Logic and Computation II

Part 4. Formal arithmetic and Gödel's incompleteness theorems

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March 25, 2023

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Logic and Computation II

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

Part 4. Schedule

- Mar. 7, (1) First-order logic
- Mar. 9, (2) Arithmetical formulas
- Mar.14, (3) Gödel's first incompleteness theorem
- Mar.16, (4) Gödel's second incompleteness theorem
- Mar.21, (5) Second-order logic
- Mar.23, (6) Analytical formulas

Today's topics

Recap

- In first-order logic (FO), quantifiers $\forall$ and $\exists$ range over the elements of a structure.
- Second-order logic (SO) allows quantifiers over relations and functions on the elements.
- Thus, a structure of SO is a pair of a first-order structure and a second-order domain. The **standard** structure of SO equips its second-order domain with all relations and functions (in the naïve sense).
- Theorem (Gödel): The validity of SO in terms of standard structures is not axiomatizable (CE).
- A **general** structure of SO has a second-order domain of relations and functions which satisfies some conditions (comprehension, choice, etc.).
- Monadic second-order logic (MSO) uses quantification over the sets of elements. The theory of (i.e., the set of sentences true in) a certain MSO (general) structure is computable, e.g., S1S = MSO$(\mathbb{N}, S(x))$, S2S = MSO$(2^{<\omega}, x^\frown 0, x^\frown 1)$.
- Lindström theorem: FO is the strongest logic that satisfies both the compactness theorem and the downward LS theorem.

Second-order arithmetic

- **Second-order arithmetic** Z_2 is a monadic second-order theory, or a two-sorted first-order theory dealing with natural numbers and sets of natural numbers under the condition of full comprehension.
- We denote numeric variables by lowercase $x, y, \ldots$, and set variables by uppercase $X, Y, \ldots$, and also use the relation symbol $\in$.
- When Z_2 is used as the base system of mathematics, it is common to formalize the first-order arithmetical part in the language $\mathcal{L}_{\mathrm{OR}}$, which includes addition and multiplication, similar to Peano arithmetic.
- Since multiplication can be defined from addition in monadic second-order logic (exercise), it is not necessary to include multiplication in the axiom. However, it is common to formalize the first-order part of Z_2 in the language $\mathcal{L}_{\mathrm{OR}}$ as in Peano arithmetic.

The formulas of second-order arithmetic

- The language $\mathcal{L}^2_{\mathrm{OR}}$ of second-order arithmetic is the language of first-order arithmetic $\mathcal{L}_{\mathrm{OR}}$ plus the membership relation symbol $\in$.
- The formulas of second-order arithmetic are constructed from atomic formulas $(t_1 = t_2,\ t_1 < t_2,\ t \in X)$ by propositional operators, numerical quantifiers $\forall x$, $\exists x$ and set quantifiers $\forall X$, $\exists X$.
- A formula can be rewritten in the prenex normal form by shifting quantifiers to the head of formula like in first-order.
- Moreover, all second-order quantifiers can be placed outside of the scopes of any first-order quantifier. For instance, in a very weak theory, the following transformation is possible.

$$\forall x \exists Y \varphi(x, Y) \Leftrightarrow \forall X \exists Y (\exists! x (x \in X) \to \forall x (x \in X \to \varphi(x, Y))).$$

- However, in a stronger theory, the axiom of choice is often used to exchange the places of quantifiers, e.g.,

$$\forall x \exists Y \varphi(x, Y) \Leftrightarrow \exists Y' \forall x \varphi(x, Y'_x),$$

where Y' is a set-valued choice function such that $Y'_x = Y'(x) = \{y : (x, y) \in Y'\}$.

Analytical Hierarchy

We inductively define the **analytical hierarchy** of $\mathcal{L}^2_{\mathrm{OR}}$-formulas, Σ^i_j and Π^i_j $(i = 0, 1,\ j \in \mathbb{N})$.

Definition

- The **Bounded** formulas are constructed from atomic formulas $(t_1 = t_2,\ t_1 < t_2,\ t \in X)$ by propositional operators and bounded quantifiers $\forall x < t$, $\exists x < t$. The class of such formulas is written as Π^0_0 or Σ^0_0.
- For each $j \geq 0$, if $\varphi \in \Sigma^0_j$, then $\forall x_1 \cdots \forall x_k \varphi \in \Pi^0_{j+1}$;
  if $\varphi \in \Pi^0_j$, then $\exists x_1 \cdots \exists x_k \varphi \in \Sigma^0_{j+1}$.
- All formulas in Σ^0_j and Π^0_j are called **arithmetical**.
  The class of arithmetical formulas is denoted as Π^1_0 or Σ^1_0.
- For each $j \geq 0$, if $\varphi \in \Sigma^1_j$, then $\forall X_1 \cdots \forall X_k \varphi \in \Pi^1_{j+1}$;
  if $\varphi \in \Pi^1_j$ then $\exists X_1 \cdots \exists X_k \varphi \in \Sigma^1_{j+1}$.

- $\Sigma^0_i(\Pi^0_i)$ formulas without set variables are nothing but $\Sigma_i(\Pi_i)$ formulas of first-order arithmetic.
- A formula that is equivalent to a Σ^i_j (or Π^i_j) formula on a given basic system is often called Σ^i_j (or Π^i_j).
- Furthermore, if a Σ^i_j formula is equivalent to a Π^i_j formula, each of them is called a Δ^i_j formula. Since the equivalence of formulas depends on a base theory T, Δ^i_j is strictly expressed as $(\Delta^i_j)^T$.
- When dealing with arithmetical hierarchies Σ^0_i Π^0_i, a system of second-order arithmetic RCA_0 is often used as a base theory. When dealing with analytical hierarchies, a stronger system ACA_0 is often assumed.
- These two systems are also suitable for the foundation of a wide range of mathematical discussions, and thus are important subsystems of Z_2 in the foundational program, so-called **Reverse Mathematics**.

First, we define the system RCA_0 of recursive comprehension axioms.

Definition (System of recursive comprehension axioms)

The system of second-order arithmetic RCA_0 consists of the following axioms.
(1) Basic Axioms of Arithmetic: Same as $\mathsf{Q}_<$.
(2) Δ^0_1 comprehension axiom (Δ^0_1-CA):

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ is Σ^0_1, $\psi(x)$ is Π^0_1, and X is not included as a free variable. This axiom roughly guarantees the existence of the set $X = \{n : \varphi(n)\}$.
(3) Σ^0_1 induction: For any Σ^0_1 formula $\varphi(x)$,

$$\varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x).$$

- Since the Δ^0_1 comprehension axiom asserts the existence of recursive sets (=computable sets) in the standard model $\mathbb{N}$, it is called the **recursive comprehension axiom**.
- More precisely, since $\psi(x)$ and $\varphi(x)$ in the axiom may include set variables (other than X) as parameters, this axiom indeed asserts that there exists a set that can be computed in a parameter set as an oracle. But notice that it does not assert the non-existence of a non-recursive set.
- RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$. That is, a sentence of $\mathcal{L}_{\mathrm{OR}}$ that is provable in RCA_0 is already provable in $\mathsf{I}\Sigma_1$.

Definition (System of arithmetical comprehension axioms)

The **system of arithmetical comprehension axioms** ACA_0 is obtained from RCA_0 by replacing the Δ^0_1 comprehension with the Σ^0_1 comprehension [1].

- ACA_0 is a conservative extension of first-order arithmetic PA.

[1] Arithmetical comprehension can be achieved by repeatedly applying the Σ^0_1 comprehension axiom to the parameters.

- In RCA_0, we encode the ordered pair of natural numbers (m, n) by $\frac{(m+n)(m+n+1)}{2} + m$.
- The **Cartesian product** $X \times Y$ is the set of all (codes of) pairs of an element of X and an element of Y:

$$n \in X \times Y \leftrightarrow \underbrace{\exists x \le n \exists y \le n(x \in X \wedge y \in Y \wedge (x, y) = n)}_{\Sigma^0_0}.$$

  So the existence of $X \times Y$ is guaranteed by RCA_0.
- A **function** $f : X \to Y$ is a unique set $F \subseteq X \times Y$ such that

$$\forall x \forall y_0 \forall y_1((x, y_0) \in F \wedge (x, y_1) \in F \to y_0 = y_1) \text{ and } \forall x \in X \exists y \in Y (x, y) \in F.$$

  If $(x, y) \in F$, we write $f(x) = y$.
- A function is called a **total function** if its domain is $\mathbb{N}$.
- In RCA_0, we can prove that the total functions are closed by primitive recursion [2].

[2]For more details, please refer to section 7.1 in my book `https://www.shokabo.co.jp/mybooks/ISBN978-4-7853-1575-7.htm`.

In RCA_0, we can not only handle functions, but also use the function quantifiers $\exists f$, $\forall f$ for a unary function f. These quantifiers can be regarded as special set quantifiers $\exists X_f$, $\forall X_f$, which presuppose that X_f denotes a function, namely, $\forall x \exists ! y\ (x, y) \in X_f$.
Now, we consider the hierarchy of formulas only with function quantifiers.

Definition (Analytical hierarchy $\mathrm{fnc}\text{-}\Sigma^1_n$, $\mathrm{fnc}\text{-}\Pi^1_n$, by function quantifiers)

Arithmetical formulas are $\mathrm{fnc}\text{-}\Sigma^1_0$ and $\mathrm{fnc}\text{-}\Pi^1_0$. For each $i \geq 0$, if a formula φ is $\mathrm{fnc}\text{-}\Pi^1_i$, then $\exists f \varphi$ is $\mathrm{fnc}\text{-}\Sigma^1_{i+1}$. If a formula φ is $\mathrm{fnc}\text{-}\Sigma^1_i$, then $\forall f \varphi$ is $\mathrm{fnc}\text{-}\Pi^1_{i+1}$.

In the following lemma, we show $\mathrm{fnc}\text{-}\Sigma^1_i$ (or $\mathrm{fnc}\text{-}\Pi^1_i$) and Σ^1_i (or Π^1_i) are equivalent. Thus "fnc-" may be omitted.

Lemma

For each $i \geq 1$, for any Σ^1_i formula (or Π^1_i formula), there exists an equivalent $\mathrm{fnc}\text{-}\Sigma^1_i$ formula (or $\mathrm{fnc}\text{-}\Pi^1_i$ formula). The converse also holds.

Proof

- First, we show that in a Σ^1_i formula (or a Π^1_i formula), a block of set quantifiers of the same kind can be unified into one. That is, in RCA_0, the following holds.

$$\exists X_0 \cdots \exists X_{n-1} \varphi \Leftrightarrow \exists X \varphi',$$

  where φ' is obtained from φ by replacing each atomic formula $t \in X_k$ $(k < n)$ in it with $(t, k) \in X$. The equivalence should be clear. Similarly for universal quantifiers $\forall X_k$.
- Next, we replace each set quantifier $\exists X$ $(\forall X)$ with a function quantifier $\exists f_X$ $(\forall f_X)$, and an atomic formula $t \in X$ with $f_X(t) > 0$. Thus, we obtain an equivalent formula with only functional quantifiers.
- Therefore, a Σ^1_i (or Π^1_i) formula can be expressed as fnc-Σ^1_i (or fnc-Π^1_i).

- Conversely, suppose fnc-Σ^1_i or fnc-Π^1_i formula φ are given.
- First, we replace the functional quantifier $\exists f$ $(\forall f)$ of φ with the set quantifier $\exists X_f$ $(\forall X_f)$ and denote the resulting formula as φ'.
- Next, consider how to eliminate f using X_f in the arithmetical part θ of φ'. For example, an atomic formula $s = t$ where t is expressed as $u(f(v))$ can be rewritten as $\exists y((v, y) \in X_f \land s = u(y))$. If t contains multiple occurrences of f, eliminate them from the inner. Similarly for $s < t$.
- Let θ' be an arithmetical formula obtained by repeating this process and eliminating all function quantifiers.
- For each f, let $\Psi(f)$ be $\forall x \exists! y\ (x, y) \in X_f$, to express the condition "X_f represents a function". Finally, we define an arithmetical formula θ'' as follows.

$$\theta'' \equiv \bigwedge_{f \text{ s.t. } \varphi \text{ contains } ``\forall f"} \Psi(f) \to (\theta' \land \bigwedge_{f \text{ s.t. } \varphi \text{ contains } ``\exists f"} \Psi(f))$$

- By replacing the arithmetical part θ of φ' with θ'', we obtain a Σ^1_i or Π^1_i formula which is equivalent to φ. □

The above lemma can also be proved in RCA_0, but the following normal form theorem requires ACA_0.

Theorem (Normal form theorem for analytical formulas)

For each $i \geq 1$, for any Σ^1_i formula (or Π^1_i formula), there exists an equivalent $\mathrm{fnc}\text{-}\Sigma^1_i$ formula (or $\mathrm{fnc}\text{-}\Pi^1_i$ formula) whose arithmetical part is Σ^0_1 or Π^0_1.

Proof.

- Any Σ^1_i formula (or Π^1_i formula) must have an equivalent fnc-Σ^1_i formula (or fnc-Π^1_i formula) as shown in the above lemma.
- To begin with, we will observe that consecutive quantifiers of the same type can be unified as one of such. First note that if x encodes a pair (x_0, x_1), x_i is obtained from x as a primitive recursive function $\pi_i(x)$ $(i = 0, 1)$. Then $\exists x_0 \exists x_1 \varphi(x_0, x_1)$ can be rewritten as $\exists x \varphi(\pi_0(x), \pi_1(x))$. Also $\exists f_0 \exists f_1 \varphi(f_0, f_1)$ can be rewritten as $\exists f \varphi(\pi_0 \circ f, \pi_1 \circ f)$. Similar for universal quantifiers $\forall x_0 \forall x_1$ and $\forall f_0 \forall f_1$. We remark that the graph of a primitive recursive function can be expressed as a Δ^0_1 formula in RCA_0 essentially by the strong representation lemma.

- Let φ be a fnc-Σ^1_i (or fnc-Π^1_i) formula. Suppose that its last function quantifier is $\exists f$.
- First, consider the case that the first quantifier of the arithmetical part of φ is $\exists x$. Then we change $\exists x$ by a function quantifier $\exists f_x$, and replace x inside with $f_x(0)$. Finally, merge the two function quantifiers $\exists f \exists f_x$ into one.
- Next, consider the case that the first arithmetical quantifier is $\forall x$. If the arithmetical part is Π^0_1, we are done.
- Otherwise, the arithmetical part is of the form $\forall x \exists y \theta(x, y)$.
- If $\forall x \exists y \theta(x, y)$ holds, there exists an arithmetical function $g(x) = y$ that takes the smallest y that satisfies $\varphi(x, y)$ for x. So, it can be rewritten as $\exists g \forall x \varphi(x, g(x))$ (in ACA_0). Finally, merge the two function quantifiers $\exists f \exists g$ into one.
- The same is true if the last function quantifier is $\forall f$.
- By repeating the above procedure, the arithmetical part becomes Σ^0_1 or Π^0_1. $\square$

Summary

- Second-order arithmetic Z_2 is a monadic second-order theory, or a two-sorted first-order theory dealing with natural numbers and sets of natural numbers under the condition of full comprehension.
- The language $\mathcal{L}^2_{\mathrm{OR}}$ of second-order arithmetic is the language of first-order arithmetic $\mathcal{L}_{\mathrm{OR}}$ plus the membership relation symbol $\in$.
- The **analytical hierarchy** of $\mathcal{L}^2_{\mathrm{OR}}$-formulas, Σ^i_j and Π^i_j: For each $j \geq 0$, if $\varphi \in \Sigma^1_j$, then $\forall X_1 \cdots \forall X_k \varphi \in \Pi^1_{j+1}$; if $\varphi \in \Pi^1_j$ then $\exists X_1 \cdots \exists X_k \varphi \in \Sigma^1_{j+1}$.
- **Analytical hierarchy** $\text{fnc-}\Sigma^1_n$**,** $\text{fnc-}\Pi^1_n$**, by function quantifiers**: For each $i \geq 0$, if φ is $\text{fnc-}\Pi^1_i$, then $\exists f \varphi$ is $\text{fnc-}\Sigma^1_{i+1}$. If φ is $\text{fnc-}\Sigma^1_i$, then $\forall f \varphi$ is $\text{fnc-}\Pi^1_{i+1}$.
- For any Σ^1_i (or Π^1_i) formula , there exists a $\text{fnc-}\Sigma^1_i$ (or $\text{fnc-}\Pi^1_i$) formula and vice versa.
- **Normal form theorem for analytical formulas**: For each $i \geq 1$, for any Σ^1_i (or Π^1_i) formula , there exists an equivalent $\text{fnc-}\Sigma^1_i$ (or $\text{fnc-}\Pi^1_i$) formula whose arithmetical part is Σ^0_1 or Π^0_1.

Thank you for your attention!