Logic and Computation II

Part 7. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
- **Part 6. Automata on infinite objects**
- **Part 7. Recursion-theoretic hierarchies**

Part 7. Schedule (tentative)

- May 20, (1) Oracle computation and relativization
- May 22, (2) m-reducibility and simple sets
- May 27, (3) T-reducibility and Post's problem
- May 29, (4) Miscellaneous

Recap

- $A \leq_{\mathrm{m}} B$, if there exists a computable function f such that for any x,

$$x \in A \quad \Leftrightarrow \quad f(x) \in B.$$

- $A \leq_{\mathrm{T}} B$, if A is computable in oracle B (i.e., recursive in χ_B).
- A set A is said to be **(T-)complete**/**m-complete** (with respect to CE) if A is CE and $B \leq_{\mathrm{T}} A$ / $B \leq_{\mathrm{m}} A$ for any CE set B.

Theorem 7.12 (Post's theorem, 1944)

There exists a CE set that is neither computable nor m-complete.

- **Post's problem**: Is there a CE set that is neither computable nor (T-)complete.
- To challenge this problem, various notions of CE set (such as immune sets, simple sets, and productive sets) were introduced. A simple set satisfies Post's theorem.

Definition 7.13

An infinite set $B \subset \mathbb{N}$ that does not contain an infinite CE subset is called an **immune set**. A CE set $A \subset \mathbb{N}$ whose complement is an immune set is called a **simple set**.

- A simple set is a CE set that has a nonempty intersection with any infinite CE set and whose complement is an infinite set.
- Simple sets are not computable. This is because if it were computable, then its complement would be an infinite CE set.

Lemma 7.14 (Dekker)

Let $f : \mathbb{N} \to \mathbb{N}$ be a computable injection. Then

$$\mathrm{Range}(f) \equiv_{\mathrm{T}} \{n : \exists m > n \ \ (f(m) < f(n))\}.$$

The set on the right-hand side is called the **deficiency set** of f, denoted by $\mathrm{Dfc}(f)$.

Lemma 7.15

$f : \mathbb{N} \to \mathbb{N}$ is a computable injection, and if $\mathrm{Range}(f)$ is not computable, then $\mathrm{Dfc}(f)$ is a simple set.

Incompleteness theorems and simple sets (1/2)

- For any CE set C, in a proper arithmetic system T, there exists Σ_1 formula $\varphi(x)$

$$n \in C \Leftrightarrow T \vdash \varphi(\overline{n}).$$

  Now suppose C is a simple set. Since $\{n : T \vdash \neg\varphi(\overline{n})\}$ is CE, if it is an infinite set, it has non-empty intersection with C, which implies the inconsistency of T.
- Thus, if T is a consistent system, $\{n : T \vdash \neg\varphi(\overline{n})\}$ is finite.
- On the other hand, since C^c is an infinite set, there are infinitely many n such that neither $\varphi(\overline{n})$ nor $\neg\varphi(\overline{n})$ can be proved in T.

Incompleteness theorems and simple sets (2/2)

- Various concrete examples of C or $\varphi(x)$ have been studied in relation to the incompleteness theorem. One of them is the set of non-random numbers.
- For $n \in \mathbb{N}$, $\mu e(\{e\}(0) = n)$ can be regareded as a minimal program that outputs n, and such e is called the **Kolmogorov complexity** of n, represented by $\mathrm{K}(n)$.
- When $\mathrm{K}(n) \geq n$, n is called **random**.
- Then the set $\{n : \mathrm{K}(n) < n\}$ of non-random numbers is a simple set.
- It turns out that there are only finitely many numbers that can be proven to be random in an appropriate system of arithmetic.

Homework

Show that $\{n : \mathrm{K}(n) < n\}$ is a simple set.

Definition 7.16

A set $A \subset \mathbb{N}$ that satisfies the following condition is called **productive**:

- There exists a computable function f such that $f(x) \in A - W_x$ for each $W_x \subset A$.

Such an f is called a **productive function** for A.

Productive sets are not CE sets.

Example

The complement K^c of K is a productive set whose productive function is the identity map $\lambda x.x$, where $x \mapsto f(x)$ is represented as $\lambda x.f(x)$.
To show this, suppose $W_x \subset \mathrm{K}^c$. By the definition of K, $x \in W_x \Leftrightarrow x \in \mathrm{K}$. Then either $x \in W_x \wedge x \in K$ or $x \notin W_x \wedge x \notin K$, where the former contradicts with $W_x \subset K^c$. So, only the latter case holds, that is, $x \in \mathrm{K}^c - W_x$.

Lemma 7.17

A productive set contains an infinite CE subset. Hence the complement of a simple set is not a productive set.

Proof Let C be a productive set with a productive function f. We will construct an infinite CE subset of C by applying f repeatedly from $\varnothing$.
First, let i_0 be the index of the empty set. That is,

$$W_{i_0} = \varnothing \subset C.$$

Suppose now that $W_{i_n} \subset C$ has been constructed. Then, since $f(i_n) \in C - W_{i_n}$, by putting

$$W_{i_{n+1}} \quad := \quad W_{i_n} \cup \{f(i_n)\},$$

we have $W_{i_{n+1}} \subset C$.
Here, since i_{n+1} is computable in i_n, the set $\{f(i_0), f(i_1), f(i_2), \ldots\}$ is an infinite CE subset of C. The second half follows from the definition of simple sets. □

A CE set A such that $B \leq_{\mathrm{m}} A$ for any CE set B is called an m-complete CE set. In particular, K is an m-complete CE set.

Lemma 7.18

If A is an m-complete CE set, then A^c is a productive set.

Proof Let A be an m-complete CE set. Then there exists a computable function f such that for any x

$$x \in \mathrm{K} \Leftrightarrow f(x) \in A.$$

Thus $\mathrm{K}^c = f^{-1}(A^c)$.
Now let $\tau(e)$ be the index of $\lambda x.\varphi_e(f(x))$. That is,

$$W_{\tau(e)} = \{x \mid \varphi_e(f(x))\downarrow\}$$

Then, for $W_e \subset A^c$,

$$W_{\tau(e)} = \{x \mid f(x) \in W_e\} = f^{-1}(W_e) \subset f^{-1}(A^c) = \mathrm{K}^c.$$

From the example in page 7, the identity map $\lambda x.x$ is a productive function on K^c, so

$$\tau(e) \in \mathrm{K}^c - W_{\tau(e)} = f^{-1}(A^c) - f^{-1}(W_e) = f^{-1}(A^c - W_e).$$

That is, $f(\tau(e)) \in A^c - W_e$. Thus $f \circ \tau$ is a productive function for A^c. $\square$

Now we are ready to show the post theorem.

Theorem 7.12 (Post theorem, 1944)

There exists an incomputable CE set that is not m-complete.

Proof By Lemma 7.15, there exists a simple set A. By Lemma 7.17, A^c is not productive. From the definition of simple sets, A is CE. By Lemma 7.18, A is not m-complete. $\square$

Homework

If $A \leq_{\mathrm{m}} B$ and A is productive, show that B is also productive.

Further Reading

- Kozen, D. C. (2006). *Theory of computation* (Vol. 170). Heidelberg: Springer.
- Soare, R. I. (2016). *Turing computability. Theory and Applications of Computability*. Springer.

Introduction to Post's problem

- Post's problem was independently solved by Friedberg (1957) and Muchnik (1956). Their proof technique is now called the **finite injury priority argument**.
- Although this proof method is already common in the study of computability, it is still difficult for a novice to grasp the argument. So, it may be a good idea to start with a quick look at its outline, and then gradually deepen your understanding by reading the proof repeatedly.
- Now, if $A \leq_{\mathrm{T}} B$ but not $B \leq_{\mathrm{T}} A$, we write $A <_{\mathrm{T}} B$. Then, Post's problem can be expressed as follows.

Theorem 7.19 (Friedberg, Mucinik)

There exists a set A such that $\varnothing <_{\mathrm{T}} A <_{\mathrm{T}} \mathrm{K}$.

Low sets

- We proved Post's theorem by showing the existence of a simple set, which is incomputable CE set that is not m-complete. Now, we introduce the notion of **low sets** to extend from "non-m-complete" to "non-T-complete".
- Fix a set $A \subset \mathbb{N}$, and let $\{\varphi_e^A\}$ be a Gödel numbering of partial recursive functions $\varphi_0, \varphi_1, \ldots$ in A. Suppose W_x^A and K^A are also defined naturally as follows:

$$W_x^A := \{z \mid \varphi_x^A(z)\downarrow\},$$

$$\mathrm{K}^A \quad := \quad \{x \mid \varphi_x^A(x)\downarrow\} \quad = \quad \{x \mid x \in W_x^A\}.$$

- We can prove that K^A is not computable in A, etc., in the same way as $A = \varnothing$.
- K^A is also written as A' and called A-**jump**.

Definition 7.20

A set A such that $A' \leq_{\mathrm{T}} \mathrm{K}$ is called **low**.

Lemma 7.21

$A <_{\mathrm{T}} \mathrm{K}$ if A is a low set.

Proof. If A is a low set, $A <_{\mathrm{T}} A' \leq_{\mathrm{T}} \mathrm{K}$, and so $A <_{\mathrm{T}} \mathrm{K}$. □

Thus, to solve Post's problem, it is sufficient to prove the following:

Lemma 7.22 (main lemma for Post's problem)

There exists a simple low set.

- We introduce some notations related to oracle computations.
- By "$\varphi^A_{e,s}(x) = y$", we denote the computation of $\varphi^A_e(x) = y$ will be completed within s steps, and if it exceeds s steps, we denote it as $\varphi^A_{e,s}(x)\uparrow$.
- For a given s, it is decidable whether or not the computation terminates within s steps. Thus, "$\varphi^A_{e,s}(x) = y$" is a function computable in A (in fact, primitive recursive in A). Also, $\uparrow$ can be regarded as a finite value.
- It doesn't matter how you measure the number of steps. What we essentially need is
$$\varphi^A_e(x) = y\ (<\infty) \Leftrightarrow \exists \sigma \subset A\ \exists s\ \forall \tau \supseteq \sigma\ \forall t \geq s\ \varphi^\tau_{e,t}(x) = y.$$
- Here $\sigma \subset A$ means σ is an initial segment of χ_A. Let $W^A_{e,s} := \mathrm{dom}\varphi^A_{e,s}$.

Proof

- In the finite injury priority argument, a desired CE set A is constructed as the infinite sum $\bigcup_s A_s$ of finite sets A_s, where $A_0 = \varnothing$ and A_s is "the (finite) set of numbers that are verified to be members of A within s step". Once an element is determined to be a member of A, it is never removed. Thus $A_s \subset A_{s+1}$ for each s.
- To ensure that A is low and simple, we construct A_s to satisfy several requirements.
- A **positive requirement** is satisfied by adding some elements to a desired set A and a **negative requirement** is by excluding some elements from A.
- Satisfying one requirement may **injure** another requirement that is already satisfied. So, **priorities** are set to all requirements, so that a requirement will be injured by only a finite number of requirements (with higher-priority).

- A is low and simple if all of the following are satisfied.
  (i) A is CE,
  (ii) A^c is infinite,
  (iii) A has a common element with each infinite CE set, and
  (iv) $\mathrm{K}^A \leq_{\mathrm{T}} \mathrm{K}$.
- In the above, condition (i) naturally holds from the inductive construction of A. Condition (ii) is also easily satisfied.
- The essential ones are the positive condition (iii) and the negative condition (iv). Rewriting these into *requirements* for each e, we have

$$\begin{array}{rcl} P_e & : & |W_e| = \infty \Rightarrow A \cap W_e \neq \varnothing \\ N_e & : & \exists^\infty s \ \varphi_{e,s}^{A_s}(e) \downarrow \Rightarrow \varphi_e^A(e) \downarrow . \end{array}$$

  Here, $\exists^\infty$ means "exists infinitely many".

- It is clear that (iii) holds if P_e holds for each e.
- Next, we show that (iv) holds if N_e holds for each e. First, assume that $s \mapsto A_s$ is computable.
  If N_e holds, then

$$\exists^\infty s\ \varphi_{e,s}^{A_s}(e)\downarrow \Rightarrow\ \varphi_e^A(e)\downarrow \Rightarrow\ \exists t \forall s > t\ \varphi_{e,s}^{A_s}(e)\downarrow$$

$$\Rightarrow\ \forall t \exists s > t\ \varphi_{e,s}^{A_s}(e)\downarrow \equiv\ \exists^\infty s\ \varphi_{e,s}^{A_s}(e)\downarrow.$$

- Thus, $\mathrm{K}^A = \{e : \varphi_e^A(e)\downarrow\}$ is a Δ_2 set.

Corollary 7.11 (Revisited)

A is Δ_2 if and only if $A \leq_T \mathrm{K}$.

- By the above fact, we have $\mathrm{K}^A \leq_{\mathrm{T}} \mathrm{K}$.

- Now we explain why N_e is a negative requirement.
- We define the following computable function r as a tool to control N_e:

$$r(e, s) = u(A_s, e, e, s).$$

  Here, the right-hand side is called the **use function**, which is 1 + the maximum number used in the computation of $\varphi_{e,s}^{A_s}(e)$, and 0 if the computation never halts.
- If $s \mapsto A_s$ is assumed to be computable, then r is also computable, which is called the **restraint function**.
- That is, given A_s, if $\varphi_{e,s}^{A_s}(e)\downarrow$, then by not adding elements x less than $r(e, s)$ to A, we have $A \upharpoonright r = A_s \upharpoonright r$, so $\varphi_e^A(e)\downarrow$, and thus N_e is fullfilled as a negative requirement.

- Among all P_e and N_e, set the priority as
$$P_0 > N_0 > P_1 > N_1 > P_2 > N_2 > \dots$$
- Note that for any requirement there are only a finitely many requirements with higher priorities. So, numbers below $r(e, s)$ may be added to A only for P_i with $i < e$.
- Now, we show the construction of A.
- Step $s = 0$: Set $A_0 = \varnothing$.
  Step $s + 1$: Assume that A_s is obtained.
  If there is an $i \leq s$ which satisfies (i) $W_{i,s} \cap A_s = \varnothing$, and
  (ii) $\exists x \in W_{i,s}(x > 2i \wedge \forall e \leq i \; r(e, s) < x)$,
  then take the smallest such i and choose the smallest x that satisfies (ii) and set $A_{s+1} = A_s \cup \{x\}$.

  Then the requirement P_i is satisfied, and after $s + 1$ it will never receive attention.

  If there is no such $i \leq s$, put $A_{s+1} = A_s$.

- When $A_{s+1} = A_s \cup \{x\}$, for e such that $x \leq r(e,s)$, N_e is **injured** by x at $s+1$. However, we have

Claim 1

For every e, N_e is injured at most finitely many times.

$(\because)$ N_e can be injured only by P_i for $i < e$.

Claim 2

For all e, $r(e) = \lim_s r(e,s)$ exists and hence N_e holds.

$(\because)$ Fix any e. From Claim 1, there exists a step s_e such that N_e is not injured after s_e. But if $\varphi_{e,s}^{A_s}(e)\downarrow$ for $s > s_e$, then for $t \geq s$, $r(e,t) = r(e,s)$ and so $r(e) = \lim_s r(e,s)$ exists. Hence $A_s \restriction r = A \restriction r$ and $\varphi_e^A(e)\downarrow$, which implies N_e holds.

Claim 3

P_i holds for all i.

($\because$) Suppose that W_i is an infinite set. From Claim 2, we take such an s that

$$\forall t \geq s \; \forall e \leq i \; r(e,t) = r(e).$$

We may assume that no P_j with $j < i$ receives attention after $s'(\geq s)$, In addition, take $t > s'$ such that

$$\exists x \in W_{i,t}(x > 2i \wedge \forall e \leq i \; r(e) < x).$$

Then we already have $W_{i,t} \cap A_t \neq \varnothing$ or P_i receives attention at $t+1$. In either case, $W_{i,t} \cap A_{t+1} \neq \varnothing$, and so P_i holds.

From the above, $A = \bigcup_{s \in \mathbb{N}} A_s$ is a simple low set. Also, A^c is infinite, since from condition (ii) that $x > 2i$, we have $| \{x \in A : x \leq 2i\} | \leq i$. $\square$

Friedberg and Mucinik actually proved the following assertion.

Theorem 7.23 (Friedberg, Muchnik)

There exist CE sets A, B such that $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$.

It is clear that A, B in this theorem are neither computable nor complete. By the finite injury priority argument, these sets are constructed as $A = \bigcup_s A_s$ and $B = \bigcup_s B_s$ with the following requirements:

$$\begin{aligned} R_{2e} &: \quad A \neq W_e^B \\ R_{2e+1} &: \quad B \neq W_e^A \end{aligned}$$

Among many generalizations of the above theorem, the following theorem is particularly important.

Theorem 7.24 (G. E. Sacks)

Let C be an incomplete CE set.
(1) There is a simple set A such that $C \not\leq_{\mathrm{T}} A$.
(2) There exists low CE sets A, B s.t. $A \not\leq_{\mathrm{T}} B$ and $B \not\leq_{\mathrm{T}} A$ with $C = A \cup B$ and $A \cap B = \varnothing$.

Thank you for your attention!