(\xi)$ are often used to denote the primitive recursive relations in ξ in some literature. Then, for the empty oracle ($\xi \equiv 0$), they are simply the primitive recursive relations, which contradicts with our formal definition: Σ_0, Π_0 represent bounded formulas or sets defined by them.
- Therefore, no formal definition is given. But a similar statement would hold whatever $\Sigma_0(\xi), \Pi_0(\xi)$ are defined, since $\Delta_1(\xi)$ is well-defined and large.

Lemma

A is $\Sigma_{k+1}(\xi)$ if and only if there exists some $\Pi_k(\xi)$ set B such that A is χ_B-CE, where χ_B is the characteristic function of B. For $k = 0$, consider $\Pi_0(\xi)$ as the primitive recursive relations in ξ.

Proof

- ($\Rightarrow$) Suppose A is $\Sigma_{k+1}(\xi)$. By definition, there exists a $\Pi_k(\xi)$ predicate $B(x_1, \ldots, x_n, y_1)$ such that

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 B(x_1, \ldots, x_n, y_1).$$

- Therefore,

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 \chi_B(x_1, \ldots, x_n, y_1) = 1,$$

and the right-hand side is χ_B-CE.

- ($\Leftarrow$) Let B be a $\Pi_k(\xi)$ set and A be χ_B-CE.
- By relativized Kleene's normal form theorem, we have,

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y T(e, x_1, \ldots, x_n, y, \chi_B \restriction y).$$

- Furthermore,

$$w = \chi_B \restriction y \Leftrightarrow \forall i < y(i \in B \Leftrightarrow w(i) = 1) \wedge \mathrm{leng}(w) = y,$$

and the right side is $\Delta_{k+1}(\xi)$. Combining both formulas, A is $\Sigma_{k+1}(\xi)$. □

In the above lemma, even if B is $\Sigma_k(\xi)$, the class of χ_B-CE does not change.

Theorem (Post)

A is $\Delta_{k+1}(\xi)$ if and only if there exists some $\Sigma_k(\xi)$ set B such that A is computable in χ_B $(A \leq_T B)$.

Corollary

A is Δ_2 if and only if $A \leq_T \mathrm{K}$.

Homework

Prove Post's theorem by using the last lemma in page 20.

Further Reading

- Kozen, D. C. (2006). *Theory of computation* (Vol. 170). Heidelberg: Springer.
- Soare, R. I. (2016). *Turing computability. Theory and Applications of Computability*. Springer.

Thank you for your attention!