Logic and Computation II, Spring 2023

Homework No.11

Name:

Problem 1
In a Σ_1 complete theory T, show that 1-consistency (Σ_1-soundness) of T is equivalent to the following: for any Σ_0 formula $\varphi(x)$, if $\varphi(\overline{n})$ is provable in T for all n, then $\exists x \neg\varphi(x)$ is not provable in T.

Solution:

Problem 2
Let A, B be two disjoint CE sets. Assume a theory T is Σ_1-complete. Show that there exists a Σ_1 formula $\psi(x)$ such that

$$n \in A \Rightarrow T \vdash \psi(\overline{n}), \quad n \in B \Rightarrow T \vdash \neg\psi(\overline{n}).$$

From this, deduce that $\{\ulcorner\sigma\urcorner : T \vdash \sigma\}$ and $\{\ulcorner\sigma\urcorner : T \vdash \neg\sigma\}$ are computably inseparable. (See Part 1-6, Slide p.25.) In particular, $\{\ulcorner\sigma\urcorner : T \vdash \sigma\}$ is not computable.

Solution:

Problem 3
Complete the proof of Gödel-Rosser's theorem.

- Let $A = \{\ulcorner\sigma\urcorner : T \vdash \sigma\}$, $B = \{\ulcorner\sigma\urcorner : T \vdash \neg\sigma\}$. If T is consitent CE theory, then A, B are disjoint CE sets.
- Similarly to the proof of the strong representation theorem for computable sets, costruct a formula $\psi(x)$ such that $A \subset \{n : T \vdash \psi(\overline{n})\}$ and $B \subset \{n : T \vdash \neg\psi(\overline{n})\}$.
- Considering the sentence σ such that $T \vdash (\sigma \leftrightarrow \neg\psi(\ulcorner\sigma\urcorner))$, prove that $\ulcorner\sigma\urcorner \notin A \cup B$.

Also, notice that if A, B were computably separable, we could construct a formula $\psi(x)$ such that $\{n : T \vdash \psi(\overline{n})\} \cup \{n : T \vdash \neg\psi(\overline{n})\} = \mathbb{N}$

Solution:

Problem 4
Show that there is a consistent theory T that proves its own contradiction $\neg\mathrm{Con}(T)$.

Solution:

Problem 5
Let $\mathrm{Bew}_T^{\#}(x) \equiv (\mathrm{Bew}_T(x) \wedge x \neq \overline{\ulcorner 0 = 1 \urcorner})$. For any true proposition σ,

$$\mathrm{Bew}_T^{\#}(\overline{\ulcorner \sigma \urcorner}) \leftrightarrow \mathrm{Bew}_T(\overline{\ulcorner \sigma \urcorner})$$

and

$$T \vdash \neg \mathrm{Bew}_T^{\#}(\overline{\ulcorner 0 = 1 \urcorner}).$$

Does it contradict with the second incompleteness theorem?

<u>Solution:</u>