Logic and Computation II

Part 5. Automata on infinite objects

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Mar.28, (1) Automata on infinite strings
- Mar.30, (2) ω-automata and S1S
- Apr.04, (3) Tree automata
- Apr.06, (4) S2S
- Apr.11, (5) Finite model theory
- Apr.13, (6) Parity games

Today's topics

1. Recap
2. Decidability of S1S
3. Introduction to S1S
4. S1S vs. NBA

Recap

Büchi automata

- For an infinite run σ, the set of states that appear infinitely in σ is denoted by $\mathrm{Inf}(\sigma)$. In other words, if $\sigma = q_0 q_1 q_2 \cdots$,

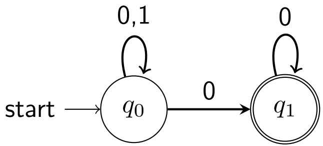
$$\mathrm{Inf}(\sigma) = \bigcap_{n \geq 0} \{q_i \mid i \geq n\}.$$

- An infinite run σ is said to be accepted by NBA if $\mathrm{Inf}(\sigma) \cap F \neq \varnothing$, that is, if a state of F occurs infinitely many times in σ.
- An input word α is accepted by NBA M if there is an accepted run on α.
- Thus, the ω-language $L(M) \subset \Omega^\omega$ accepted by M is defined as

$$L(M) = \{\alpha \in \Omega^\omega \mid \text{there is a run } \sigma \text{ of } M \text{ on } \alpha \text{ such that } \mathrm{Inf}(\sigma) \cap F \neq \varnothing\}.$$

Example

The following NBA accepts the set $(0+1)^*0^\omega$, where "1" appears finitely times.



where $Q = \{q_0, q_1\}$, $\Omega = \{0, 1\}$, $F = \{q_1\}$.

- Note that non-determinism of the Büchi automaton is necessary to guess when the last "1" appears so that the automaton can move to loop in q_1 with input always 0.
- In fact, this language cannot accepted by any DBA.

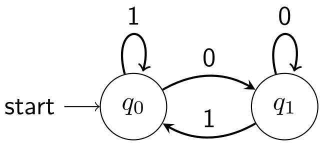
Muller condition

- The acceptance condition of a **Muller automaton** is given by $\mathcal{F} \subseteq \mathcal{P}(Q)$, and a run is accepted iff $\mathrm{Inf}(\sigma) \in \mathcal{F}$.
- Büchi condition ($\mathrm{Inf}(\sigma) \cap F \neq \varnothing$) can be expressed in terms of the Muller condition

$$\mathcal{F} = \{A \subseteq Q \mid A \cap F \neq \varnothing\}.$$

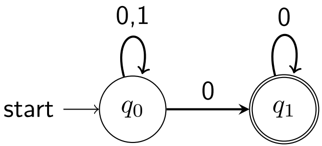
- Non-deterministic / deterministic Muller automata are abbreviated as NMA / DMA.

An DMA accepting $L = (0+1)^*0^\omega$



where $\mathcal{F} = \{\{q_1\}\}$.

An equivalent NBA

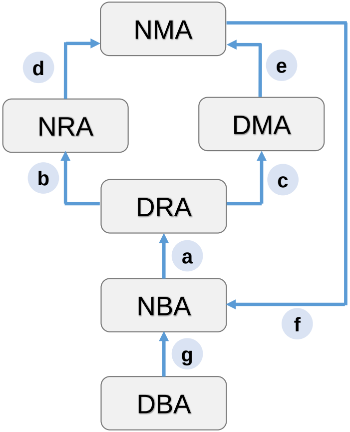


where $F = \{q_1\}$.

Rabin condition

- The acceptance condition of a **Rabin automaton** is given by
$$\mathcal{F} = \{(G_i, R_i) \mid 1 \leq i \leq k\},$$
where $G_i, R_i \subset Q$.
- A run σ is **accepted**, if there exists i such that $\mathrm{Inf}(\sigma) \cap G_i \neq \varnothing$ and $\mathrm{Inf}(\sigma) \cap R_i = \varnothing$.
- Non-deterministic / deterministic Rabin automata are abbreviated as NRA / DRA.
- When a G_i/R_i state is visited, we say that the i-th green/red signal is on. A green signal is expected to turn on infinitely many times but a red signal only finitely many.
- A Büchi automaton can be simulated by a Rabin automaton with
$$k = 1,\ G_1 = F,\ R_1 = \varnothing.$$
- A Rabin automaton turns into a Muller automaton if
$$\mathcal{F} = \{A \subseteq Q \mid \bigvee_i (A \cap G_i \neq \varnothing \wedge A \cap R_i = \varnothing)\}$$

- (b) and (e) are obvious. (c) and (d) have been explained above.
- To show (f). Please refer to the examples in Page 6.
  - Let M be an NMA with an accepting set $\mathcal{F}$. Goal: construct an NBA N to simulate M.
  - For input x, N mimics M by nondeterministically guessing a run σ of M on x.
  - At some point, N nondeterministically predicts that all states of M not in $\mathrm{Inf}(\sigma)$ have appeared and also guesses that $\mathrm{Inf}(\sigma)$ is a certain set $A \in \mathcal{F}$.
  - Then check if A is indeed $\mathrm{Inf}(\sigma)$ as follows:
    - Any state of σ (from that point) is in A, and
    - Let s be the state of N representing that every state of A appeared at least once. Then N accepts the input if s appears infinite many times.



"automaton $M_1 \to$ automaton M_2" means $L(M_1) \subset L(M_2)$.

- (a): NBA→ DRA is the most difficult to prove.
- It was first prove by McNaughton in 1966, but his construction was doubly exponential. Safra proposed a more efficient exponential construction in 1988.

NBA

Given $B = (Q, \Omega, \delta, Q_0, F)$ with $\mid Q \mid = n$

DRA

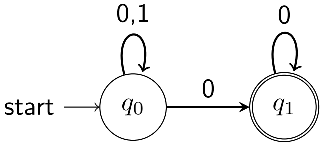
We want to construct a deterministic Rabin automaton

$$R = (S, \Omega, \delta', S_0, \{(G_1, R_1), (G_2, R_2) \cdots (G_n, R_n)\})$$

that accepts the same language.

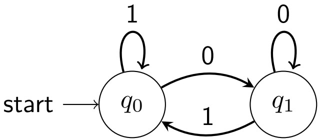
Consider $L = (0+1)^*0^\omega$, where 1 appears finitely times.

An NBA accepting L



L cannot accepted by any DBA.

An DRA accepting L



where $\mathcal{F} = \{(G_1, R_1)\} = \{(\{q_1\}, \{q_0\})\}$

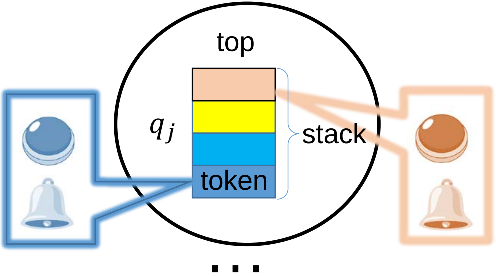
Strategy 1: Determinizing the non-determinism.

- We consider the state transition of NBA B as a graph. Their vertices representing the states are also called **positions** where a **token** is placed. All the tokens move from one position to another simultaneously, and some are removed.
- A board with a token on some positions is a state of the new deterministic automaton.
- The simulation starts with a board with a token at each initial state of NBA B.
- Suppose the next input symbol is a. Erase the token at each position q and put (a copy of) the token at each $p \in \delta(q, a)$. If multiple tokens are put in p, choose one.
- At time t, the state of the new automaton represents the possible states of B at time t as its tokens.
- For a finite automaton, by defining a final state of the new automata as a graph with at least one token in a final state of B, they accept the same language.
- This does not work for the Büchi condition. For instance, as for the above example of NBA on $(01)^\omega$, a token is placed on q_1 infinitely many times, but $(01)^\omega$ is not in L. For the correct simulation, the same token must be placed on a final state infinitely many times.

Strategy 2: Last visiting record

- Each token should have some partial history of visiting final positions. Indeed, such a token is expressed as a pile of colored tokens, which we call a **stack**.
- A stack is not ony moved according to the transition of B, but it may get another colored token on the top (at a final state position). Also, an upper part of a stack may be removed by a certain rule explained below.
- A token or its color on the board at time t is said to be **in play** at time t.
- The colors in play at time t are ordered by their **age**, namely the last time they appeared in play. Tokens of the same color in play come into play at the same time. In each stack at any time, the tokens are ordered from bottom to top as oldest to youngest. A color is **visible** if it is the color of the token on top of some stack.
- On the stacks, we define a reverse lexicographic linear order $\sigma \ll_t \tau$ as follows.
  - σ is a proper extension of τ (τ is obtained by removing the top of σ), or
  - Neither σ nor τ is an extension of the other, and at the lowest position where σ and τ differ in color, the color of σ is older.

- We consider the state transition of automaton B as a graph. Their vertices representing the states are called **positions**.
- A **stack** (or pile) of colored tokens is placed on some positions. A stack moves from one position to another, sometimes changes its contents, and sometimes gets removed.
- A board with some stacks on some positions is a state of the automaton R.
- The board is connected to different **bells** and **buzzers** for each color.
- The **height** of the stack σ is written as $|\sigma|$.



Buzzer and bell for each color

- The simulation starts with a board with one white token at each initial state position.
- At each time, the three steps are all executed in this order: Move, Cover, Remove

Move

- Suppose the next input symbol is a. Erase the stack at each position q and put (a copy of) the stack that was at q at each $p \in \delta(q, a)$.
- If there are multiple stacks to put in p, put the smallest stack with respect to $\ll_t$.
- If a certain color disappears in this process, sound the buzzer for that color.

Cover

- For each final state $q \in F$, put a token of a color not in play on the top of the stack at that position.
- Stacks with the same visible color are then covered with tokens of the same new color.
- Thus, if color c is placed directly above color d in a stack, then all tokens of color c in play are placed directly above tokens of color d.

Remove

- For any invisible color c in play, remove all tokens above tokens of color c, sound the buzzer of the removed color, and ring the bell of the visible color c.
- Note that when a token is removed in this process, all tokens of that color are removed. The order of removal is not important.

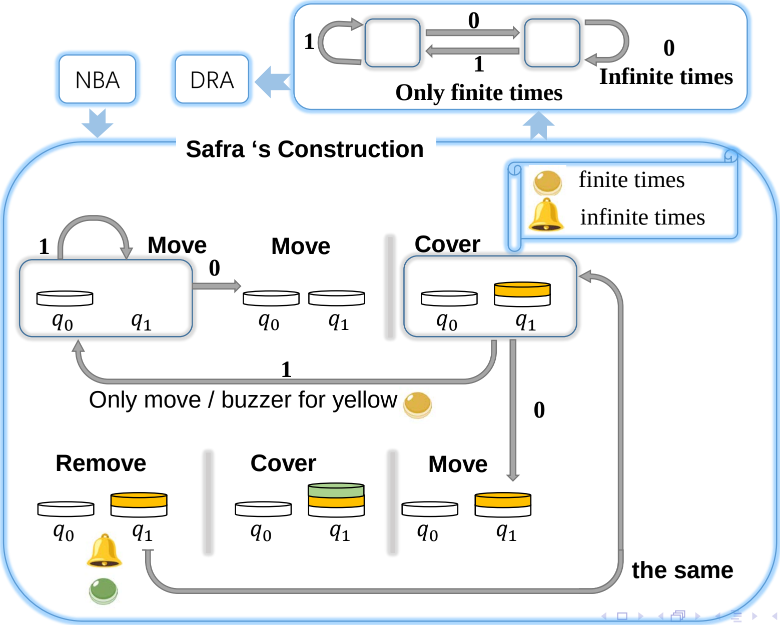
After performing these three steps, there are at most n ($=$ the number of the states) colors left in play. Otherwise, there must be at least one invisible color, then repeat the remove step.

Lemma

The following are equivalent

(1) B accepts x.

(2) There is a color that rings the bell infinitely many times but sounds the buzzer only finitely many times.


NBA
DRA
1
0
1
0
Only finite times
Infinite times
Safra 's Construction
finite times
infinite times
1
Move
0
Move
Cover
q_0
q_1
q_0
q_1
q_0
q_1
1
Only move / buzzer for yellow
0
Remove
Cover
Move
q_0
q_1
q_0
q_1
q_0
q_1
the same

Proof.

To show $(2) \Rightarrow (1)$

- Suppose that there exists a color, say yellow, that rings the bell infinitely many times but the buzzer a finitely many times.
- Let $t_0, t_1, \dots$ be the time when the yellow bell rings after the last buzzer.
- From time t_0, yellow continues to be in play. Otherwise the buzzer will sound.
- To get a yellow bell at each time t_i, all yellow tokens must be covered immediately. In other words, no matter how to move from a state with a yellow token at t_i to a state with a yellow token at t_{i+1}, it visits some state of F.
- Therefore, there exists a run where the state of F appears infinitely.

Proof.

To show $(1) \Rightarrow (2)$

- Conversely, suppose that there is an accepted run ρ of B for x.
- Let σ_t be the stack following ρ at time t. So, set $m = \liminf |\sigma_t|$.
- In other words, after a certain time t_0, the minimum stack height is m, and it reaches the height m infinitely many times.
- White (the oldest color) is always in play, so $m \geq 1$, and there are at most n colors in play, so $m \leq n$.
- After time t_0, colors in stacks at heights at most m may be replaced by smaller stacks on $\ll_t$ by translations, which only happens finite times by the definition of $\ll_t$.
- Therefore, the colors in the stack below m from a certain time t_1 can be assumed to remain unchanged.
- At this time, the color attached to the height m is assumed to be black.
- Since this sequence of actions is an accepted run, the state of F is visited infinitely many times from now on, and the stack gets a new token each time.
- Then the stack height returns to m again, but a black bell rings again.
- Therefore, the black bell rings infinite times and the buzzer rings only finite times. □

A state of the simulating Rabin automaton R consists of the state of B and the stack (which may be empty). The number of combinations of stacks is roughly n^n. The treatment of bells and buzzers and auxiliary machineries needs n^n at most. So, the states of R roughly $n^{kn} = 2^{O(n \log n)}$. The acceptance condition consists of n pairs, one for each color.

Therefore, we have

Theorem (Safra)

Any NBA with n states can be simulated with a DRA consisting of $2^{O(n \log n)}$ states and n pairs of acceptance conditions. Therefore, it can also be simulated with a DMA with the same number of states.

Corollary

The class of ω-regular languages is closed with Boolean operations.

Proof.

- We already known that the class of languages accepted by NBA is closed with $\cup$ and $\cap$.
- The closure of complement follows from the above theorem, classes of languages accepted by NBA and DMA are the same.
- In fact, a DMA that accepts the complement of the language of a DMA $M = (Q, \Omega, \delta, q_0, \mathcal{F})$ by replacing the acceptance condition $\mathcal{F}$ of M with $\mathcal{P}(Q) - \mathcal{F}$.

$\square$

Homework

Prove that $L = \{u^\omega : u \in \{0,1\}^+\}$ is not an ω-regular language.

Decidability of S1S

- We showed in Lecture-03-05 that the FO theory of $(\mathbb{N}, +, 0)$ is decidable, but the MSO theory of $(\mathbb{N}, +, 0)$ is undecidable since multiplication is definable there.
- Then, how far should the first-order part be weakened to make the MSO theory decidable?
- One of the answers is $(\mathbb{N}, x + 1, 0)$, and this MSO theory is S1S[1].
- Today we show that S1S and NBA (Nondeterministic Buchi Automata) have equivalent expressive power.
- Thus, the decision problem of S1S can be reduced to the emptiness problem of NBA.

[1] The first "S" stands for "second-order", and the next "1S" stands for "One Successor". Since it is different from the general second-order theory dealt with in the last homework of lecture04-05, it is more suitable to call it MS1S, but by convention it is called S1S .

- The original S1S is the MSO theory (or all the true MSO formulas) of the standard structure $(\mathbb{N} \cup \mathcal{P}(\mathbb{N}), x+1, \in)$.
- To show the decidability of S1S is to show its axiomatizability.
- In the following, S1S is treated as a two-sorted first-order theory, where numerical variables are represented by lower case letters $x, y, \ldots$, and set variables are represented by upper case letters $X, Y, \ldots$.
- It should be noted that a S1S formula holds iff it is true with the ordinary mathematical sense.

In S1S, the equality symbol $=$, the inequality symbol $\leq$, and the constant 0 are defined as follows, and they have their usual meanings.

- "$x = y$" : $\forall X(x \in X \leftrightarrow y \in X)$.
- "$X \subseteq Y$" : $\forall x(x \in X \to x \in Y)$.
- "$X = Y$" : $X \subseteq Y \land Y \subseteq X$.
- "$x = 0$" : $\forall y \neg(x = y + 1)$; x has no predecessor.
- "$x = 1$" : $x = 0 + 1$. Since 0 is defined above, 0 is treated like a given symbol. In terms of the original symbols, we can write $\exists y(x = y + 1 \land \forall z \neg(y = z + 1))$.
- "$x \leq y$" : $\forall X(x \in X \land \forall z(z \in X \to z + 1 \in X)) \to y \in X$. That is, any set X that contains x and is closed under successor also contains y.
- "X is finite" : $\exists x \forall y(y \in X \to y \leq x)$.

Homework

(1) Express the following predicates with S1S formulas.

(a) X is the set of even numbers.

(b) X is finite with even number of elements.

(2) Explain why "X and Y have the same cardinality" cannot be expressed by an S1S formula.

For a set $A \subseteq \mathbb{N}$, the infinite sequence $\alpha \in \{0,1\}^{\mathbb{N}}$ such that $\alpha(i) = 1 \Leftrightarrow i \in A$ is called the **characteristic function** of A.

In the following, a number $a \in \mathbb{N}$ is identified with the singleton set $\{a\}$. Then the characteristic function of a tuple $(a_1, \ldots, a_n, A_1, \ldots, A_m) \in \mathbb{N}^n \times (\mathcal{P}(\mathbb{N}))^m$ can be expressed as an infinite sequence (called a characteristic sequence) over the alphabet $\Omega = \{0,1\}^{m+n}$. This sequence is divided into $m+n$ tracks, where each track is the characteristic function of a_i or A_i.

Example 7 The characteristic sequence of $(3, 5, \{\text{even numbers}\}, \{\text{prime numbers}\})$ is described as follows.

0	0	0	1	0	0	0	0	0	0	0	0	...	(⇐ 3)	
0	0	0	0	0	1	0	0	0	0	0	0	...	(⇐ 5)	$\in (\{0,1\}^4)^{\mathbb{N}}$
1	0	1	0	1	0	1	0	1	0	1	0	...	(⇐ even numbers)	
0	0	1	1	0	1	0	1	0	0	0	1	...	(⇐ prime numbers)	

The following theorem asserts that S1S and NBA have equivalent expressive power. In the proof, the equivalence of NBA, NMA, and DMA is used freely .

Theorem (The equivalence of S1S and NBA)

The following holds.

(1) Let $\varphi(\vec{x}, \vec{X})$ be an S1S formula with free numerical variables $\vec{x} = (x_1, \ldots, x_m)$ and free set variables $\vec{X} = (X_1, \ldots, X_n)$. Then there exists an equivalent NBA M_φ on $\Omega = \{0,1\}^{m+n}$ such that

$$L(M_\varphi) = \{\text{the characteristic sequence of } (\vec{a}, \vec{A}) \ : \ \varphi(\vec{a}, \vec{A}) \text{ is true}\},$$

where $\vec{a} = (a_1, \ldots, a_m), \vec{A} = (A_1, \ldots, A_n)$.

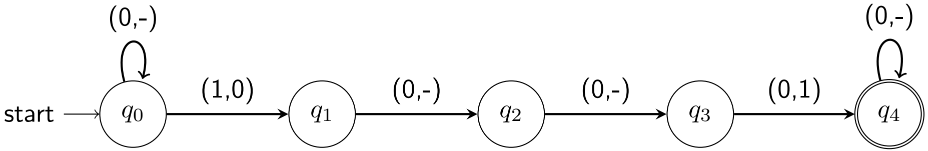
(2) Let M be a NBA on $\Omega = \{0,1\}$. There is a S1S formula $\varphi_M(X)$ such that

$$L(M) = \{\text{the characteristic sequence of } A : \varphi_M(A) \text{ is true}\}.$$

Proof. We show (1) by induction on the construction of the formula φ.
① The atomic formula is in the form of

$$\overbrace{\mathrm{SS}\cdots\mathrm{S}}^{k}\, x \in X.$$

An NBA that accepts a characteristic sequence of (a, A) should check that there is a unique 1, say in the a-th position in the track for x, and there is a 1 in the $a+k$-th position of the track for X.
For example, the figure below is an NBA (possibly, a DBA) for $k = 3$.



The edge label (b, c) represents the input $\begin{bmatrix} b \\ c \end{bmatrix}$, where b and c are 0 or 1 on the track for x and X, respectively. Also, - indicates both of 0 and 1 at the same time.

② Next, consider a formula of the form $\varphi_1(\vec{x}, \vec{X}) \land \varphi_2(\vec{x}, \vec{X})$. We may assume that there is a DMA $M_i = (Q_i, \Sigma, \delta_i, (q_0)_i, \mathcal{F}_i)$ for each φ_i.
DMA M_3 for $\varphi_1(\vec{x}, \vec{X}) \land \varphi_2(\vec{x}, \vec{X}) = (Q_3, \Sigma, \delta_3, (q_0)_3, \mathcal{F}_3)$ is constructed as follows.

$$\begin{aligned} Q_3 &= Q_1 \times Q_2 \\ \delta_3((q_1, q_2), a) &= (\delta_1(q_1, a), \delta_2(q_2, a)) \\ (q_0)_3 &= ((q_0)_1, (q_0)_2) \\ \mathcal{F}_3 &= \{A \subseteq Q_3 \mid \pi_1(A) \in \mathcal{F}_1 \text{ and } \pi_2(A) \in \mathcal{F}_2\} \end{aligned}$$

where π_1 and π_2 are the projections from $Q_1 \times Q_2$ to Q_1 and Q_2, respectively.

③ If $M = (Q, \Sigma, \delta, q_0, \mathcal{F})$ is a DMA for $\varphi(\vec{x}, \vec{X})$, then a DMA for $\neg\varphi$ can be constructed by taking the acceptance condition as $\mathcal{P}(Q) - \mathcal{F}$.

④ Automata for $\lor, \rightarrow$ cases are constructed similarly.

⑤ For $\exists X_1\varphi(\vec{x}, X_1, \ldots, X_m)$, suppose a DMA M_φ for $\varphi(\vec{x}, X_1, \ldots, X_m)$.
A NMA M of $\exists X_1\varphi$ takes $a_1, \ldots, a_n, A_2, \ldots, A_m$ together with a nondeterministic guess A_1 as input and mimic M_φ on $a_1, \cdots, a_n, A_1, A_2, \ldots, A_m$.

⑥ An automaton for $\forall X\varphi$ can be constructed as $\neg\exists X\neg\varphi$.

Thus, we can construct an NBA M_φ that accepts the set of characteristic sequences of $\vec{a}, \vec{A}$ satisfying φ.

(Note: The NBA's above may use some working tracks in addition to the input trucks. Especially when φ is a sentence, it is appropriate to arrange for a meaningless track. See the proof of decidability of S1S below.)

We show (2). Let $M = (Q, \{0,1\}, \delta, q_0, \mathcal{F})$ be a DMA. Let X be a set variable of the input sequence, and Y_q be the set of times when q is visited. A run $\vec{Y} = \{Y_q\}$ on input X is defined as follows.

$$\begin{aligned} \mathrm{run}(X, \vec{Y}) \quad = \quad & 0 \in Y_{q_0} \\ & \wedge \forall n \bigwedge_q (n \in Y_q \wedge n \notin X \to \mathrm{S}(n) \in Y_{\delta(q,0)}) \\ & \wedge \forall n \bigwedge_q (n \in Y_q \wedge n \in X \to \mathrm{S}(n) \in Y_{\delta(q,1)}) \\ & \wedge \forall n \bigwedge_{p \neq q} \neg (n \in Y_p \wedge n \in Y_q) \end{aligned}$$

Furthermore, "a run $\vec{Y}$ is accepted" can be defined:

$$\mathrm{accept}(\vec{Y}) = \bigvee_{F \in \mathcal{F}} \left(\bigwedge_{q \in F} Y_q \text{ is infinite} \wedge \bigwedge_{q \notin F} Y_q \text{ is finite} \right)$$

Finally, the desired formula is

$$\varphi_M(X) = \exists \vec{Y} (\mathrm{run}(X, \vec{Y}) \wedge \mathrm{accept}(\vec{Y}))$$

Corollary

S1S is decidable.

Proof

Let σ a S1S sentence. Its truth can be determined by the emptiness of an NBA that is equivalent to $\sigma \wedge (X = X)$, which is decidable by the above theorem. □

- We have treated Pressburger arithmetic on the natural numbers as a regular language.
- Then we can expect that addition of real numbers as infinite decimals can be handled in the ω regular language, and indeed it is possible.
- Represent real numbers as infinite binary decimals. Treat the decimal point as $\star$ as an infinite sequence over the language $\{0, 1, \star\}$.
- Furthermore, the first bit represents the $\pm$ sign, i.e., 0 is positive and 1 is negative.
- For example, 3.5 can be represented by the infinite sequence $011 \star 10^\omega$ or $011 \star 01^\omega$.
- Note that we can also define the equality $=$ between these two notations.

Further readings

- Infinite Words. Automata, Semigroups, Logic and Games. Dominique Perrin and Jean-Éric Pin. Pure and Applied Mathematics Vol 141. Elsevier, 2004.
- Automata, Logics, and Infinite Games: A Guide to Current Research. Editors: Erich Grädel, Wolfgang Thomas, Thomas Wilke. Lecture Notes in Computer Science (LNCS, volume 2500), Springer Berlin, Heidelberg, 2002.

Thank you for your attention!