Logic and Computation II

Part 6. Automata on infinite objects

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Logic and Computation II

- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
- **Part 6. Automata on infinite objects**
- **Part 7. Recursion-theoretic hierarchies**

Part 6. Schedule (tentative)

- Apr.15, (1) Second-order arithmetic and analytical hierarchy
- Apr.17, (2) Büchi automata
- Apr.22, (3) Safra's theorem
- Apr.24, (4) The decidability of S1S
- May 6, (5) Tree automata
- May 8, (6) Tree automata and parity games
- May 13, (7) The decidability of S2S
- May 15, (8) Positional determinacy of parity games

Recap

The outline of the proof of the main lemma.

Lemma 6.21

For any PTA M, there is a PTA M' that accepts the complement of $L(M)$.

PTA M does not accept t. $\Leftrightarrow$ II has a winning strategy σ for t in the game $\mathcal{G}(M,t)$.

$\Updownarrow$

All the paths through the $\Omega \times S_{\mathrm{II}}$-labeled tree $T^{t,\sigma}$ negates the parity condition.

$\Updownarrow$

The ω-language $L(t,\sigma)$ on $\Omega' = \Omega \times S_{\mathrm{II}} \times \{0,1\}$ consists of ω-words negating the parity condition.

$\Updownarrow$

PTA M' accepts t. $\Leftrightarrow$ $L(t,\sigma) \cap L(A) = \emptyset$.

Let A be an NPA which accepts all ω-words on Ω' satisfying the parity cond.

Let A' be a DPA which accepts the complement of $L(A)$. Let M' be a PTA constructed form A'.

S2S and MTA

- Now we will show the equivalence of S2S and MTA.
- First, to translate an S2S formula $\varphi(\vec{x}, \vec{X})$ into a tree language, we need something like the characteristic sequence we defined to translate S1S.
- For simplicity, we replace the first-order variable x with second-order variable X representing the singleton set, and consider the translation of the formula $\varphi(\vec{X})$ with no free occurrences of first-order variables.
- Let $\vec{T} = (T_1, \ldots, T_n)$ be an n-tuple of subsets of $\{0,1\}^*$. Letting $\Omega = \{0,1\}^n$, we express $\vec{T}$ by an Ω-labeled tree $t : \{0,1\}^* \to \{0,1\}^n$ such that for each $i = 1, \ldots, n$,

$$T_i = \{d \in \{0,1\}^* : i\text{-th element of } t(d) \text{ is } 1\}$$

Then, such a t is called the **characteristic representation tree** (representation tree, in short) of $\vec{T}$.

Lemma 6.23

Given an S2S formula $\varphi(\vec{X})$, there exists an MTA M_φ on $\Omega = \{0,1\}^n$ such that,

$$L(M_\varphi) = \{\text{The representation tree of } \vec{T} : \varphi(\vec{T}) \text{ holds}\}.$$

Proof. The atomic formula of S2S has a form

$$S_{b_1} S_{b_2} \dots S_{b_k} x \in X \ (\text{where } b_i = 0,1).$$

Then (d, T) satisfies the above relation iff the word $db_k \dots b_2 b_1$ belongs to T. So, it is easy to construct a PTA M that accepts the set of the representation trees of such (d, T)'s. Furthermore, since the class of languages accepted by MTA's is closed under Boolean operations and projections, any S2S formula has an equivalent MTA. □

Conversely, let $\{P_a : a \in \Omega\}$ $(P_a = t^{-1}(a))$ be the partition of $\{0,1\}^*$ determined by the Ω- labeled tree t. If an S2S formula φ holds in the structure

$$(\{0,1\}^* \cup \mathcal{P}(\{0,1\}^*), S_0(x), S_1(x), \in, P_a)_{a\in\Omega},$$

φ is said to holds in t. Then,

Lemma 6.24

Given an MTA M on Ω, there exists an S2S formula φ_M containing $P_a(a \in \Omega)$ as a set constant such that

$$t \in L(M) \Leftrightarrow \varphi_M \text{ holds in } t.$$

Proof. The idea of constructing the S2S formula φ_M from MTA M is almost the same as the proof of the lemma for S1S. First, the basic predicates of S1S can be used in S2S. For example, "$x = y$", "$X \subseteq Y$", "$X = Y$" etc. can be used. In addition, we define

- "$x = \epsilon$" : $\neg\exists y(S_0 y = x \vee S_1 y = x)$.
- "$\mathrm{Path}(X)$" : $\exists x \in X(x = \epsilon) \wedge \forall x \in X(x \neq \epsilon \rightarrow \exists y \in X(S_0 y = x \vee S_1 y = s))\wedge$ $\forall x \in X\exists! y(S_0 x = y \vee S_1 x = y)$.

Now, let $M = (Q, \Omega, \delta, Q_0, \mathcal{F})$ be a *complete* (no dead ends in state transitions) MTA . Then, if the input tree is represented by $\{P_a : a \in \Omega\}$, the run-tree $\vec{Y} = \{Y_q\}$ (Y_q is the set of vertices with label q) is expressed as follows.

$$
\begin{aligned}
\mathrm{run}(\vec{Y}) \quad = \quad & \bigvee_{q \in Q_0} \epsilon \in Y_q \\
& \wedge \forall x \bigvee_{(q,a,q_0,q_1) \in \delta} (x \in Y_q \wedge P_a(x) \wedge x0 \in Y_{q_0} \wedge x1 \in Y_{q_1}) \\
& \wedge \forall x \bigwedge_{p \neq q} \neg(x \in Y_p \wedge x \in Y_q)
\end{aligned}
$$

Furthermore, the Muller acceptance condition is expressed as

$$
\begin{aligned}
\varphi_M \quad = \quad & \exists \vec{Y} (\mathrm{run}(\vec{Y}) \\
& \wedge \forall X (\mathrm{Path}(X) \to \bigvee_{F \in \mathcal{F}} (\bigwedge_{q \in F} Y_q \cap X \text{ is infinite} \wedge \bigwedge_{q \notin F} Y_q \cap X \text{ is finite})
\end{aligned}
$$

Obviously, this satisfies the lemma.

Corollary 6.25

S2S is decidable.

Proof. Let σ be an S2S sentence. Its truth can be determined by checking whether or not the emptiness problem of the MTA language equivalent to $\sigma \wedge (X = X)$. This problem is decidable by the lemma in Page **??** of this slides. □

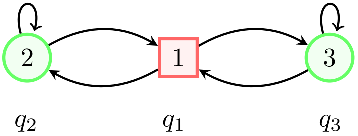
§6.7. Parity games

- A parity game $G = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$ is a game on a directed graph $(V_{\mathsf{I}} \cup V_{\mathsf{II}}, E)$ with a priority function $\pi : V_{\mathsf{I}} \cup V_{\mathsf{II}} \to \{0, 1, \cdots, k\}$ and $V_{\mathsf{I}} \cap V_{\mathsf{II}} = \varnothing$.
- Two players, player I and II, move a token along the edges of the graph. At a vertex $v \in V_{\mathsf{I}}$ (V_{II}), it is player I (II)'s turn to choose some v' such that $(v, v') \in E$.
- For an infinite resulting path $\rho = \rho_0\rho_1\cdots$ (called a **play**), let $\pi(\rho) := \pi(\rho_0)\pi(\rho_1)\cdots$. Player I **wins** in ρ iff the smallest number appearing infinitely often in $\pi(\rho)$ is even.
- A strategy for player I is a mapping $\sigma : (V_{\mathsf{I}} \cup V_{\mathsf{II}})^{<\omega} V_{\mathsf{I}} \to V_{\mathsf{I}} \cup V_{\mathsf{II}}$.
  A play ρ is **consistent** with σ if for all i, $\rho_i \in V_{\mathsf{I}} \Rightarrow \sigma(\rho_0\rho_1\cdots\rho_i) = \rho_{i+1}$.
- σ is a **winning strategy** for player I if Player I wins in any play consistent with σ.
- A (winning) strategy for player II can be defined similarly.
- A game is said to be **determined** if one of the players has a winning strategy.
- Martin proved that Borel games (including parity games) are determined.

- A **memoryless strategy** for player I is a mapping $\sigma : V_{\mathsf{I}} \to V_{\mathsf{I}} \cup V_{\mathsf{II}}$.
- A **memoryless strategy** for player II is a mapping $\tau : V_{\mathsf{II}} \to V_{\mathsf{I}} \cup V_{\mathsf{II}}$.
- From now on, by a strategy we mean a memoryless strategy.
- A play ρ is **consistent** with such a σ if for all i, $\rho_i \in V_{\mathsf{I}} \Rightarrow \sigma(\rho_i) = \rho_{i+1}$. Similar for τ.
- σ (τ) is a **winning strategy** if player I (II) wins in any play consistent with σ (τ) .
- Let $W_{\mathsf{I}}(G, \sigma)$ be the set of starting points $\rho_0 \in V$ such that σ is a winning strategy for player I. Let
$$W_{\mathsf{I}}(G) = \bigcup_{\mathsf{I}'s \text{ winning strategy } \sigma} W_{\mathsf{I}}(G, \sigma).$$
- Similarly, $W_{\mathsf{II}}(G, \tau)$ and $W_{\mathsf{II}}(G)$ are defined.
- Clearly, $W_{\mathsf{I}}(G) \cap W_{\mathsf{II}}(G) = \varnothing$.
- When $W_{\mathsf{I}}(G) \cup W_{\mathsf{II}}(G) = V$, the game G is said to have **memoryless determinacy**.

Example (revisit)

Consider the following parity game $G = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$, where $V_{\mathsf{I}} = \{q_2, q_3\}$ and $V_{\mathsf{II}} = \{q_1\}$, $\pi(q_i) = i$ for $i = 1, 2, 3$.



- $W_{\mathsf{I}}(G) = \{q_2\}$
- $W_{\mathsf{II}}(G) = \{q_1, q_3\}$
- Since $W_{\mathsf{I}}(G) \cup W_{\mathsf{II}}(G) = V$, the above game G has memoryless determinacy.

Lemma 6.26

In any parity game G, there exists a strategy σ for player I such that $W_{\mathsf{I}}(G,\sigma) = W_{\mathsf{I}}(G)$. Similarly, there exists a II's strategy τ such that $W_{\mathsf{II}}(G,\tau) = W_{\mathsf{II}}(G)$.

Proof

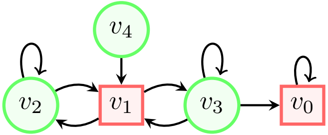
- By the well-ordering theorem, let $W_{\mathsf{I}}(G) = \{v_\beta\}_{\beta<\alpha}$ (α, β are ordinals).
- For each $\beta < \alpha$, let σ_β be a winning strategy of player I starting from v_β.
- Then, we define a function $f : W_{\mathsf{I}}(G) \to \alpha$ as follows: for $v \in W_{\mathsf{I}}(G)$, let $f(v)$ be the smallest $\beta < \alpha$ such that $v \in W_{\mathsf{I}}(G, \sigma_\beta)$.
- Finally, we define a strategy σ as $\sigma(v) := \sigma_{f(v)}(v)$. We want to show that $W_{\mathsf{I}}(G,\sigma) = W_{\mathsf{I}}(G)$. Since $W_{\mathsf{I}}(G,\sigma) \subseteq W_{\mathsf{I}}(G)$, it is sufficient to show any play consistent with σ starting from a vertex of $W_{\mathsf{I}}(G)$ is a winning play for I.

- Now, let ρ be a play consistent with σ, starting from vertex ρ_0 of $W_{\mathsf{I}}(G)$.
- If ρ is also consistent with $\sigma_{f(\rho_0)}$, then player I wins in ρ, which completes the proof. Otherwise, we can get the smallest k such that $\rho_k \in V_{\mathsf{I}}$ and $\rho_{k+1} \neq \sigma_{f(\rho_0)}(\rho_k)$.
- Since $\rho\upharpoonright(k+1)$ is consistent with $\sigma_{f(\rho_0)}$, player I can win the game from ρ_k following $\sigma_{f(\rho_0)}$, that is, $\rho_k \in W_{\mathsf{I}}(G, \sigma_{f(\rho_0)})$. But $\rho_{k+1} = \sigma(\rho_k) = \sigma_{f(\rho_k)}(\rho_k) \neq \sigma_{f(\rho_0)}(\rho_k)$, so $f(\rho_k) < f(\rho_0)$.
- Player I wins if ρ obeys $\sigma_{f(\rho_k)}$ from ρ_k onwards.
- Othewise, some k' appears such that $\rho_{k'} \in V_{\mathsf{I}}$ and $\rho_{k'+1} \neq \sigma_{f(\rho_k)}(\rho_{k'})$, then $f(\rho_{k'}) < f(\rho_k) < f(\rho_0)$.
- By repeating this, the descending sequence of ordinal numbers ends in finite steps. So there exists some $l \in \omega$ such that ρ is consistent with $\sigma_{f(\rho_l)}$ from ρ_l, and hence player I wins.
- Therefore, σ is I's winning strategy starting from any vertex of $W_{\mathsf{I}}(G)$. That is, $W_{\mathsf{I}}(G, \sigma) = W_{\mathsf{I}}(G)$.
- $W_{\mathsf{II}}(G, \tau) = W_{\mathsf{II}}(G)$ can be shown similarly.

- If there exist σ and τ such that $W_{\mathsf{I}}(G,\sigma) \cup W_{\mathsf{II}}(G,\tau) = V$, game G is said to have **uniform memoryless determinacy**.
- From the above lemma, if a parity game has memoryless determinacy, it also has uniform memoryless determinacy.
- We say that $v \in V$ is an **absorbing vertex** if no edges exit from v, i.e., $\{w : (v,w) \in E\} = \{v\}$. Note that we assume that no deadlocks exist.
- We say that $v \in V$ is a **vanishing vertex** if no edges enter v, i.e., $\{w : (w,v) \in E\} = \varnothing$.
- Vertices that are neither absorbing nor vanishing are called **relevant vertices**, and the set of such vertices is denoted by V_{r}.
- $\pi(v)$ for $v \in V_{\mathrm{r}}$ is called a **relevant priority**.

Example 2

Consider the following parity game $G = (V_{\text{I}}, V_{\text{II}}, E, \pi)$, where $V_{\text{I}} = \{v_2, v_3, v_4\}$ and $V_{\text{II}} = \{v_0, v_1\}$, $\pi(v_i) = i$ for $i = 0, 1, 2, 3, 4$.



- $W_{\text{I}}(G) = \{v_0, v_1, v_2, v_3, v_4\}$
- $W_{\text{II}}(G) = \varnothing$
- The above game G is uniform memoryless determined.
- v_0 is absorbing, v_4 is vanishing, v_1,v_2 and v_3 are relevant.
- $\{1, 2, 3\}$ is the set of relevant priorities.

Theorem 6.27

Any parity game G has uniform memoryless determinacy.

Proof

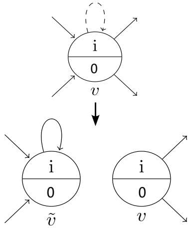
Consider a parity game $G = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$. We prove by induction on the number of relevant priorities $\pi(V_{\mathrm{r}})$.

Base case:

- If there are no relevant points, all vertices are absorbing or vanishing.
- From an absorbing vertex v, $v \in W_{\mathsf{I}}(G, \sigma)$ for any σ (if $\pi(v)$ is even), or $v \in W_{\mathsf{II}}(G, \tau)$ for any τ (otherwise).
- From a vanishing vertex v, each edge goes to an absorbing vertex, where the winner is determined regardless of the strategy. So, by selecting an appropriate $\sigma(v)$ or $\tau(v)$, we have $v \in W_{\mathsf{I}}(G, \sigma) \cup W_{\mathsf{II}}(G, \tau)$, where the values of σ and τ at other vertex than v are not irrelevant.
- Thus, there exist σ and τ such that $W_{\mathsf{I}}(G, \sigma) \cup W_{\mathsf{II}}(G, \tau) = V$.

Induction case:

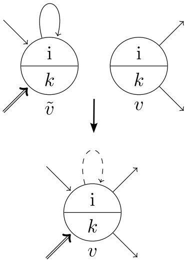
- Suppose the number of relevant priorities is $k > 0$. We first prove a weak claim $W_{\mathsf{I}}(G) \cup W_{\mathsf{II}}(G) \neq \varnothing$.
- For simplicity, assume that the minimum of the relevant priorities is 0.
- We will modify the game G so that the vertices with priority 0 are changed to non-relevant vertices. Such a modified game is called G^+, to which we will apply the induction hypothesis.
- Let D be the set of relevant vetices with priority 0 in G.
- Make a copy of D and put $\tilde{D} := \{\tilde{v} : v \in D\}$.
- $G^+ = (V_{\mathrm{I}}^+, V_{\mathsf{II}}^+, E^+, \pi^+)$ is defined as follows.
- $V_{\mathrm{I}}^+ := V_{\mathrm{I}} \cup \{\tilde{v} : v \in D \cap V_{\mathrm{I}}\}$,
- $V_{\mathsf{II}}^+ := V_{\mathsf{II}} \cup \{\tilde{v} : v \in D \cap V_{\mathsf{II}}\}$,
- $E^+ := \{(u, v) \in E : v \notin D\} \cup \{(u, \tilde{v}) : (u, v) \in E \wedge v \in D\} \cup \{(\tilde{v}, \tilde{v}) : v \in D\}$
- $\pi^+ := \pi \cup \{(\tilde{v}, 0) : v \in D\}$.



i=I, II

G^+ is obtained by separating each vertex v of D into vanishing vertex v and absorbing vertex $\tilde{v}$.

- Therefore, the number of relevant priorities of G^+ is less than that of G.
- By induction hypothesis, there exist σ^+ and τ^+ such that $W_{\mathrm{I}}(G^+, \sigma^+) \cup W_{\mathrm{II}}(G^+, \tau^+) = V^+ = V^+_{\mathrm{I}} \cup V^+_{\mathrm{II}}$.
- The strategies $\sigma^\pm : V_{\mathrm{I}} \to V$ and $\tau^\pm : V_{\mathrm{II}} \to V$ in G can be derived from $\sigma^+ : V^+_{\mathrm{I}} \to V^+$ and $\tau^+ : V^+_{\mathrm{II}} \to V^+$ by restricting it to V.
- That is, $\sigma^\pm$ restricts the domain of σ^+ to V_{I}, and when the value is $\tilde{v} \in \tilde{D}$, change it to v. $\tau^\pm$ can be obtained similarly.



- First, consider the case $W_{\mathrm{I}}(G^+, \sigma^+) = V^+$.
- Take any play ρ consistent with $\sigma^\pm$ in G.
- If a vertex of D appears infinitely many times in ρ, then player I wins in ρ.
- Otherwise, from some vertex in ρ, its remaining play ρ' does not visit D, and since ρ' also obeys $\sigma^\pm$ in G, ρ' obeys σ^+ in G^+, which means that player I wins in G^+, and thus also wins with ρ' in G. Therefore, player I wins even with ρ in G, because any finite part of the play makes no difference to the parity condition.
- That is, $W_{\mathrm{I}}(G, \sigma^\pm) = V$.

- Next, consider the case $W_{\mathrm{I}}(G^+, \sigma^+) \neq V^+$.
- Then we have $v \in W_{\mathrm{II}}(G^+, \tau^+) = V^+ - W_{\mathrm{I}}(G^+, \sigma^+)$.
- Consider a play starting from v consistent with τ^+. If an absorbing vertex $\tilde{v} \in \tilde{D}$ appears in the middle, then after that, it just repeats $\tilde{v}$, and so priority 0 appears infinitely often, which means player I wins, which contradicts with $v \in W_{\mathrm{II}}(G^+, \tau^+)$.
- Therefore, in a play of G^+ from v consistent with τ^+, a vanishing vertex may appear only at the start, and no vertex in $D \cup \tilde{D}$ appear in the middle.
- Thus, any play of G starting from v and consistent with $\tau^\pm$ does not enter D in the middle, and so it is also consistent with τ^+, which means player II wins. That is, $v \in W_{\mathrm{II}}(G, \tau^\pm)$.
- Combining the above two cases, we can say at least $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G) \neq \varnothing$.

- Next we show $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G) = V$. By the way of contradiction, assume $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G) \neq V$.

- Let $V^- := V - (W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G))$ and consider the game G^- by restricting G to V^-.

- Note that for every $v \in V^-$ there is a $u \in V^-$ such that $(v, u) \in E$. Because if every u such that $(v, u) \in E$ belongs to $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G)$, so is v, which contradicts $v \in V^-$. Therefore, the game G^- is a correct parity game.

- In the following, we will show that $W_{\mathrm{I}}(G^-) \cup W_{\mathrm{II}}(G^-) = \varnothing$, which contradicts with the previous claim $W_{\mathrm{I}}(G) \cup W_{\mathrm{II}}(G) \neq \varnothing$, since the number of the relevant priorities of G^- is not larger than that number k of G.

- Let $v \in W_{\mathrm{I}}(G^-)$ and σ^- be a winning strategy for I starting from v in G^-.

- Now consider a play ρ starting at v consistent with σ^- in G.

- At $u \in V_{\mathrm{II}} \cap V^-$ in the middle of play, no vertex of $W_{\mathrm{II}}(G)$ will be chosen in the next move. Because if it were selected, we would have $u \in W_{\mathrm{II}}(G)$, which contradicts $u \in V^-$. Thus, ρ is always in G^-.

- Since ρ is also consistent with σ^- in G^-, player I wins, that is, $v \in W_{\mathrm{I}}(G)$.
- But since $V^- \cap W_{\mathrm{I}}(G) = \varnothing$, we must have $W_{\mathrm{I}}(G^-) = \varnothing$.
- Similarly, $W_{\mathrm{II}}(G^-) = \varnothing$. Hence, $W_{\mathrm{I}}(G^-) \cup W_{\mathrm{II}}(G^-) = \varnothing$.
- Since G^- is a parity game with at most k relevant priorities, $W_{\mathrm{I}}(G^-) \cup W_{\mathrm{II}}(G^-) \neq \varnothing$, which denies the assumption of $W_{\mathrm{I}}(G, \sigma) \cup W_{\mathrm{II}}(G, \tau) \neq V$. $\square$

Further readings

The above proof is based on S. Le Roux's paper:
"Memoryless determinacy of infinite parity games: Another simple proof", *Info. Proc. Letters* 143 (2019).
Le Roux's proof also relies on Haddad's paper: " Memoryless determinacy of finite parity games: another simple proof", *Info. Proc. Letters* 132 (2018) 19–21.
which in turn refers to many previous studies.

- In a parity game G over a finite graph, it can be checked in polynomial time whether a given memoryless strategy is a winning strategy. So $W_{\mathrm{I}}(G)$ is NP.
- Similarly $W_{\mathrm{II}}(G)$ is also NP and $W_{\mathrm{I}}(G) = V - W_{\mathrm{II}}(G)$, so $W_{\mathrm{I}}(G) \in$ NP$\cap$ co-NP.
- However, it is not yet known whether it will be in P, and currently it is $O(|G|^{\log n+6})$ (where n is the number of priorities), due to Calude-Jain-Khoussainov-Li-Stephan results (STOC 2017).

DECIDING PARITY GAMES IN QUASI-POLYNOMIAL TIME*

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Abstract. It is shown that the parity game can be solved in quasi-polynomial time. The parameterized parity game—with n nodes and m distinct values (a.k.a. colors or priorities)—is proven to be in the class of fixed parameter tractable problems when parameterized over m. Both results improve known bounds, from runtime $n^{O(\sqrt{n})}$ to $O(n^{\log(m)+6})$ and from an **XP** algorithm with runtime $O(n^{\Theta(m)})$ for fixed parameter m to a fixed parameter tractable algorithm with runtime $O(n^5 + 2^{m\log(m)+6m})$. As an application, it is proven that colored Muller games with n nodes and m colors can be decided in time $O((m^m \cdot n)^5)$; it is also shown that this bound cannot be improved to $2^{o(m\cdot\log(m))} \cdot n^{O(1)}$ in the case that the exponential time hypothesis is true. Further investigations deal with memoryless Muller games and multidimensional parity games.

Thank you for your attention!