# Logic and Computation II

## Part 4. Formal arithmetic and Gödel's incompleteness theorems

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March 7, 2023

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# Outline of the Course

1. This is an introductory graduate-level course in **mathematical logic** and **theory of computation**. Its first part delivered in the last semester covered the basic topics of the two fields and their interactions. In this semester, we discuss more advanced topics emphasizing on decidability and definability.
2. Each week, there are two lectures, in Tuesday and Thursday. Every Thursday, we will assign simple homework problems or questionnaires to registered students, who are motivated to attend the class continuously. Normally, homeworks are due next Monday.
3. TA (Dr. Li) will handle homeworks as well as questions and comments from students via WeChat. We may assign harder problems to students, who will presumably go to the research level with us in the following years.
4. Lecture slides will be uploaded on the lecture page at BIMSA.

**Logic and Computation II**

- **Part 4. Formal arithmetic and Gödel's incompleteness theorems**
- **Part 5. Automata on infinite objects**
- **Part 6. Recursion-theoretic hierarchies**
- **Part 7. Admissible ordinals and second order arithmetic**

**Part 4. Schedule**

- Mar. 7, (1) First-order logic
- Mar. 9, (2) Arithmetical formulas
- Mar.14, (3) Gödel's first incompleteness theorem
- Mar.16, (4) Gödel's second incompleteness theorem
- Mar.21, (5) Second-order logic
- Mar.23, (6) Analytical formulas

# Today's topics

# Introduction

- Propositional logic is the study of logical connections $\neg, \wedge, \vee, \rightarrow$ between propositions.
- First order logic is obtained from propositional logic by adding logical symbols: $\forall, \exists$.
  - ⋆ the quantifier $\forall x$ expresses "for every element $x$ (of the underlying set)", and
  - ⋆ the quantifier $\exists x$ expresses "there exists an element $x$ (of the underlying set)".
- Historically, first order logic was tailored by D. Hilbert from Russell's type theory to capture mathematical theories in algebraic formulations. And, he asked if his formulation is complete (i.e., sufficient to prove all the valid formulas). Gödel immediately showed that it is complete.
- Hilbert also proposed the decision problem (such as satisfiability) of first-order logic as "**the main problem of mathematical logic** (Hauptproblem)" (1928). Then, Gödel showed the undecidability of first-order arithmetic. Subsequently, Church and Turing proved that first-order logic is undecidable.

# First order logic

- In order to develop a formal argument, we first specify the symbols involved.

**Symbols**

- Common logical symbols of first-order logic
  1. **propositional connectives**: $\neg$ (not $\cdots$), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies),
  2. **quantifiers**: $\forall$ (for all $\cdots$), $\exists$ (there exists $\cdots$).
  3. **variables**: $x_0, x_1, \cdots$
  4. auxiliary symbols such as equality $=$, parentheses $($, $)$.
- Mathematical symbols of a specific theory:
  **constants** $\mathrm{c}, \cdots$; **function symbols** $\mathrm{f}, \cdots$; and **relation symbols** $\mathrm{R}, \cdots$.

- The latter set of symbols is called the **language**[1] $\mathcal{L}$ of the theory. Note that $\mathcal{L}$ may be infinite, though in an ordinary theory, at most five or six symbols are used.

[1] "Langauge" here is different from that in Part 1 and 2 of this course.

- A **structure** in language $\mathcal{L}$ (simply, an $\mathcal{L}$-structure) is defined as a non-empty set $A$ equipped with an interpretation of the symbols in $\mathcal{L}$, denoted as

$$\mathcal{A} = (A, \mathrm{c}^{\mathcal{A}}, \cdots, \mathrm{f}^{\mathcal{A}}, \cdots, \mathrm{R}^{\mathcal{A}}, \cdots).$$

- $A$ is called the **domain** of the structure $\mathcal{A}$. We do not make a strict distinction between the set $A$ and the structure $\mathcal{A}$ if it is clear from the context.
- Each function symbol has a predetermined number of arguments, called its **arity**. If the arity of $\mathrm{f}$ is $n$, then $\mathrm{f}^{\mathcal{A}} : A^n \to A$.
- Each relation symbol also has an **arity**. If the arity of $\mathrm{R}$ is $n$, then $\mathrm{R}^{\mathcal{A}} \subseteq A^n$.
- A **constant** could be regarded as a function symbol with no argument ($0$-ary function), but here a constant plays a special role distinct from a function.

**Example 1**

- The ordered field of real numbers $\mathcal{R} = (\mathbb{R}, 0, 1, +, \cdot, <)$ is a structure in the language $\mathcal{L}_{\mathrm{OR}} = \{0, 1, +, \cdot, <\}$, where $0$ and $1$ are constants, $+$ and $\cdot$ are binary function symbols, and $<$ is a binary relation symbol.
- Rigorously, $\mathcal{R}$ should be written as $(\mathbb{R}, 0^{\mathcal{R}}, 1^{\mathcal{R}}, +^{\mathcal{R}}, \cdot^{\mathcal{R}}, <^{\mathcal{R}})$. For simplicity, we often omit a superscipt such as ${}^{\mathcal{R}}$ unless a serious confusion might occur.
- The subscript $\mathrm{OR}$ of $\mathcal{L}_{\mathrm{OR}}$ stands for ordered rings, since a typical structure in this language is an ordered ring (e.g., integers). However, a structure in $\mathcal{L}_{\mathrm{OR}}$ is not necessarily an ordered ring. E.g., $(\mathbb{N}, 0, 1, +, \cdot, <)$ is not a ring.

Fix a language $\mathcal{L}$ and define a "term" of $\mathcal{L}$ to denote a specific element of $\mathcal{L}$-structure $\mathcal{A}$.

## Definition (Terms)

The **terms** of the language $\mathcal{L}$ are symbol strings defined inductively as follows.

1. variables and constants in $\mathcal{L}$ are terms of $\mathcal{L}$.
2. If $t_0, \cdots, t_{n-1}$ are terms and $\mathrm{f}$ is an $n$-ary function symbol of $\mathcal{L}$, then $\mathrm{f}(t_0, \cdots, t_{n-1})$ is a term of $\mathcal{L}$.

For a term $t$ with no variable, its **value** in a structure $\mathcal{A}$, denoted $t^{\mathcal{A}}$, is defined inductively as follows.

1. the value of constant $\mathrm{c}$ in $\mathcal{L}$ is $\mathrm{c}^{\mathcal{A}}$.
2. the value of term $\mathrm{f}(t_0, \cdots, t_{n-1})$ is $\mathrm{f}^{\mathcal{A}}(t_1^{\mathcal{A}}, \cdots, t_{n-1}^{\mathcal{A}})$.

**Example 2t**

In language $\mathcal{L}_{\mathrm{OR}} = \{0, 1, +, \cdot, <\}$, symbol strings such as $x$, $x+1$, $(x+1)\cdot y$ are terms. $(1+1)\cdot(1+1)$ is a term without variables, sometimes called a closed term, which has a unique value in an $\mathcal{L}_{\mathrm{OR}}$-structure. $((1+1)\cdot(1+1))^{\mathcal{N}}$ is "4" in $(\mathbb{N}, 0, 1, +, \cdot, <)$.

A formula is introduced as a symbol string to describe a property of a structure.

## Definition (Formulas)

A **formula** of language $\mathcal{L}$ is a sequence of symbols inductively defined as follows.

(1) $s, t, t_0, \cdots, t_{n-1}$ are terms of $\mathcal{L}$, and $\mathrm{R}$ is an $n$-ary relation symbol of $\mathcal{L}$, then

$$s = t \quad \text{and} \quad \mathrm{R}(t_0, \cdots, t_{n-1})$$

are formulas of $\mathcal{L}$, which are called **atomic** formulas.

(2) If $\varphi, \psi$ are formulas of $\mathcal{L}$, then so are the followings: for any variable $x$,

$$\neg(\varphi),\ (\varphi) \wedge (\psi),\ (\varphi) \vee (\psi),\ (\varphi) \to (\psi),\ \ \forall x(\varphi),\ \ \exists x(\varphi).$$

As in propositional logic, parentheses in a formula are appropriately omitted.

**Example 2f**

In $(\mathbb{N}, 0, 1, +, \cdot, <)$, the following formula $\varphi(x)$ denotes "$x$ is prime".

$$\varphi(x) \equiv \forall y \forall z (x = y \cdot z \to (y = 1 \vee z = 1)) \wedge x > 1.$$

- To promote in-depth discussion on formulas, we must clarify the role of variables in formulas.
- Let $Q$ denote $\exists$ or $\forall$. Assume $\varphi$ contains a subformula of the form $Qx(\psi)$, where no quantifier of the form $Qx$ appears in $\psi$. Then each occurrence of $x$ in ($Qx$ and $\psi$) is said to be **bound** in $\varphi$. An occurrence of the variable $x$ in the formula $\varphi$ is said to be **free** when it is not bound.
- A variable may have both bound and free occurrences in a formula. For example, in

$$(\forall x(x \leq y)) \to (\exists y(x \leq y)),$$

  the first two of the three occurrences of $x$ are bound, and last one is free.
- If a variable occurs both bound and free in a formula, we often automatically replace the bound occurrence with another variable to avoid unnecessary misreading.
- For example, the above formula can be rewritten as

$$(\forall w(w \leq y)) \to (\exists z(x \leq z)).$$

- The variables in a formula can be separated into free variables and bound variables.

- A formula without free variables is called a **sentence**.
- For a formula $\varphi$ with free variables, a sentence of the form $\forall x_1 \cdots \forall x_n \varphi$ (i.e. all free variables appearing in $\varphi$ are in $\{x_1, \ldots, x_n\}$) is called the **universal closure** of $\varphi$.

- We often add new constants to a given language $\mathcal{L}$ to handle some elements of a structure. We prepare a name (constant) $c_a$ for each element $a$ of structure $\mathcal{A}$. Then for $B \subseteq A$, by $\mathcal{L}_B$ we denote the language $\mathcal{L}$ extended with the new constant $c_a$ for each element $a$ of $B$.
- An $\mathcal{L}$-structure $\mathcal{A}$ is naturally extended to the structure in $\mathcal{L}_B$ by interpreting $c_a$ as $a$, denoted $\mathcal{A}_B$.
- This kind of expansion is often made implicitly. Unless a serious confusion occurs, we may write $\mathcal{A}$ for $\mathcal{A}_B$, and $a$ and $c_a$ are indiscriminate.

## Definition (Tarski's truth definition clauses)

For a sentence $\varphi$ in $\mathcal{L}_A$, "$\varphi$ is **true** in $\mathcal{A}$ (written as $\mathcal{A} \models \varphi$)" is defined as follows.

$$\begin{aligned}
\mathcal{A} \models s = t &\Leftrightarrow s^{\mathcal{A}} = t^{\mathcal{A}}, \\
\mathcal{A} \models \mathrm{R}(s_0, \cdots, s_{n-1}) &\Leftrightarrow \mathrm{R}^{\mathcal{A}}(s_0^{\mathcal{A}}, ..., s_{n-1}^{\mathcal{A}}), \\
\mathcal{A} \models \neg\varphi &\Leftrightarrow \mathcal{A} \models \varphi \text{ does not hold}, \\
\mathcal{A} \models \varphi \land \psi &\Leftrightarrow \mathcal{A} \models \varphi \text{ and } \mathcal{A} \models \psi, \\
\mathcal{A} \models \varphi \lor \psi &\Leftrightarrow \mathcal{A} \models \varphi \text{ or } \mathcal{A} \models \psi, \\
\mathcal{A} \models \varphi \to \psi &\Leftrightarrow \text{ if } \mathcal{A} \models \varphi, \text{ then } \mathcal{A} \models \psi, \\
\mathcal{A} \models \forall x \varphi(x) &\Leftrightarrow \text{for any constant } a, \mathcal{A} \models \varphi(a), \\
\mathcal{A} \models \exists x \varphi(x) &\Leftrightarrow \text{there exists a constant } a \text{ s.t. } \mathcal{A} \models \varphi(a).
\end{aligned}$$

The truth of a formula with free variables is defined by the truth of its universal closure.

- If $\mathcal{L}$-structures $\mathcal{A}$, $\mathcal{B}$ are isomorphic, $\mathcal{A} \cong \mathcal{B}$, then it can be shown by simple induction that,

$$\underbrace{\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi \quad \text{for any formula } \varphi.}_{\mathcal{A} \equiv \mathcal{B}, \text{ elementary equivalence}}$$

- However, the converse, namely $\mathcal{A} \equiv \mathcal{B} \Rightarrow \mathcal{A} \cong \mathcal{B}$, does not hold in general (See the Löwenheim-Skolem theorem in the next lecture)

## Definition

- The set $T$ of sentences in the language $\mathcal{L}$ is called a **theory**.
- $\mathcal{A}$ is a **model** of $T$, denoted by $\mathcal{A} \models T$, if all the sentence of $T$ are true in $\mathcal{A}$.
- A theory is said to be **satisfiable** if it has a model.
- We say that $\varphi$ holds in $T$, written as $T \models \varphi$, if any model $\mathcal{A}$ of $T$ is also a model of $\varphi$.
- In particular, given $T = \varnothing$, $\varphi$ satisfying $\models \varphi$ is said to be **valid**.

# Recap from Lecture 03-02

- $\varphi$ can be transformed into an equivalent **prenex normal form**, abbreviated as $\mathrm{PNF}$:
$$\varphi' \equiv Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \theta.$$
For example, a formula $\varphi \equiv \theta \wedge \forall x \xi(x)$ is equivalent to $\forall x(\theta \wedge \xi(x))$, if $\theta$ does not have $x$ as a free variable. If $\theta$ has $x$ as a free variable, we replace the bound variable $x$ of $\forall x \xi(x)$ by a new variable $y$ and then obtain the equivalent transformation $\varphi' \equiv \forall y(\theta \wedge \xi(y))$.
- Next, by repeating the following operations as much as possible, we obtain a **Skolem normal form** of $\varphi$, abbreviated as $\mathrm{SNF}$.
  - Let $Q_i$ be the outermost (leftmost) existential symbol in $\varphi'$. Remove $Q_i x_i$ and replace all occurrences of $x_i$ on its right side (inside) of $Q_i x_i$ with $\mathrm{f}(x_1, \ldots, x_{i-1})$, where $\mathrm{f}$ is a new function symbol and is called a **Skolem function**.

  In the above definition, when $Q_1$ is existential, $x_1$ is replaced by a "constant" (or a $0$-ary function symbol).

**Example 3**

For a $\mathrm{PNF}$ $\forall w \exists x \forall y \exists z \theta(w, x, y, z)$, we obtain a $\mathrm{SNF}$ $\varphi^S \equiv \forall w \forall y \theta(w, \mathrm{f}(w), y, \mathrm{g}(w, y))$.

# Recap II from Lecture 03-02

- For a formula $\varphi$ of $\mathcal{L}$ (i.e., not containing a Skolem function), $T \models \varphi \Leftrightarrow T^S \models \varphi$. Namely, $T^S = \{\sigma^S : \sigma \in T\}$ is a **conservative extension** of $T$.
- Löwenheim-Skolem's downward theorem.
  For a structure $\mathcal{A}$ in a countable language $\mathcal{L}$, there exists a countable substructure $\mathcal{A}' \subset \mathcal{A}$ s.t. $\mathcal{A}' \models \varphi \Leftrightarrow \mathcal{A} \models \varphi$ for any $\mathcal{L}_{A'}$-sentence $\varphi$. Such $\mathcal{A}'$ is called an **elementary substructure** of $\mathcal{A}$, denoted as $\mathcal{A}' \prec \mathcal{A}$.
- Herbrand's theorem (Skolem version). In first-order logic (without equality), $\exists$-formula $\exists\vec{x}\varphi(\vec{x})$ is valid if and only if
  - there exist $n$-tuples of terms, $\vec{t}_1, \ldots, \vec{t}_k$, from $\mathcal{L}(\varphi)$ and
  - $\varphi(\vec{t}_1) \vee \cdots \vee \varphi(\vec{t}_k)$ is a tautology.

# What happens if equality "=" is considered?

- Suppose we are given an arbitrary sentence $\sigma$ with an equality "=".
- Let $\mathrm{D}(\sigma)$ be the conjunction of the following axioms for all symbols $\mathrm{f}, \mathrm{R} \in \mathcal{L}(\sigma)$,

$$\forall \vec{x}\ \forall \vec{y}\ (\vec{x} = \vec{y} \to \mathrm{f}(\vec{x}) = \mathrm{f}(\vec{y})), \quad \forall \vec{x}\ \forall \vec{y}\ (\vec{x} = \vec{y} \to \mathrm{R}(\vec{x}) \leftrightarrow \mathrm{R}(\vec{y})).$$

- Let $\mathrm{Eq}(\sigma)$ be the conjunction of $\mathrm{D}(\sigma)$ with the reflexivity, symmetricity, and transitivity of "=", which can be expressed as a whole by a $\forall$ sentence.
- Therefore, an $\exists$-sentence $\sigma$ is valid in first-order logic with "=" iff

$$\mathrm{Eq}(\sigma) \to \sigma$$

is valid without the equality axioms.
- Since the above expression is a $\exists$ statement, applying the Herbrand's theorem to this, we obtain the equivalent condition as a tautology.

# Formal system of first-order logic

- Before introducing Gödel's completeness theorem, we define the the formal system of first-order logic.
- Among the various formal systems, we consider an formal system by extending that of propositional logic in part 2 of this course.

**Axiom system**

P1. $\varphi \to (\psi \to \varphi)$

P2. $\big(\varphi \to (\psi \to \theta)\big) \to \big((\varphi \to \psi) \to (\varphi \to \theta)\big)$

P3. $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

P4. $\forall x \varphi(x) \to \varphi(t)$ (the quantification axiom)

**Inference rules**

(1) If $\varphi$ and $\varphi \to \psi$ are theorems, so is $\psi$

(2) If $\psi \to \varphi(x)$ (where $\psi$ does not include $x$) is a theorem, then so is $\psi \to \forall x \varphi(x)$ (the generalization rule)

- The existential quantifiers $\exists x\varphi(x) :\equiv \neg\forall x\neg\varphi(x)$.
- In languages with equality, the axiom $\mathrm{Eq}$ is assumed (reflexive, symmetrical, transitive, and for each symbol $\mathrm{f}$ or $\mathrm{R}$, its value is preserved with equality).
- If a sentence $\sigma$ can be proved from the set of sentences $T$, then $\sigma$ is called a **theorem** of $T$, and written as $T \vdash \sigma$.
- The quantification axiom and the equality axiom hold trivially in any structure, and the generalization rule also clearly preserves truth (because the free variable $x$ of a formula is interpreted by universal closure).
- So if $T \vdash \sigma$ then $T \models \sigma$. This means that the deductive system does not derive any strange theorems, and is called the **soundness theorem**.
- The completeness theorem also asserts the opposite, that the system derives all valid sentences.

## Theorem (Completeness theorem (a weak version))

For any sentence $\sigma$, $\models \sigma$ iff $\vdash \sigma$.

- We only need to show that for any sentence $\sigma$, if $\models \sigma$ then $\vdash \sigma$. So, assuming $\models \neg\sigma$, we will show $\vdash \neg\sigma$.
- By Herbrand's Theorem (Skolem's Fundamental Theorem), let $\forall\vec{x}\varphi(\vec{x})$ be the $\mathrm{SNF}\sigma^S$ of $\sigma$. If $\neg\sigma$ is valid, there are $n$ pairs of terms $\vec{t_i}$ such that $\neg\varphi(\vec{t_1}) \vee \cdots \vee \neg\varphi(\vec{t_k})$ is a tautology.
- By the completeness theorem of propositional logic, the tautology is a theorem of propositional logic. So, it is also a theorem of first-order logic, by regarding the atomic propositions as atomic formulas of first-order logic.
- Since $\neg\varphi(\vec{t_i}) \to \exists\vec{x}\neg\varphi(\vec{x})$ can be proved in first-order logic, we can deduce $\exists\vec{x}\neg\varphi(\vec{x})$ from the theorem $\neg\varphi(\vec{t_1}) \vee \cdots \vee \neg\varphi(\vec{t_k})$. Thus, $\neg\sigma$ is provable.
- To show the completeness theorem, Gödel introduced new relation symbols instead of Skolem functions, and transformed any sentence into a $\forall\exists$ sentence.

- Subsequently, L. Henkin introduced a constant $c_{\exists x\varphi(x)}$ (**Henkin constant**) for each sentence $\exists x\varphi(x)$, and assume the following formula $\exists x\varphi(x) \to \varphi(c_{\exists x\varphi(x)})$ as an additional axiom, called the Henkin axiom. By the Henkin axioms, any sentence can be rewritten as a formula without quantifiers.

**Henkin Theorem**

If $T$ is consistent, $T \cup \{\exists x\varphi(x) \to \varphi(c_{\exists x\varphi(x)})\}$ is also consistent.

**Proof**

By way of contradiction, assume $T \cup \{\exists x\varphi(x) \to \varphi(c_{\exists x\varphi(x)})\}$ were inconsistent. By the deduction theorem, we have $T \vdash \exists x\varphi(x) \wedge \neg\varphi(c_{\exists x\varphi(x)})$. Since the constant $c_{\exists x\varphi(x)}$ does not appear in theory $T$, we can replace all its occurrences by a new variable $y$ in its proof and then obtain a proof for $T \vdash \exists x\varphi(x) \wedge \neg\varphi(y)$. Thus, we have $T \vdash \exists x\varphi(x) \wedge \forall y\neg\varphi(y)$, equivalently $T \vdash \exists x\varphi(x) \wedge \neg\exists x\varphi(x)$, which means that $T$ is inconsistent. $\square$

- For a consistent theory $T$ in a language $\mathcal{L}$, there are a set $C$ of constants such that any sentence of the form $\exists x \varphi(x)$ in $\mathcal{L} \cup C$ has its Henkin constant in $C$.
- Let $T_H$ be a consistent extension of $T$ with all Henkin axioms $H$. Let $\hat{S}$ a maximal consistent set of $T_H$ (with the equality axioms). Then, we define a model $\mathcal{A}$ of $T$ as follows.
- First define an equivalence relation $\approx$ on $C$ by $\mathrm{c} \approx \mathrm{d} \Leftrightarrow (\mathrm{c} = \mathrm{d}) \in \hat{S}$.
  Let $A = C/\approx$. Then define the interpretations of f and R on $A$ as follows

$$\mathrm{f}^{\mathcal{A}}([\mathrm{c}_1], [\mathrm{c}_2], \ldots, [\mathrm{c}_n]) = [\mathrm{d}] \Leftrightarrow (\mathrm{f}(\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_n) = \mathrm{d}) \in \hat{S}$$
$$\mathrm{R}^{\mathcal{A}}([\mathrm{c}_1], [\mathrm{c}_2], \ldots, [\mathrm{c}_n]) \Leftrightarrow \mathrm{R}(\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_n) \in \hat{S}$$

  Here, a structure $\mathcal{A}$ is well-defined since $\hat{S}$ includes all the equality axioms. Then, we can also show by induction that $\mathcal{A} \models \varphi([\mathrm{c}_1], \ldots, [\mathrm{c}_n]) \Leftrightarrow \varphi(\mathrm{c}_1, \ldots, \mathrm{c}_n) \in \hat{S}$. Thus, $\mathcal{A}$ is a model of $T$.

## Theorem (Model existence theorem)

If a set $T$ of sentences of first order logic is consistent, then there exists a model of $T$.

# Summary

- **Formal system** of first-order logic: formal system of propositional logic + $\forall x \varphi(x) \to \varphi(t)$ (the quantification axiom) + the generalization inference rule
- **Herbrand's theorem** (Skolem version). In first-order logic (without equality), $\exists$-formula $\exists \vec{x} \varphi(\vec{x})$ is valid if and only if
  - there exist $n$-tuples of terms, $\vec{t}_1, \ldots, \vec{t}_k$, from $\mathcal{L}(\varphi)$ and
  - $\varphi(\vec{t}_1) \vee \cdots \vee \varphi(\vec{t}_k)$ is a tautology.
- **Löwenheim-Skolem's downward theorem**.
  For a structure $\mathcal{A}$ in a countable language $\mathcal{L}$, there exists a countable substructure $\mathcal{A}' \subset \mathcal{A}$ s.t. $\mathcal{A}' \models \varphi \Leftrightarrow \mathcal{A} \models \varphi$ for any $\mathcal{L}_{A'}$-sentence $\varphi$. Such $\mathcal{A}'$ is called an **elementary substructure** of $\mathcal{A}$, denoted as $\mathcal{A}' \prec \mathcal{A}$.
- **Henkin axiom** $\exists x \varphi(x) \to \varphi(c_{\exists x \varphi(x)})$, by which any sentence can be rewritten as a formula without quantifiers.
- **Gödel's completeness theorem** (a weak version). In first order logic, $\vdash \varphi \Leftrightarrow \models \varphi$.

# Thank you for your attention!