Logic and Computation II

Part 7. Recursion-theoretic hierarchies

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Logic and Computation II

- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
- **Part 6. Automata on infinite objects**
- **Part 7. Recursion-theoretic hierarchies**

Part 6. Schedule (tentative)

- May 20, (1) Oracle computation and relativization
- May 22, (2) m-reducibility and simple sets
- May 27, (3) T-reducibility and Post's problem
- May 29, (4) Arithmetical hierarchy

Recap

- Fix a function $\xi : \mathbb{N} \to \mathbb{N}$. Then, a function $f : \mathbb{N}^n \to \mathbb{N}$ is said to be **computable in oracle ξ** if there exists an algorithm that computes f using ξ as a database.
- The three classes of functions (primitive recursive / recursive / partial recursive) are extended as **primitive recursive in ξ** / **recursive in ξ** / **partial recursive in ξ**, by adding ξ to the initial functions in each definition. (Definition 7.1)
- Almost all the theorems of recursion theory can be extended to statements with oracle ξ, which are called **relativizations** of the original theorems. (Theorems 7.3-5).
- **Relativized Kleene normal form theorem** 7.2: A partial recursive function in ξ can be expressed as $U(\mu y T^{\xi}(e, x_1, \cdots, x_n, y))$, also denoted $\{e\}^{\xi}(x_1, \cdots, x_n)$.
- A **partial recursive functional** $F : \mathbb{N}^n \times (\mathbb{N}^{\mathbb{N}})^k \to \mathbb{N}$ is represented as

$$F(x_1, \cdots, x_n, \xi_1, \cdots, \xi_k) = U(\mu y T(e, x_1, \cdots, x_n, y, \xi_1 \restriction y, \cdots, \xi_k \restriction y)).$$

Definition 7.6 (Relativized arithmetic hierarchy)

Given a $\xi : \mathbb{N} \to \mathbb{N}$ and $k \geq 0$, the following set A is said to be $\Sigma_{2k+1}(\xi)$ (with index e).

$$(x_1, \dots, x_n) \in A \Leftrightarrow \exists y_1 \forall y_2 \cdots \exists y_{2k-1} \forall y_{2k} \{e\}^{\xi}(x_1, \dots, x_n, y_1, \dots, y_{2k}) \downarrow .$$

The following set A is a $\Sigma_{2k+2}(\xi)$ set (with index e).

$$(x_1, \dots, x_n) \in A \Leftrightarrow \exists y_1 \forall y_2 \cdots \forall y_{2k} \exists y_{2k+1} \{e\}^{\xi}(x_1, \dots, x_n, y_1, \dots, y_{2k}) \uparrow .$$

$\Pi_k(\xi)$ is the complement of $\Sigma_k(\xi)$. $\Delta_k(\xi)$ is $\Sigma_k(\xi)$ and $\Pi_k(\xi)$.

- Here, $\downarrow$ / $\uparrow$ means that the function is defined / undefined.
- A $\Sigma_1(\xi)$ set is also called a ξ-CE set.
- We fix the arity n of set $A \subset \mathbb{N}^n$ arbitrarily so that $\Sigma_k(\xi)$ and $\Pi_k(\xi)$ sets are treated complementary. In fact, it is enough to consider the case $n = 1$ using the sequence code $c(x, i)$, i.e., a primitive recursive function that extracts the i-th element x_i in the sequence with code x. For instance, $A \subset \mathbb{N}^n$ is identified with the following set

$$\{x \in \mathbb{N} : (c(x, 0), \cdots, c(x, n-1)) \in A\}.$$

Theorem 7.7 (Relativized arithmetical enumeration theorem)

For each $k \geq 1$, there exists $\Sigma_k(\xi)$ (or $\Pi_k(\xi)$) subset U of $\mathbb{N}^{n+1}$ with the following property (U is called a universal set). For any $\Sigma_k(\xi)$ (or $\Pi_k(\xi)$) subset R of $\mathbb{N}^n$, there exists some e such that

$$R(x_1, \cdots, x_n) \Leftrightarrow U(e, x_1, \cdots, x_n).$$

Proof.

- In the case of $\Sigma_1(\xi)$, it follows from the relativized enumeration theorem. For the $\Pi_1(\xi)$ set, take the complement of universal set U for $\Sigma_1(\xi)$.
- For $k > 1$, a $\Sigma_k(\xi)$ formula is obtained from a $\Sigma_1(\xi)$ or $\Pi_1(\xi)$ formula by adding appropriate arithmetical quantifiers in the front. Since there is a universal set (or formula) for $\Sigma_1(\xi)$ or $\Pi_1(\xi)$, the formula obtained from it by adding appropriate arithmetical quantifiers is universal for $\Sigma_k(\xi)$. Similarly for $\Pi_k(\xi)$. □

Theorem 7.8 (Relativized arithmetical hierarchy theorem)

For every $k \geq 1$,

$$\Sigma_k(\xi) \cup \Pi_k(\xi) \subsetneq \Delta_{k+1}(\xi).$$

Proof.

- keys: relativized arithmetical enumeration theorem and diagonalization argument.
- By the relativized arithmetical enumeration theorem, there exists a universal $\Sigma_k(\xi)$ subset U of $\mathbb{N}^2$. Then consider the $\Pi_k(\xi)$ subset $V(e)$ of $\mathbb{N}^1$ defined by $\neg U(e,e)$.
- Then, $V(e)$ is not $\Sigma_k(\xi)$. Suppose $V(e)$ were $\Sigma_k(\xi)$. There would exist some e_0 such that $V(e) \Leftrightarrow U(e_0, e)$. By substituting $e = e_0$, we have $\neg U(e_0, e_0) \Leftrightarrow V(e_0) \Leftrightarrow U(e_0, e_0)$, which is a contradiction.
- Furthermore, by setting $W(e) \Leftrightarrow \neg V(e)$, $W(e)$ is not $\Pi_k(\xi)$, but a $\Sigma_k(\xi)$ set.
- So, if we set $Z(e,d) \Leftrightarrow (V(e) \wedge d = 0) \vee (W(e) \wedge d > 0)$, then $Z(e,d)$ is clearly a $\Delta_{k+1}(\xi)$ subset of $\mathbb{N}^2$, which is neither $\Sigma_k(\xi)$ nor $\Pi_k(\xi)$. □

Comments on $k = 0$

- Note that we have not defined $\Sigma_0(\xi), \Pi_0(\xi)$. To define $\Sigma_0(\xi), \Pi_0(\xi)$ in the formal arithmetical hierarchy, ξ must also be a formal object such as a formula.
- However, $\Sigma_0(\xi), \Pi_0(\xi)$ are often used to denote the primitive recursive relations in ξ in some literature. Then, for the empty oracle $(\xi \equiv 0)$, they are simply the primitive recursive relations, which contradicts with our formal definition: Σ_0, Π_0 represent bounded formulas or sets defined by them.
- Therefore, no formal definition is given. But a similar statement would hold whatever $\Sigma_0(\xi), \Pi_0(\xi)$ are defined, since $\Delta_1(\xi)$ is well-defined and large.

Lemma 7.9

A is $\Sigma_{k+1}(\xi)$ if and only if there exists some $\Pi_k(\xi)$ set B such that A is χ_B-CE, where χ_B is the characteristic function of B. For $k=0$, consider $\Pi_0(\xi)$ as the primitive recursive relations in ξ.

Proof

- ($\Rightarrow$) Suppose A is $\Sigma_{k+1}(\xi)$. By definition, there exists a $\Pi_k(\xi)$ predicate $B(x_1, \ldots, x_n, y_1)$ such that

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 B(x_1, \ldots, x_n, y_1).$$

- Therefore,

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y_1 \chi_B(x_1, \ldots, x_n, y_1) = 1,$$

and the right-hand side is χ_B-CE.

- ($\Leftarrow$) Let B be a $\Pi_k(\xi)$ set and A be χ_B-CE .
- By relativized Kleene's normal form theorem, we have,

$$(x_1, \ldots, x_n) \in A \Leftrightarrow \exists y T(e, x_1, \ldots, x_n, y, \chi_B \upharpoonright y).$$

- Furthermore,

$$w = \chi_B \upharpoonright y \Leftrightarrow \forall i < y(i \in B \Leftrightarrow w(i) = 1) \wedge \mathrm{leng}(w) = y,$$

and the right side is $\Delta_{k+1}(\xi)$. Combining both formulas, A is $\Sigma_{k+1}(\xi)$. □

In the above lemma, even if B is $\Sigma_k(\xi)$, the class of χ_B-CE does not change.

Theorem 7.10 (Post)

A is $\Delta_{k+1}(\xi)$ if and only if there exists some $\Sigma_k(\xi)$ set B such that A is computable in χ_B $(A \leq_T B)$.

Corollary 7.11

A is Δ_2 if and only if $A \leq_T \mathrm{K}$.

Homework

Prove the above theorem by using the last lemma in page 8.

§7.2. m-reducibility and simple sets

- The early concern in recursion theory or computability theory was to understand the structure of the m-degrees and T-degrees of CE sets.
- The **m-degree** of a set A is the equivalence class of A in the many-to-one reducibility $\leq_{\mathrm{m}}$. The **T-degree of** A is the equivalence class of A in the Turing reducibility $\leq_{\mathrm{T}}$.
- Obviously, there are at least two CE T-degrees. That is, the degree of the computable sets (or the degree of $\varnothing$) and the degree of the complete CE sets (or the degree of the halting problem).
- Since any m-degree is a subset of a T-degree, there are at least two CE m-degrees (except for $\varnothing$ and $\mathbb{N}$).
- Furthermore, Post showed that there are more than two m-degrees of CE sets, and raised the corresponding question about T-degrees (1944).
- **Post's problem** was independently solved by Friedberg and Muchnik. The technique used in their proof is called the **finite injury priority method**.

- First, let us review the basic concepts and results in part 1 of last semester.

Recall

- A sequence (or set) of partial computable functions $\varphi_0, \varphi_1, \varphi_2, \ldots$ (with repetition) is called a CE numbering, if $\varphi(e, x) := \varphi_e(x)$ is a partial computable function.
- A sequence of CE sets $A_0, A_1, A_2, \ldots$, is called a CE numbering if $\{\langle e, x\rangle : x \in A_e\}$ is CE.
- The CE numbering of partial computable functions $\varphi_0, \varphi_1, \varphi_2, \ldots$ is called a Gödel numbering if for any CE numbering $\psi_0, \psi_1, \psi_2, \ldots$, there exists a computable function σ such that for any e,

$$\psi_e(x) \sim \varphi_{\sigma(e)}(x).$$ (both undefined, or both defined and the same value)

- A typical Gödel numbering is $\{\{e\} : e \in \mathbb{N}\}$, where $\{e\}$ is Kleene's bracket notation.
- For a Gödel numbering $\varphi_0, \varphi_1, \ldots$, let $W_e = \{x \mid \varphi_e(x)\downarrow\}$, where $\varphi_e(x)\downarrow$ means that φ_e is defined at x. Then, $W_0, W_1, \ldots$ is a CE numbering.
- A typical incomputable CE set is the halting problem K defined as follows.

$$\mathrm{K} := \{x \mid \varphi_x(x)\downarrow\} = \{x \mid x \in W_x\}.$$

- For $A, B \subset \mathbb{N}$, if there exists a computable function f, for any x,

$$x \in A \quad \Leftrightarrow \quad f(x) \in B$$

  then we write $A \leq_{\mathrm{m}} B$.
- $\varnothing$ and $\mathbb{N}$ are minimal with respect to $\leq_{\mathrm{m}}$. K is an m-complete CE set. Also, there is the degree of computable sets (except for $\varnothing$ and $\mathbb{N}$).

Theorem 1.60 (Lecture01-06)

For any $A \subset \mathbb{N}$, the following statements are equivalent.
(1) $A \leq_{\mathrm{m}} \mathrm{K}$.
(2) $A \leq_1 \mathrm{K}$.
(3) A is CE.

Definition 1.61 (Lecture01-06)

We say that a set A is **m-complete** (with respect to CE) if A is CE and $B \leq_{\mathrm{m}} A$ for any CE set B.

- If A is computable in oracle B (or recursive in χ_B), we write $A \leq_{\mathrm{T}} B$.
- If $A \leq_{\mathrm{m}} B$ then $A \leq_{\mathrm{T}} B$.
- $\mathrm{K}^c \leq_{\mathrm{T}} \mathrm{K}$ is obvious, where K^c is the complement of K. But not $\mathrm{K}^c \leq_{\mathrm{m}} \mathrm{K}$.
- If $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} A$, we write $A \equiv_{\mathrm{m}} B$. If $A \leq_{\mathrm{T}} B$ and $B \leq_{\mathrm{T}} A$, then $A \equiv_{\mathrm{T}} B$.
- The m-degree of A is $\{B : B \equiv_m A\}$. The T-degree of A is $\{B : B \equiv_T A\}$

Homework

Show that $\mathrm{K}^c \leq_{\mathrm{m}} \mathrm{K}$ does not hold.

- The minimum degree with respect to $\leq_{\mathrm{m}}$ (except for $\varnothing$ and $\mathbb{N}$) is the equivalence class consisting of all computable sets.
- The maximum CE m-degree is the class of m-complete CE sets.
- Post showed that there exists a CE m-degree between these two.

Theorem 7.12 (Post theorem, 1944)

There exists a CE set that is neither computable nor m-complete.

- Following the above theorem, Post also sought an intermediate T-degree, which is known as Post's problem. To challenge it, various notions of CE sets (such as immune sets, simple sets, and productive sets) were introduced.

Definition 7.13

An infinite set $B \subset \mathbb{N}$ that does not contain an infinite CE subset is called an **immune set**. A CE set $A \subset \mathbb{N}$ whose complement is an immune set is called a **simple set**.

- A simple set is a CE set that has a nonempty intersection with any infinite CE set and whose complement is an infinite set.
- Simple sets are not computable. This is because if it were computable, then its complement would be an infinite CE set. As we will see later, this set is closely related to the incompleteness theorem.
- First, we must show the existence of simple sets, which is easily derived from the following lemma.

Lemma 7.14 (Dekker)

Let $f : \mathbb{N} \to \mathbb{N}$ be a computable injection. Then

$$\mathrm{Range}(f) \equiv_{\mathrm{T}} \{n : \exists m > n \ \ (f(m) < f(n))\}.$$

The set on the right-hand side is called the **deficiency set** of f, denoted by $\mathrm{Dfc}(f)$.

proof

- Let $f : \mathbb{N} \to \mathbb{N}$ be a computable injection and $A = \mathrm{Range}(f)$ and $B = \mathrm{Dfc}(f)$.
- To show $B \leq_{\mathrm{T}} A$.
- $n \in B$ is equivalent to $\exists k < f(n)(k \in A \land f^{-1}(k) > n)$, the latter of which is computable in A. Note that if $k \in A$ is known, it is easy to compute the value $f^{-1}(k) = \mu x(f(x) = k)$.
- To show $A \leq_{\mathrm{T}} B$.
- The complement B^c of B is always infinite. Because for any x, if we take y_x such that $f(y_x) = \min\{f(y) : y \geq x\}$, then $y_x \in B^c \land y_x \geq x$.
- Obviously $n \in A$ is equivalent to $\exists l \leq k(f(l) = n))$ for a sufficiently large k. So it is also equivalent to a Σ_1 formula $\exists k > n(k \in B^c \land f(k) > n \land \exists l \leq k(f(l) = n))$.
- Also, if $k \in B^c \land f(k) > n$, then $\forall l > k(f(l) > f(k) > n)$, and so $n \in A$ is equivalent to a Π_1 formula $\forall k > n(k \in B^c \land f(k) > n \to \exists l \leq k(f(l) = n))$.
- Therefore, A is computable in B. □

In part 1, we prove that any nonempty CE set A is represented as the range of computable injection f. Then, if A is not computable, then Dfc(f) is a simple set. For example, setting $A = \mathrm{K}$ gives a simple set.

Lemma 7.15

$f : \mathbb{N} \to \mathbb{N}$ is a computable injection, and if $\mathrm{Range}(f)$ is not computable, then $\mathrm{Dfc}(f)$ is a simple set.

Proof

Since $B = \mathrm{Dfc}(f)$ is also not computable, it is clear that its complement is not finite.
By way of contradiction, we assume that B is simple, that is, there exists an infinite CE set $C \subset B^c$.
Then by the second half of the proof for the lemma in page 16, $n \in \mathrm{Range}(f)$ is equivalent to

$$\exists k > n(k \in C \wedge f(k) > n \wedge \exists l \leq k(f(l) = n))$$

and

$$\forall k > n(k \in C \wedge f(k) > n \to \exists l \leq k(f(l) = n)),$$

which is computable and contradicts the assumption. □

Incompleteness theorems and simple sets (1/2)

- For any CE set C, in a proper arithmetic system T, there exists Σ_1 formula $\varphi(x)$

$$n \in C \Leftrightarrow T \vdash \varphi(\overline{n}).$$

  Now suppose C is a simple set. Since $\{n : T \vdash \neg\varphi(\overline{n})\}$ is CE, if it is an infinite set, it has non-empty intersection with C, which implies the inconsistency of T.
- Thus, if T is a consistent system, $\{n : T \vdash \neg\varphi(\overline{n})\}$ is finite.
- On the other hand, since C^c is an infinite set, there are infinitely many n such that neither $\varphi(\overline{n})$ nor $\neg\varphi(\overline{n})$ can be proved in T.

Incompleteness theorems and simple sets (2/2)

- Various concrete examples of C or $\varphi(x)$ have been studied in relation to the incompleteness theorem. One of them is the set of non-random numbers.
- For $n \in \mathbb{N}$, $\mu e(\{e\}(0) = n)$ can be regareded as a minimal program that outputs n, and such e is called the **Kolmogorov complexity** of n, represented by $\mathrm{K}(n)$.
- When $\mathrm{K}(n) \geq n$, n is called **random**.
- Then the set $\{n : \mathrm{K}(n) < n\}$ of non-random numbers is a simple set.
- It turns out that there are only finitely many numbers that can be proven to be random in an appropriate system of arithmetic.

Homework

Show that $\{n : \mathrm{K}(n) < n\}$ is a simple set.

Thank you for your attention!