Logic and Computation II

Part 5. Automata on infinite objects

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Mar.28, (1) Automata on infinite strings
- Mar.30, (2) The decidability of S1S
- Apr. 4, (3) Tree automata
- Apr. 6, (4) The decidability of S2S
- Apr.11, (5) Finite model theory
- Apr.13, (6) Parity games

Today's topics

Recap

The outline of the proof of the main lemma.

Lemma

For any PTA M, there is a PTA M' that accepts the complement of $L(M)$.

PTA M does not accept t $\Leftrightarrow$ II has a winning strategy σ for t in the game $G(M,t)$.

$\Updownarrow$

The $\Omega \times S_{\mathrm{II}}$ labeled $T^{t,\sigma}$ has no path satisfying the parity conditions.

$\Updownarrow$

The ω-language $L(t,\sigma)$ on $\Omega' = \Omega \times S_{\mathrm{II}}$ has no ω-sequence of states satisfying the parity condition.

$\Updownarrow$

PTA M' accepts t. $\Leftrightarrow$ $L(t,\sigma) \cap L(A) = \emptyset$. ← Let A be an NPA which accepts all ω-words on Ω'.

↑ Let A' be a DPA which accepts the complement of $L(A)$. Let M' be a PTA constructed form A'.

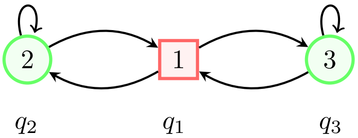
Parity games

- A parity game $G = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$ is a game on a directed graph $(V_{\mathsf{I}} \cup V_{\mathsf{II}}, E)$ with a priority function $\pi : V_{\mathsf{I}} \cup V_{\mathsf{II}} \to \{0, 1, \cdots, k\}$ and $V_{\mathsf{I}} \cap V_{\mathsf{II}} = \varnothing$.
- Two players, player I and II, move a token along the edges of the graph. At a vertex $v \in V_{\mathsf{I}}$ (V_{II}), it is player I (II)'s turn to choose some v' such that $(v, v') \in E$.
- For an infinite resulting path $\rho = \rho_0 \rho_1 \cdots$ (called a **play**), let $\pi(\rho) := \pi(\rho_0)\pi(\rho_1)\cdots$. Player I **wins** in ρ iff the smallest number appearing infinitely often in $\pi(\rho)$ is even.
- A strategy for player I is a mapping $\sigma : (V_{\mathsf{I}} \cup V_{\mathsf{II}})^{<\omega} V_{\mathsf{I}} \to V_{\mathsf{I}} \cup V_{\mathsf{II}}$.
  A play ρ is **consistent** with σ if for all i, $\rho_i \in V_{\mathsf{I}} \Rightarrow \sigma(\rho_0 \rho_1 \cdots \rho_i) = \rho_{i+1}$.
- σ is a **winning strategy** for player I if Player I wins in any play consistent with σ.
- A (winning) strategy for player II can be defined similarly.
- A game is said to be **determined** if one of the players has a winning strategy.
- Martin proved that Borel games (including parity games) are determined.

- A **memoryless strategy** for player I is a mapping $\sigma : V_{\mathsf{I}} \to V_{\mathsf{I}} \cup V_{\mathsf{II}}$.
- A **memoryless strategy** for player II is a mapping $\tau : V_{\mathsf{II}} \to V_{\mathsf{I}} \cup V_{\mathsf{II}}$.
- From now on, by a strategy we mean a memoryless strategy.
- A play ρ is **consistent** with such a σ if for all i, $\rho_i \in V_{\mathsf{I}} \Rightarrow \sigma(\rho_i) = \rho_{i+1}$. Similar for τ.
- σ (τ) is a **winning strategy** if player I (II) wins in any play consistent with σ (τ) .
- Let $W_{\mathsf{I}}(G, \sigma)$ be the set of starting points $\rho_0 \in V$ such that σ is a winning strategy for player I. Let
$$W_{\mathsf{I}}(G) = \bigcup_{\mathsf{I}'s \text{ winning strategy } \sigma} W_{\mathsf{I}}(G, \sigma).$$
- Similarly, $W_{\mathsf{II}}(G, \tau)$ and $W_{\mathsf{II}}(G)$ are defined.
- Clearly, $W_{\mathsf{I}}(G) \cap W_{\mathsf{II}}(G) = \varnothing$.
- When $W_{\mathsf{I}}(G) \cup W_{\mathsf{II}}(G) = V$, the game G is said to have **memoryless determinacy**.

Example (revisit)

Consider the following parity game $G = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$, where $V_{\mathsf{I}} = \{q_2, q_3\}$ and $V_{\mathsf{II}} = \{q_1\}$, $\pi(q_i) = i$ for $i = 1, 2, 3$.



- $W_{\mathsf{I}}(G) = \{q_2\}$
- $W_{\mathsf{II}}(G) = \{q_1, q_3\}$
- Since $W_{\mathsf{I}}(G) \cup W_{\mathsf{II}}(G) = V$, the above game G has memoryless determinacy.

Lemma

In any parity game G, there exists a strategy σ for player I such that $W_{\mathsf{I}}(G, \sigma) = W_{\mathsf{I}}(G)$. Similarly, there exists a II's strategy τ such that $W_{\mathsf{II}}(G, \tau) = W_{\mathsf{II}}(G)$.

Proof

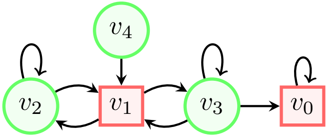
- By the well-ordering theorem, let $W_{\mathsf{I}}(G) = \{v_\beta\}_{\beta<\alpha}$ (α, β are ordinals).
- For each $\beta < \alpha$, let σ_β be a winning strategy of player I starting from v_β.
- Then, we define a function $f : W_{\mathsf{I}}(G) \to \alpha$ as follows: for $v \in W_{\mathsf{I}}(G)$, let $f(v)$ the smallest $\beta < \alpha$ such that $v \in W_{\mathsf{I}}(G, \sigma_\beta)$.
- Finally, we define a strategy σ as $\sigma(v) := \sigma_{f(v)}(v)$. We want to show that $W_{\mathsf{I}}(G, \sigma) = W_{\mathsf{I}}(G)$. Since $W_{\mathsf{I}}(G, \sigma) \subseteq W_{\mathsf{I}}(G)$, it is sufficient to show any play consistent with σ starting from a vertex of $W_{\mathsf{I}}(G)$ is a winning play for I.

- Now, let ρ be a play consistent with σ, starting from vertex ρ_0 of $W_{\mathsf{I}}(G)$.
- If ρ is also consistent with $\sigma_{f(\rho_0)}$, then player I wins in ρ, which completes the proof. Otherwise, we can get the smallest k such that $\rho_k \in V_{\mathsf{I}}$ and $\rho_{k+1} \neq \sigma_{f(\rho_0)}(\rho_k)$.
- Since $\rho \restriction (k+1)$ is consistent with $\sigma_{f(\rho_0)}$, player I can win the game from ρ_k following $\sigma_{f(\rho_0)}$, that is, $\rho_k \in W_{\mathsf{I}}(G, \sigma_{f(\rho_0)})$. But $\rho_{k+1} = \sigma(\rho_k) = \sigma_{f(\rho_k)}(\rho_k) \neq \sigma_{f(\rho_0)}(\rho_k)$, so $f(\rho_k) < f(\rho_0)$.
- Player I wins if ρ obeys $\sigma_{f(\rho_k)}$ from ρ_k onwards.
- Othewise, some k' appears such that $\rho_{k'} \in V_{\mathsf{I}}$ and $\rho_{k'+1} \neq \sigma_{f(\rho_k)}(\rho_{k'})$, then $f(\rho_{k'}) < f(\rho_k) < f(\rho_0)$.
- By repeating this, the descending sequence of ordinal numbers ends in finite steps. So there exists some $l \in \omega$ such that ρ is consistent with $\sigma_{f(\rho_l)}$ from ρ_l, and hence player I wins.
- Therefore, σ is I's winning strategy starting from any vertex of $W_{\mathsf{I}}(G)$. That is, $W_{\mathsf{I}}(G, \sigma) = W_{\mathsf{I}}(G)$.
- $W_{\mathsf{II}}(G, \tau) = W_{\mathsf{II}}(G)$ can be shown similarly. □

- If there exist σ and τ such that $W_{\mathsf{I}}(G,\sigma) \cup W_{\mathsf{II}}(G,\tau) = V$, game G is said to have **uniform memoryless determinacy**.
- From the above lemma, if a parity game has memoryless determinacy, it also has uniform memoryless determinacy.
- We say that $v \in V$ is an **absorbing vertex** if no edges exit from v, i.e., $\{w : (v,w) \in E\} = \{v\}$. Note that we assume that no deadlocks exist.
- We say that $v \in V$ is a **vanishing vertex** if no edges enter v, i.e., $\{w : (w,v) \in E\} = \varnothing$.
- Vertices that are neither absorbing nor vanishing are called **relevant vertices**, and the set of such vertices is denoted by V_{r}.
- $\pi(v)$ for $v \in V_{\mathrm{r}}$ is called a **relevant priority**.

Example 2

Consider the following parity game $G = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$, where $V_{\mathsf{I}} = \{v_2, v_3, v_4\}$ and $V_{\mathsf{II}} = \{v_0, v_1\}$, $\pi(v_i) = i$ for $i = 0, 1, 2, 3, 4$.



- $W_{\mathsf{I}}(G) = \{v_0, v_1, v_2, v_3, v_4\}$
- $W_{\mathsf{II}}(G) = \varnothing$
- The above game G is uniform memoryless determined.
- v_0 is absorbing, v_4 is vanishing, v_1,v_2 and v_3 are relevant.
- $\{1, 2, 3\}$ is the set of relevant priorities.

Theorem

Any parity game G has uniform memoryless determinacy.

Proof

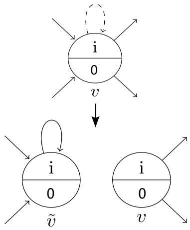
Consider a parity game $G = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$. We prove by induction on the number of relevant priorities $\pi(V_{\mathrm{r}})$.

Base case:

- If there are no relevant points, all vertices are absorbing or vanishing.
- From an absorbing vertex v, $v \in W_{\mathsf{I}}(G, \sigma)$ for any σ (if $\pi(v)$ is even), or $v \in W_{\mathsf{II}}(G, \tau)$ for any τ (otherwise).
- From a vanishing vertex v, each edge goes to an absorbing vertex, where the winner is determined regardless of the strategy. So, by selecting an appropriate $\sigma(v)$ or $\tau(v)$, we have $v \in W_{\mathsf{I}}(G, \sigma) \cup W_{\mathsf{II}}(G, \tau)$, where the values of σ and τ at other vertex than v are not irrelevant.
- Thus, there exist σ and τ such that $W_{\mathsf{I}}(G, \sigma) \cup W_{\mathsf{II}}(G, \tau) = V$.

<u>Induction case</u>:

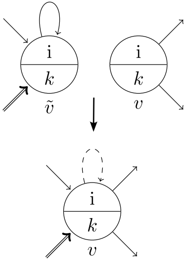
- Suppose the number of relevant priorities is $k > 0$. We first prove a weak claim $W_{\mathsf{I}}(G) \cup W_{\mathsf{II}}(G) \neq \varnothing$.
- For simplicity, assume that the minimum of the relevant priorities is 0.
- We will modify the game G so that the vertices with priority 0 are changed to non-relevant vertices. Such a modified game is called G^+, to which we will apply the induction hypothesis.
- Let D be the set of relevant vetices with priority 0 in G.
- Make a copy of D and put $\tilde{D} := \{\tilde{v} : v \in D\}$.
- $G^+ = (V_{\mathrm{I}}^+, V_{\mathsf{II}}^+, E^+, \pi^+)$ is defined as follows.
- $V_{\mathrm{I}}^+ := V_{\mathrm{I}} \cup \{\tilde{v} : v \in D \cap V_{\mathrm{I}}\}$,
- $V_{\mathsf{II}}^+ := V_{\mathsf{II}} \cup \{\tilde{v} : v \in D \cap V_{\mathsf{II}}\}$,
- $E^+ := \{(u, v) \in E : v \notin D\} \cup \{(u, \tilde{v}) : (u, v) \in E \wedge v \in D\} \cup \{(\tilde{v}, \tilde{v}) : v \in D\}$
- $\pi^+ := \pi \cup \{(\tilde{v}, 0) : v \in D\}$.



i=I, II

G^+ is obtained by separating each vertex v of D into vanishing vertex v and absorbing vertex $\tilde{v}$.

- Therefore, the number of relevant priorities of G^+ is one less than that of G.
- By induction hypothesis, there exist σ^+ and τ^+ such that $W_{\mathrm{I}}(G^+, \sigma^+) \cup W_{\mathrm{II}}(G^+, \tau^+) = V^+ = V_{\mathrm{I}}^+ \cup V_{\mathrm{II}}^+$.
- The strategies $\sigma^{\pm} : V_{\mathrm{I}} \to V$ and $\tau^{\pm} : V_{\mathrm{II}} \to V$ in G can be derived from $\sigma^+ : V_{\mathrm{I}}^+ \to V^+$ and $\tau^+ : V_{\mathrm{II}}^+ \to V^+$ by restricting it to V.
- That is, $\sigma^{\pm}$ restricts the domain of σ^+ to V_{I}, and when the value is $\tilde{v} \in \tilde{D}$, change it to v. $\tau^{\pm}$ can be obtained similarly.



- First, consider the case $W_{\mathrm{I}}(G^+, \sigma^+) = V^+$.
- Take any play ρ consistent with $\sigma^{\pm}$ in G.
- If a vertex of D appears infinitely many times in ρ, then player I wins in ρ.
- Otherwise, from some vertex in ρ (written as ρ') does not visit D, and so since ρ' obeys $\sigma^{\pm}$ in G, ρ' obeys σ^+ in G^+, which means that player I wins in G^+, and I also wins with ρ' in G. Finally, player I wins even with ρ in G, because any finite part of the play makes no difference to the parity condition.
- That is, $W_{\mathrm{I}}(G, \sigma^{\pm}) = V$.

- Next, consider the case $W_{\mathrm{I}}(G^+, \sigma^+) \neq V^+$.
- Then we have $v \in W_{\mathsf{II}}(G^+, \tau^+) = V^+ - W_{\mathrm{I}}(G^+, \sigma^+)$.
- Consider a play starting from v consistent with τ^+. If an absorbing vertex $\tilde{v} \in \tilde{D}$ appears in the middle, then after that it just repeats $\tilde{v}$, and so priority 0 appears infinitely often, which means player I wins. This contradicts with $v \in W_{\mathsf{II}}(G^+, \tau^+)$.
- Therefore, in such a play of G^+, a vanishing vertex may appear only at the start, and no vertex in $D \cup \tilde{D}$ appear in the middle.
- Thus, any play of G starting from v and consistent with $\tau^{\pm}$ does not enter D in the middle, and so it is also consistent with τ^+, which means player II wins. That is, $v \in W_{\mathsf{II}}(G, \tau^{\pm})$.
- Combining the above two cases, we can say at least $W_{\mathrm{I}}(G) \cup W_{\mathsf{II}}(G) \neq \varnothing$.

- Next we show $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G) = V$. By the way of contradiction, assume $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G) \neq V$.
- Let $V^- := V - (W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G))$ and consider the game G^- by restricting G to V^-.
- Note that for every $v \in V^-$ there is a $u \in V^-$ such that $(v, u) \in E$. Because if every u such that $(v, u) \in E$ belongs to $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G)$, so is v, which contradicts $v \in V^-$.
- Therefore, the game G^- is a correct parity game.
- Let $v \in W_{\mathrm{I}}(G^-)$ and σ^- be a winning strategy for I starting from v in G^-.
- Now consider a play ρ starting at v consistent with σ^- in G.
- At $u \in V_{\mathrm{II}} \cap V^-$ in the middle of play, no vertex of $W_{\mathrm{II}}(G)$ will be chosen in the next move. Because if it were selected, we would have $u \in W_{\mathrm{II}}(G)$, which contradicts $u \in V^-$.

- Therefore, ρ is a play consistent with σ^- in G^-, and so player I wins. That is, $v \in W_{\mathrm{I}}(G)$.
- But since $V^- \cap W_{\mathrm{I}}(G) = \varnothing$, $W_{\mathrm{I}}(G^-) = \varnothing$.
- Similarly, $W_{\mathrm{II}}(G^-) = \varnothing$, so $W_{\mathrm{I}}(G^-) \cup W_{\mathrm{II}}(G^-) = \varnothing$.
- Since G^- is a parity game with at most k relevant priorities, $W_{\mathrm{I}}(G^-) \cup W_{\mathrm{II}}(G^-) \neq \varnothing$, which contradicts the assumption of $W_{\mathrm{I}}(G, \sigma) \cup W_{\mathrm{II}}(G, \tau) \neq V$. □

Further readings

The above proof is based on S. Le Roux's paper:
"Memoryless determinacy of infinite parity games: Another simple proof", *Info. Proc. Letters* 143 (2019).
Le Roux's proof also relies on Haddad's paper: " Memoryless determinacy of finite parity games: another simple proof", *Info. Proc. Letters* 132 (2018) 19–21.
which in turn refers to many previous studies.

- In a parity game G over a finite graph, it can be checked in polynomial time whether a given memoryless strategy is a winning strategy. So $W_{\mathrm{I}}(G)$ is NP.
- Similarly $W_{\mathrm{II}}(G)$ is also NP and $W_{\mathrm{I}}(G) = V - W_{\mathrm{II}}(G)$, so $W_{\mathrm{I}}(G) \in$ NP$\cap$ co-NP.
- However, it is not yet known whether it will be in P, and currently it is $O(|G|^{\log n+6})$ (where n is the number of priorities), due to Calude-Jain-Khoussainov-Li-Stephan results (STOC 2017).

DECIDING PARITY GAMES IN QUASI-POLYNOMIAL TIME*

CRISTIAN S. CALUDE[†], SANJAY JAIN[‡], BAKHADYR KHOUSSAINOV[†], WEI LI[§], AND FRANK STEPHAN[‡§]

Abstract. It is shown that the parity game can be solved in quasi-polynomial time. The parameterized parity game—with n nodes and m distinct values (a.k.a. colors or priorities)—is proven to be in the class of fixed parameter tractable problems when parameterized over m. Both results improve known bounds, from runtime $n^{O(\sqrt{n})}$ to $O(n^{\log(m)+6})$ and from an **XP** algorithm with runtime $O(n^{\Theta(m)})$ for fixed parameter m to a fixed parameter tractable algorithm with runtime $O(n^5 + 2^{m\log(m)+6m})$. As an application, it is proven that colored Muller games with n nodes and m colors can be decided in time $O((m^m \cdot n)^5)$; it is also shown that this bound cannot be improved to $2^{o(m\cdot\log(m))} \cdot n^{O(1)}$ in the case that the exponential time hypothesis is true. Further investigations deal with memoryless Muller games and multidimensional parity games.

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Course Announcement:

- There is no class next week = no class on April 18 and 20, 2023.
- Before Golden Week, we still have two classes on April 25 and 27, 2023.