Logic and Computation: I

Chapter 2 Propositional logic and computational complexity

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Logic and Computation I

- **Part 1. Introduction to Theory of Computation**
- **Part 2. Propositional Logic and Computational Complexity**
- **Part 3. First Order Logic and Decision Problems**

Part 2. Schedule

- Nov.17, (1) Tautologies and proofs
- Nov.22, (2) The completeness theorem of propositional logic
- Nov.24, (3) SAT and NP-complete problems
- Nov.29, (4) NP-complete problems about graph
- Dec. 1, (5) Time-bound and space-bound complexity classes
- Dec. 6, (6) PSPACE-completeness and TQBF

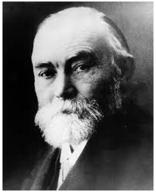
Tautologies and proofs

History

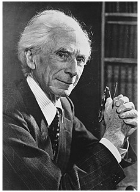
- Propositional logic is the study of logical connections between propositions.
- While Aristotle of ancient Greece pioneered predicate logic by analyzing the relationships between terms, propositional logic was founded by Chrysippus of the Stoics about a century later.
- It was G. Frege at the end of the 19th century who merged these logics and created modern formal logic, which was conceived as a comprehensive system by B. Russell.
- Subsequently, D.Hilbert recognized formal logic in the modern way such as

Propositional logic ⊂ First-order logic ⊂ Higher-order logic.

Aristotle



G. Frege



B. Russell

D. Hilbert

- Though E. Boole's aim was to formalize Aritotle's logic, he invented something like algebra of set-theoretic operations, now known as a **Boolean algebra** $(B, \wedge, \vee, \neg, 0, 1)$, which can be also seen as a model of proposional logic.
- A Boolean expression φ is said to be satisfiable if there exists an assignment of values $0, 1$ to its variables with which φ has value 1.
- φ is a valid (always takes value 1), is equivalent to the negation of φ being unsatisfiable.
- Although satisfiability and validity of Boolean expressions is computable, efficient calculation methods are not known. S.A. Cook showed that the satisfiability problem is NP-complete, and paved the way for research on computational complexity.
- Cook's argument is similar to Turing's proof on undecidability of first-order logic, which will be discussed in Part 3.

Ordinary mathematics are carried out with the following six logical operations.

- $\neg$ (not $\cdots$), $\wedge$ (and), $\vee$ (or), $\to$ (implies),
- $\forall$ (for all $\cdots$), $\exists$ (exists $\cdots$).

ε-δ argument: $f(x)$ is continuous at $x = x_0$

(1) For any $\varepsilon > 0$, there is some $\delta > 0$, for all x such that $| x - x_0 | < \delta$, $| f(x) - f(x_0) | < \varepsilon$.

Let us express this sentence using the above logical symbols. A tricky part is that "for all x such that $| x - x_0 | < \delta$" can be replaced by "for any x, if it is the case that $| x - x_0 | < \delta$". Now (1) can be expressed by logical symbols as follows.

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \; \Big(\, | x - x_0 | < \delta \to | f(x) - f(x_0) | < \varepsilon \Big).$$

which can also be rewritten as

$$\forall \varepsilon \Big(\varepsilon > 0 \to \exists \delta \big(\delta > 0 \wedge \forall x (| x - x_0 | < \delta \to | f(x) - f(x_0) | < \varepsilon) \big) \Big).$$

Logical symbols

propositional connectives: $\neg$ (not $\cdots$), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies),
quantifiers: $\forall$ (for all $\cdots$), $\exists$ (exists $\cdots$).

- In this part, we will study **propositional logic** which treats the logical relationships between propositions in terms of propositional connectives.
- In the next part, we will study **first-order logic** which examines atomic propositions by decomposing them into (subject + predicate) using quantifiers.
- The importance of first-order logic as the **foundation of mathematics** was emphasized by D. Hilbert.

- In propositional logic, propositions are constructed from atomic propositions by way of propositional connectives.
- Atomic propositions are simply symbols that can take value either true or false.

Definition (Proposition)

Let $p_0, p_1, p_2, \cdots$ be **atomic propositions**. **Propositions** are defined recursively:

(1) All atomic propositions are propositions;

(2) If φ and ψ are propositions, then $\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi)$ are also propositions.

- For example, $(\neg p_1 \wedge (p_3 \rightarrow (p_1 \vee p_2)))$ is a proposition.
- In the following, we will omit the parentheses as long as there is no misreading. The above proposition can be written as $\neg p_1 \wedge (p_3 \rightarrow p_1 \vee p_2)$.
- Strictly speaking, we omit the parentheses by assuming operator precedence
  $\neg, \quad \wedge, \quad \vee, \quad \rightarrow$.
  E.g. $\neg$ has higher precedence than $\wedge$.

When the proposition φ is true, φ has **truth value** T (meaning true), while φ is false, it has **truth value** F (meaning false).
For the atomic propositions, T and F can be freely assigned to them.
For a compound proposition, its truth value is uniquely determined from a truth value assignment to the atomic propositions it includes.

Definition

Let v be a function that assigns truth values to atomic propositions. We can define a function V that assigns truth values to all propositions uniquely. Such a function V is called a **truth value assignment** or a **truth value function**.

(1) for an atomic proposition φ, $V(\varphi) = v(\varphi)$.

(2a) $V(\neg\varphi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{F}$,

(2b) $V(\varphi \wedge \psi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{T}$ and $V(\psi) = \mathrm{T}$,

(2c) $V(\varphi \vee \psi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{T}$ or $V(\psi) = \mathrm{T}$,

(2d) $V(\varphi \to \psi) = \mathrm{T} \overset{\text{def}}{\Longleftrightarrow} V(\varphi) = \mathrm{F}$ or $V(\psi) = \mathrm{T}$.

For the four possible assignments to the propositions φ and ψ, the truth values of $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ are shown in the following table.

φ	ψ	$\neg\varphi$	$\varphi \vee \psi$	$\varphi \wedge \psi$	$\varphi \to \psi$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	F	T

- The above table is called a **truth(-value) table**.
- Given a truth-value function v for the atomic propositions, any proposition has a unique truth-value. It is easily shown by induction the construction of the "proposition" in way of the above truth table.
- Thus, the truth value function V is uniquely obtained from v.
- In the following, we use " truth value functions v for atomic propositions" and "truth-value function V" interchangeably.

Definition

If a proposition φ is always true, i.e., $V(\varphi) = \mathrm{T}$ for any truth-value function V, then φ is said to be **valid** or a **tautology**, written as $\models \varphi$.

Example 1

$\varphi \to (\psi \to \varphi)$ is a tautology, for any φ, ψ.

- For contradiction, suppose that $V(\varphi \to (\psi \to \varphi)) = \mathrm{F}$ for some truth value function V. From condition (2d) above, we have $V(\varphi) = \mathrm{T}$ and $V(\psi \to \varphi) = \mathrm{F}$.
- Then, applying condition (2d) to $V(\psi \to \varphi) = \mathrm{F}$ again, we have $V(\psi) = \mathrm{T}$ and $V(\varphi) = \mathrm{F}$, which is a contradiction, since $V(\varphi)$ cannot be both T and F at the same time.

Example 2: $(\varphi \vee \psi) \wedge (\neg\psi \vee \theta) \rightarrow (\varphi \vee \theta)$ is valid.

- There are eight assignments of truth values T and F to propositions φ, ψ and θ.
- In the table below, $\wedge$ stands for $(\varphi \vee \psi) \wedge (\neg\psi \vee \theta)$ and $\rightarrow$ for the whole formula.
- The whole is always T, so $(\varphi \vee \psi) \wedge (\neg\psi \vee \theta) \rightarrow (\varphi \vee \theta)$ is valid.

φ	ψ	θ	$(\varphi \vee \psi)$	$\wedge$	$(\neg\psi \vee \theta)$	$\rightarrow$	$(\varphi \vee \theta)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F
F	F	T	F	F	T	T	T
F	F	F	F	F	T	T	F

- For propositions with a few connectives as in Example 2, the truth table method to verify the validity is simple and easy to understand.
- However, if the length of the proposition is doubled or tripled, the truth table becomes too large to create manually.
- In some cases, a clever method like the contradiction method in Example 1 can be used. Is there any effective calculation method to determine the validity of any proposition?
- Cook's theorem shows that this is indeed difficult.

- Before discussing the efficiency issue, we consider the structure of the tautologies.
- To this end, it is not necessary to deal with all four propositional symbols at once. By setting $\varphi \vee \psi := \neg\varphi \to \psi$, $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$, we omit $\vee$ and $\wedge$.
- We consider an axiomatic system that derives all valid propositions only using $\neg, \to$. Here we take Łukasiewicz's system, which consists of only three axioms.

Łukasiewicz's system [1]

P1. $\varphi \to (\psi \to \varphi)$

P2. $\big(\varphi \to (\psi \to \theta)\big) \to \big((\varphi \to \psi) \to (\varphi \to \theta)\big)$

P3. $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

- Since φ, ψ, and θ represent arbitrary propositions, P1, P2, and P3 are collections of propositions in three pattens (called **axiom schema**). In the following, **axiom schema** are also simply referred to as axioms.

[1] This is a less popular one among his many systems

- We have already shown axiom P1 is a tautology.

Homework

Show that P2 and P3 are tautologies.

- Statements derived from axioms by proper rules are generally called "theorems".

Definition (Theorems)

The **theorems** of propositional logic are defined as follows.

(1) Axioms $\mathrm{P1}$, $\mathrm{P2}$, $\mathrm{P3}$ are theorems.

(2) If φ and $\varphi \to \psi$ are theorems, so is ψ. (detachment rule)

- **Detachment rule** is also called **modus ponens** (**MP** for short) and **cut**.

Definition (Propositions, revisited)

Let $p_0, p_1, p_2, \cdots$ be **atomic propositions**. **Propositions** are defined recursively:

(1) All atomic propositions are propositions;

(2) If φ and ψ are propositions, then $\neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi)$ are also propositions.

Propositions and Theorems

- Both Propositions and Theorems are recursively defined.
- The two definitions look similar but they have very different effect.
- Whether or not a given string is a "proposition" can be determined only by observing the string.
- However whether or not a given proposition is a "theorem" cannot be judged from the structure of the proposition alone. In fact, if a theorem ψ is obtained by the detachment rule, there are infinitely many possible choices for its assumption φ.
- For example, $p_0 \to p_0$ is a proposition, but we cannot see at a glance that it is a theorem.

We introduce the concept of "proof" as a process generating theorem.

Definition (Proof)

A sequence of propositions $\varphi_0, \varphi_1, \cdots, \varphi_n$ is called a **proof** of φ_n if it satisfies the following conditions: For $k \leq n$,

(1) φ_k is one of axioms $\mathrm{P1}$, $\mathrm{P2}$, $\mathrm{P3}$, or

(2) There exist $i, j < k$ such that $\varphi_j = \varphi_i \to \varphi_k$ (MP).

Note that a "theorem" is the last component of a "proof" and vice versa.
By $\vdash \varphi$, we mean that φ is a theorem, namely a provable proposition.

Example 3

The following sequence of propositions $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$ is a proof of $\varphi \to \varphi$.

$$\begin{aligned}
\varphi_0 &= \varphi \to ((\varphi \to \varphi) \to \varphi) && : \mathrm{P1} \\
\varphi_1 &= (\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)) && : \mathrm{P2} \\
\varphi_2 &= (\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi) && : \mathrm{MP}(\varphi_1 = \varphi_0 \to \varphi_2) \\
\varphi_3 &= \varphi \to (\varphi \to \varphi) && : \mathrm{P1} \\
\varphi_4 &= \varphi \to \varphi && : \mathrm{MP}(\varphi_2 = \varphi_3 \to \varphi_4)
\end{aligned}$$

Homework

Construct proofs of the following propositions.

(1) $\neg\varphi \to (\varphi \to \psi)$.

(2) $\neg\neg\varphi \to \varphi$.

Hints of homework problems.

(1) By way of contradiction, there exists a truth-value function V such that $V((\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))) = \mathrm{F}$.

(2) By way of contradiction, there exists a truth-value function V such that $V((\neg\psi \to \neg\varphi) \to (\varphi \to \psi)) = \mathrm{F}$.

(1) First, we show that if A is a theorem, then for any B, $B \to A$ is a theorem. By P1, $A \to (B \to A)$. By applying MP to this and A, we have $B \to A$.

(2) First we show that if $C \to D$ and $D \to E$ are theorems, then $C \to E$ is also. If $D \to E$ is a theorem, as shown in (1), $C \to (D \to E)$ is also. From P2, we have $(C \to (D \to E)) \to ((C \to D) \to (C \to E))$, and then by MP, we obtain $(C \to D) \to (C \to E)$. Since $C \to D$ is a theorem, by MP, we obtain $C \to E$.

Summary

- Propositional logic is the study of logical connections between propositions.
  $\neg$ (not $\cdots$), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies).
- If a proposition φ is always true, i.e., $V(\varphi) = \mathrm{T}$ for any truth-value function V, then φ is said to be **valid** or a **tautology**, written as $\models \varphi$.
- A theorem is the last component of a proof, which is a sequence of propositions consisting of axioms or applications of MP. If φ is a theorem, we write $\vdash \varphi$.
- In the next lecture, we will prove the completeness theorem: $\vdash \varphi \Leftrightarrow \models \varphi$.

Further readings

E. Mendelson. *Introduction to Mathematical Logic*, CRC Press, 6th edition, 2015.

Quiz 2: The blue-eyed islanders puzzle [2]

- There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces).
- If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness.
- Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).
- One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe.
- One evening, he addresses the entire tribe to thank them for their hospitality.

[2] Puzzle from `https://terrytao.wordpress.com/2008/02/05/the-blue-eyed-islanders-puzzle/?fbclid=IwAR3bgqV5KWoPgXu84_-un74ILGciL09Qu3sGGXBEtWe_I3ngR5ulsG9DUVQ`

Quiz 2: The blue-eyed islanders puzzle (continued)

However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

What happens in the following days?

A. The foreigner has no effect, because his comments do not tell the tribe anything that they do not already know.

B. 100 days after the address, all the blue eyed people commit suicide.

Please scan the following QR code to submit your answer now.

Thank you for your attention!