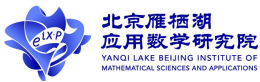
Logic and Computation: I

Chapter 1 Introduction to theory of computation

Kazuyuki Tanaka

BIMSA

November 10, 2022



Logic and Computation I

- **Part 1. Introduction to Theory of Computation**
- **Part 2. Propositional Logic and Computational Complexity**
- **Part 3. First Order Logic and Decision Problems**

Part 1. Schedule

- Oct.27, (1) Automata and monoids
- Nov. 1, (2) Turing machines
- Nov. 3, (3) Computable functions and primitive recursive functions
- Nov. 8, (4) Decidability and undecidability
- Nov.10, (5) Partial recursive functions and computable enumerable sets
- Nov.12, (6) Rice's theorem and many-one reducibility

Recap

- A TPL program $\mathcal{P}$ is a sequence of instructions with codes $c_0, c_1, \ldots, c_l$. So, it is represented by a sequence $1^{c_0}0\cdots01^{c_l}$ on $\{0,1\}^*$, and hence by a single number

$$p(0)^{c_0+1} \cdot p(1)^{c_1+1} \cdot \cdots \cdot p(l)^{c_l+1}.$$

  This number is called the Gödel number of program $\mathcal{P}$, denoted $\ulcorner\mathcal{P}\urcorner$.

- Any Turing machine $\mathcal{M}$ can be emulated by a TPL program $\mathcal{P}_\mathcal{M}$ on a universal Turing machine. So, the index (or code) of TM $\mathcal{M}$ is defined to be the Gödel number $\ulcorner\mathcal{P}_\mathcal{M}\urcorner$.

- $f : \mathbb{N}^k \longrightarrow \mathbb{N}$ is a partial computable function if $\{1^{m_1}0\cdots01^{m_k}01^{f(m_1,\ldots,m_k)} : m_1, \ldots, m_k \in \mathbb{N}\}$ is a 0-type language.

  If it is realized by a TM $\mathcal{M}$ with index e, f is denoted by $\{e\}^k$ (or simply $\{e\}$). When e is not an index of TM, $\{e\}$ is regarded as a partial function with empty domain.

- Enumeration theorem: For any $n \geq 0$, there exists a natural number e_n such that for any d, $x_1, \ldots, x_n$,

$$\{e_n\}^{n+1}(d, x_1, \ldots, x_n) \sim \{d\}^n(x_1, \ldots, x_n).$$

  $f(x_1, \ldots, x_n) \sim g(x_1, \ldots, x_n)$ means either both sides are not defined or they are defined with the same value.

- A set $X \subset \mathbb{N}^n$ is said to be computably enumerable or CE if $\{1^{x_1}0 \cdots 01^{x_n} : (x_1, \ldots, x_n) \in X\}$ is 0-type, i.e., the domain of a partial recursive function.

- X is said to be computable if both X and X^c are CE.

- If a function $f(x)$ has a finite value at $x = n$, we write $f(n)\downarrow$. Then we define a halting program K as follows.

$$\mathrm{K} = \{e : \{e\}(e)\downarrow\}.$$

  K is CE but not computable (by a diagonal argument).

Partial recursive function and CE set

Definition (Partial recursive function)

The **partial recursive functions** are defined as follows.

1. Constant 0, **Successor function** $\mathrm{S}(x) = x + 1$, **Projection** $\mathrm{P}_i^n(x_1, x_2, \cdots, x_n) = x_i \ (1 \leq i \leq n)$ are partial recursive functions.
2. **Composition**. If $g_i : \mathbb{N}^n \to \mathbb{N}, h : \mathbb{N}^m \to \mathbb{N} (1 \leq i \leq m)$ are partial recursive functions, the composed function $f = h(g_1, \cdots, g_m) : \mathbb{N}^n \to \mathbb{N}$ defined by

$$f(x_1, \cdots, x_n) \sim h(g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n))$$

is partial recursive, where

$$h(g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n)) = z$$

means that each $g_i(x_1, \cdots, x_n) = y_i$ is defined and $h(y_1, \cdots, y_m) = z$.

Note: By $f(x_1, \cdots, x_n) \sim g(x_1, \cdots, x_n)$, we mean that either both functions are undefined or defined with the same value.

Definition (Partial recursive function)

3. **Primitive recursion**.
   If $g : \mathbb{N}^n \to \mathbb{N}, h : \mathbb{N}^{n+2} \to \mathbb{N}$ are partial recursive functions, the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$\begin{aligned} f(x_1, \cdots, x_n, 0) &\sim g(x_1, \cdots, x_n) \\ f(x_1, \cdots, x_n, y+1) &\sim h(x_1, \cdots, x_n, y, f(x_1, \cdots, x_n, y)) \end{aligned}$$

   is partial recursive.

Definition (Partial recursive function)

4. **Minimization**.
   - Let $g : \mathbb{N}^{n+1} \to \mathbb{N}$ be a partial recursive function.
   - If "$g(x_1, \cdots, x_n, c) = 0$, and for $z < c$, $g(x_1, \cdots, x_n, z)$ is defined with non-zero values", let $\mu y(g(x_1, \cdots, x_n, y) = 0) = c$;
     if there is no such c, then $\mu y(g(x_1, \cdots, x_n, y) = 0)$ is undefined.
   - Then $f : \mathbb{N}^n \to \mathbb{N}$ satisfying

     $$f(x_1, \cdots, x_n) \sim \mu y(g(x_1, \cdots, x_n, y) = 0)$$

     is partial recursive.

Theorem (Kleene normal form theorem)

There are a primitive recursive function $U(y)$ and a primitive recursive relation $T_n(e, x_1, \cdots, x_n, y)$ such that any computable partial function $f(x_1, \cdots, x_n)$, there exists e such that

$$f(x_1, \cdots, x_n) \sim U(\mu y T_n(e, x_1, \cdots, x_n, y)),$$

where $\mu y T_n(e, x_1, \cdots, x_n, y)$ takes the smallest value y satisfying $T_n(e, x_1, \cdots, x_n, y)$; if there is no such y, it is undefined.

Thus, every partial recursive function can be obtained from two fixed primitive recursive functions by applying μ-operator to one of them.

Proof.

- We define a relation $T_n(e, x_1, \cdots, x_n, y)$ as follows:
  $T_n(e, x_1, \cdots, x_n, y) \Leftrightarrow$ "y is the Gödel number (code) of the whole computation process γ of a universal TM with input $(e, x_1, \cdots, x_n)$"
  Note that γ essentially includes the computation process of TM with index e on input $(x_1, \cdots, x_n)$, but the transition rules governing γ are given by universal TM, not by e.
- The whole computation process γ is a sequence of configurations $\alpha_0 \triangleright \alpha_1 \triangleright \cdots \triangleright \alpha_n$ with the initial α_0 and a final α_n, which can regarded as a word over $\Omega \cup Q \cup \{\triangleright\}$.
- In general, it is not decidable whether a whole computation process γ exists or not. But for a given γ, we can easily check that for each $i < n$, $\alpha_i \triangleright \alpha_{i+1}$ is a valid transition of universal TM, as well as α_0 and α_n are the first and last configurations.
- Thus, it is primitive recursive to check that y is a code for such a γ.
- Finally, let $U(y)$ be a primitive recursive function that extracts the output information from the last configuration α_n of γ coded by y. Then, $U(\mu y T_n(e, x_1, \cdots, x_n, y))$ is nothing but the result of inputting $(x_1, \cdots, x_n)$ to a TM with index e. □

Corollary

Every partial computable function is a partial recursive function.
Every computable function is recursive.

Corollary

There is a primitive recursive function $T_n(e, x_1, \ldots, x_n, y)$ such that for any CE set $X \subset \mathbb{N}^n$, there exists e satisfying

$$(x_1, \ldots, x_n) \in X \Leftrightarrow \exists y\, T_n(e, x_1, \ldots, x_n, y).$$

Theorem (Parameter theorem [1])

For any $m, n \geq 1$, there exists a primitive recursive function $S_n^m : \mathbb{N}^{m+1} \to \mathbb{N}$ such that

$$\{e\}^{m+n}(x_1, \cdots, x_n, y_1, \cdots, y_m) \sim \{S_n^m(e, y_1, \cdots, y_m)\}^n(x_1, \cdots, x_n).$$

Remark [2]

- This theorem says that input data $(y_1, \cdots, y_m)$ may be treated as parameters in the TPL program.
- Roughly speaking, the Enumeration and Parameter theorems are inverses to one another: the former pushes indices up as variables while the latter pulls variables down as indices (parameters).

[1] Also known as "S-m-n theorem", which comes from the symbol S_n^m in the statememt of the theorem.
[2] cf. *Recursively Enumerable Sets and Degrees* by Robert I. Soare

Proof.

- Suppose $\mathcal{P}$ is a TPL program with Gödel number $\{e\}^{m+n}$ and a sequence $(y_1, \ldots, y_m)$ is given.
- First construct a TPL program $\mathcal{P}'$ that rewrites the input string $1^{x_1}0\cdots01^{x_n}$ to $1^{x_1}0\cdots01^{x_n}01^{y_1}0\cdots01^{y_m}$. The Gödel number of $\mathcal{P}'$ can be obtained by the primitive recursive function of $(y_1, \ldots, y_m)$.
- By connecting the programs $\mathcal{P}'$ and then $\mathcal{P}$, we obtain a program $\mathcal{P}''$ with the following two modifications: Let k be the number of lines in $\mathcal{P}'$.
  - Rewrite all "**halt**" in $\mathcal{P}'$ as "**goto** $k+1$".
  - In $\mathcal{P}$, change l in "**goto** l" and "**if** ? **then goto** l" to $k+1+l$.
- Thus the Gödel number of $\mathcal{P}''$ are obtained as a primitive recursive function of e and $(y_1, \cdots, y_m)$. Denote it as S_n^m. This completes the proof. □

Theorem (Recursion theorem)

Let $f(x_1, \cdots, x_n, y)$ be a partial recursive function. There exists e such that

$$\{e\}^n(x_1, \cdots, x_n) \sim f(x_1, \cdots, x_n, e).$$

Proof.
By parameter theorem, there is a primitive recursive function $g(y)$ such that $\{g(y)\}(x_1, \cdots, x_n) \sim \{y\}(x_1, \cdots, x_n, y)$.

Let d be the index of a partial computable function $f(x_1, \cdots, x_n, g(y))$.
Then let $e = g(d)$. We have

$$\{g(d)\}(x_1, \cdots, x_n) \sim \{d\}(x_1, \cdots, x_n, d) \sim f(x_1, \cdots, x_n, g(d))$$

□

There are infinitely many e satisfying the theorem. Because there are infinitely many functions S_n^m by parameter theorem.

From the recursion theorem we can directly derive the following theorem, which is also often called the recursion theorem.

Theorem (Fixed point theorem)

Let $\sigma(x)$ be a computable function. Then, there exists an e such that

$$\{e\}^n(x_1, \cdots, x_n) \sim \{\sigma(e)\}^n(x_1, \cdots, x_n).$$

Such an e is called a fixed point of σ.

- The right-hand side of the fixed point theorem can be viewed as a special case of the right-hand side of the recursion theorem.
- Using the parameter theorem, this fixed point theorem immediately leads to the recursion theorem.

- In order to see that a fixed point e can be arbitrarily large, we take any N. Let $\{e\}$ be a partial recursive function different from any of $\{0\}, \{1\}, \cdots, \{N\}$. For a given computable function $\sigma(x)$, define

$$\sigma'(x) = \begin{cases} e & \text{if} \quad x \leq N \\ \sigma(x) & \text{if} \quad x > N \end{cases}$$

  Any fixed point of $\sigma'(x)$ is greater than N and is also a fixed point of $\sigma(x)$.

Homework 1

Let $\sigma(x, y)$ be a computable function. Prove the existence of computable function k such that

$$\{k(y)\}^n(x_1, \cdots, x_n) \sim \{\sigma(k(y), y)\}^n(x_1, \cdots, x_n).$$

Hint: Consider a computable function $h(x)$ such that $\{\{x\}(x)\} \sim \{h(x)\}$ and then $\sigma(h(x), y)$ is expressed as $\{S(y)\}(x)$ by the parameter theorem.

Example: Application of the Recursion Theorem

- Choose any recursive function f whose range is an infinite set. Then, the set $A = \{x : \forall y < x f(x) \neq f(y)\}$ is also infinite.
- Consider a recursive function g that lists the elements of A from the smallest to the largest. Using the primitive recursion, g may be defined as follows:

$$\begin{aligned} g(0) &= 0 \\ g(x) &= \mu w \forall y \leq g(x \dot{-} 1) f(w) \neq f(y). \end{aligned}$$

  But, this definition uses primitive recursion and minimization at the same time, which does not fit our definition of partial recursive functions.
- However, using the recursion theorem,

$$g(x) \sim \begin{cases} 0 & \text{if } x = 0 \\ \mu w \forall y \leq g(x \dot{-} 1) f(w) \neq f(y) & \text{otherwise} \end{cases}$$

  we have a partial recursive function g without primitive recursion. Since it is easy to prove by induction that this g is total, g is the desired recursive function.

The following function f is called the **Ackermann function**.

$$\begin{cases} f(0,y) &= y+1, \\ f(x+1,0) &= f(x,1), \\ f(x+1,y+1) &= f(x,f(x+1,y)) \end{cases}$$

Wilhelm Ackermann
(1986 - 1862)

- Ackermann function is a total computable function that is not primitive recursive.
- The values of Ackermann function grow rapidly.

- To show Ackermann function is a total recursive function, recursion theorem will be used.
- To this end, we defined the following primitive recursive function.

$$c(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}, \quad \bar{c}(x) = 1 - c(x)$$

- Then, by the recursion theorem, there is an index e such that

$$\begin{aligned} \{e\}(x,y) \quad \sim \quad & (y+1)\bar{c}(x) + \{e\}(\dot{x-}1, y)c(x)\bar{c}(y) \\ & + \{e\}(\dot{x-}1, \{e\}(x, \dot{y-}1))c(x)c(y). \end{aligned}$$

- By induction we can verify that the function $\{e\}$ is total.
- It follows from Kleene's normal form theorem that a total computable function can be defined as a recursive function.

Homework 2 (Hard)

Show that the Ackermann function is not primitive recursive.
Hint: For any primitive recursive function $g(x, y)$ there exists a c such that

$$g(x, y) < f(c, \max\{x, y\}).$$

Definition

A relation $R \subset \mathbb{N}^n$ is **recursive** if its characteristic function $\chi_R : \mathbb{N}^n \to \{0, 1\}$

$$\chi_R(x_1, \cdots, x_n) = \begin{cases} 1, & \text{if } R(x_1, \cdots, x_n) \\ 0, & \text{otherwise} \end{cases}$$

is recursive.

Lemma

For recursive n-ary relations A, B,

$$\neg A, A \wedge B, A \vee B, \forall y < z A, \exists y < z A$$

are also recursive.

Lemma

Given two recursive n-ary functions g, h and a recursive n-ary relation R, f defined below is also recursive.

$$f(x_1, \cdots, x_n) = \begin{cases} g(x_1, \cdots, x_n), & \text{if } R(x_1, \cdots, x_n) \\ h(x_1, \cdots, x_n), & \text{otherwise} \end{cases}$$

Lemma

The graph of a recursive function is recursive.

Lemma

A total function with a recursive graph is a recursive function.

Proof. A function with graph $F(x, y)$ is represented by $\mu y F(x, y)$, and more precisely, $\mu y\ (1 - \chi_F(x, y) = 0)$. So a function with a recursive graph is partial recursive. If it is total, by Kleene's normal form theorem it is a recursive function. □

Homework 3

Show that the graph of the Ackermann function is a primitive recursive set.

Definition

An n-ary relation $R \subset \mathbb{N}^n$ is **recursively enumerable**, RE for short, if the following partial function f is partial recursive.

$$f(x_1, \cdots, x_n) = \begin{cases} 1 & if R(x_1, \cdots, x_n) holds \\ \text{undefined} & \text{otherwise} \end{cases}$$

- The domain of any partial recursive function is RE. By next lemma, we will see that the range of any (partial) recursive function is RE, and the name "recursively enumerable" actually describes this property.
- As shown, "recursive" and "computable" are interchangeable, so "recursively enumerable (RE)" and "computably enumerable (CE)" are also interchangeable.
- In this course, we will use CE more often than RE.

Since there are many conditions equivalent to RE, some basic ones are summarized in the next lemma.

Lemma

For the relation $R \subset \mathbb{N}^n$, the following conditions are equivalent.

(1) R is recursively enumerable.

(2) R is an empty set or the range of some primitive recursive function.

(3) R is a finite set or the range of a some recursive injection (1-to-1 function).

(4) R is an empty set or the range of some recursive function.

(5) R is the range of some partial recursive function.

(6) There exists a primitive recursive relation S such that

$$R(x_1, \cdots, x_n) \Leftrightarrow \exists y S(x_1, \cdots, x_n, y).$$

(7) There exists a recursive relation S such that

$$R(x_1, \cdots, x_n) \Leftrightarrow \exists y S(x_1, \cdots, x_n, y).$$

Proof. To show $(1) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1)$.

- $(1) \Rightarrow (6)$. Consider a partial recursive function f

$$f(x_1, \cdots, x_n) = \begin{cases} 1 & \text{if} \quad R(x_1, \cdots, x_n) \\ \text{not defined} & \text{otherwise} \end{cases}$$

  By Kleene's normal form theorem, there exist primitive recursive U and T such that

$$f(x_1, \cdots, x_n) \sim U(\mu y T(x_1, \cdots, x_n, y)).$$

  Thus, $R(x_1, \cdots, x_n) \Leftrightarrow \exists y T(x_1, \cdots, x_n, y)$.

- $(6) \Rightarrow (7)$ is obvious.

- $(7) \Rightarrow (1)$. Let S be a partial recursive relation such that

$$R(x_1, \cdots, x_n) \Leftrightarrow \exists y S(x_1, \cdots, x_n, y).$$

  Using a primitive recursive function U with $U(y) = 1$,
  $f(x_1, \cdots, x_n) \sim U(\mu y S(x_1, \cdots, x_n, y))$, and this f satisfies (1).

Proof. To show $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (2)$.

- (2) or $(4) \Rightarrow (3)$ The example g on p.18 will be used to make a given function injective.
- $(3) \Rightarrow (4) \Rightarrow (5)$ are obvious.
- $(5) \Rightarrow (6)$. By Kleene's normal form theorem, there exist primitive recursive U and T such that

$$f(x) \sim U(\mu y T(x, y)).$$

$$\begin{aligned} z \in \mathrm{range}(f) &\Leftrightarrow \exists x \exists y (T(x, y) \wedge U(y) = z) \\ &\Leftrightarrow \exists x (T(\mathrm{c}(x, 0), \mathrm{c}(x, 1)) \wedge U(\mathrm{c}(x, 1)) = z), \end{aligned}$$

  where x codes a pair (x_0, x_1) and x_i is given by a primitive recursive function $\mathrm{c}(x, i)$.

- $(6) \Rightarrow (2)$. Suppose that S is primitive recursive and $R(x) \Leftrightarrow \exists y S(x, y)$. We may assume that R is non-empty and choose any $d \in R$. Then define the following primitive recursive function g.

$$g(x) = \begin{cases} \mathrm{c}(x, 0) & \text{if } S(\mathrm{c}(x, 0), \mathrm{c}(x, 1)) \\ d & \text{otherwise} \end{cases}$$

  Then $\mathrm{range}(g)$ is the same as R.

Lemma

If the n-term relation A, B is CE, the following relations are also CE.

$$A \wedge B,\ A \vee B,\ \forall y < z A,\ \exists y < z A, \text{ and } \exists y A.$$

Lemma

The graph of partial recursive functions is CE.

Lemma

A function whose graph is a CE set is a partial recursive.

Homework 4

Show that if a CE set is infinite, it contains computable infinite subsets.

Summary

- The set of all **partial recursive functions**: the smallest class that contains the constant 0, successor function, projection, and closed under composition, primitive recursion and minimalization.
- **Kleene normal form theorem**: every partial recursive function can be obtained from two fixed primitive recursive functions by applying μ-operator to one of them.
- **Parameter theorem**: input can be seen as parameters in the TPL program.
- **Recursion theorem**: A function to define can be used in the definition.
- Definition of **recursively enumerable relation** and its equivalent statements.

Further readings

N. Cutland. *Computability: An Introduction to Recursive Function Theory*, Cambridge University Press, 1st edition, 1980.

Thank you for your attention!