Logic and Computation II

Part 6. Automata on infinite objects

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Logic and Computation II

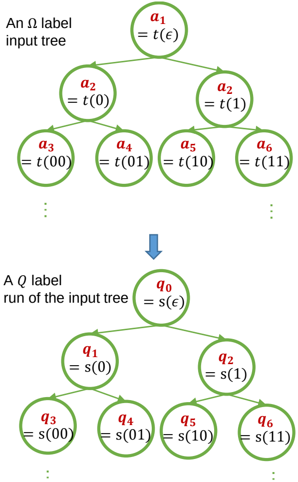
- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
- **Part 6. Automata on infinite objects**
- **Part 7. Recursion-theoretic hierarchies**

Part 6. Schedule (tentative)

- Apr.15, (1) Second-order arithmetic and analytical hierarchy
- Apr.17, (2) Büchi automata
- Apr.22, (3) Safra's theorem
- Apr.24, (4) The decidability of S1S
- May 6, (5) Tree automata
- May 8, (6) Tree automata and parity games
- May 13, (7) The decidability of S2S
- May 15, (8) Positional determinacy of parity games

Recap

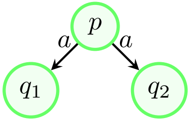
- An (Ω-)**labeled tree** is the complete binary tree $\{0,1\}^*$ with each vertex labeled by a symbol in Ω. It can be viewed as a function $t : \{0,1\}^* \to \Omega$.
- The **tree automaton** $M = (Q, \Omega, \delta, Q_0, Acc)$:
  - Q: a set of states,
  - $\delta \subseteq Q \times \Omega \times Q^2$: a transition relation,
  - $Q_0 \subseteq Q$: a set of initial states, and
  - Acc: an acceptance conditions.
- For an input Ω-labeled tree $t : \{0,1\}^* \to \Omega$, a run-tree of M is a Q-labeled tree $s : \{0,1\}^* \to Q$ such that
  - $s(\epsilon) \in Q_0$, where ϵ is empty and represents the root of the binary tree.
  - for any $u \in \{0,1\}^*$, $(s(u), t(u), s(u0), s(u1)) \in \delta$.
- To simplify the discussion, assume that for any input, a run-tree can be constructed. (Such an automaton is said to be **complete**).



- For a Q-labeled tree s and an infinite path $\alpha : \mathbb{N} \to \{0,1\}^*$, $s(\alpha)$ denotes the ω-sequence of states (labels) on a path α in s. $\mathsf{Inf}(s(\alpha))$ denotes the set of states which appears infinitely often on $s(\alpha)$.
- An input tree t is accepted by a tree automaton M ($t \in L(M)$) iff there is a run-tree s in which all its infinite paths $s(\alpha)$ satisfy the following condition.
  - For a **Büchi tree automaton** (BTA) M, the acceptance condition Acc is $F(\subseteq Q)$: $\mathsf{Inf}(s(\alpha)) \cap F \neq \varnothing$.
  - For a **Muller tree automaton** (MTA) M, Acc is $\mathcal{F}(\subseteq \mathcal{P}(Q))$: $\mathsf{Inf}(s(\alpha)) \in \mathcal{F}$.
  - For a **Rabin tree automata**(RTA) M, Acc is $\mathcal{F} = \{(G_i, R_i) \mid 1 \leq i \leq k\}$, where $G_i, R_i \subset Q$: there exists i satisfying $\mathsf{Inf}(s(\alpha)) \cap G_i \neq \varnothing$ and $\mathsf{Inf}(s(\alpha)) \cap R_i = \varnothing$.
  - For a **parity tree automaton** (PTA) M, Acc is a priority function $\pi : Q \to \{0, 1, \dots, k\}$: $\min\{\pi(q) : q \in \mathsf{Inf}(s(\alpha))\}$ is even.
- Even with nondeterminism, BTA has less expressive power than the other three. PTA $\to$ RTA $\to$ NMA is easy, and NMA $\to$ PTA was shown in the last lecture.

Express a PTA as an infinite game

- Given a PTA $M = (Q, \Omega, \delta, Q_0, \pi)$ and an input tree t, we construct an infinite game $G(M, t)$ in which two players alternately move as follows:
  (1) Player I (Automaton) chooses a next pair of states (q_1, q_2) from $\delta(p, a)$.
  (2) Player II (Path Finder) chooses either 0 or 1 for the next direction.



- The **goal of the Path Finder** is to find a path $\alpha \subseteq \{0,1\}^*$ in the run-tree s that does not satisfy the acceptance condition, whereas the **goal of the Automaton** is to find the Q labels of the run-tree so that the label sequence satisfies the acceptance conditions.
- Player I (automaton) wins in $G(M, t)$ if the label string $s(\alpha)$ produced by the two players satisfies the acceptance condition of M.
- Thus "M accepts $t \Leftrightarrow$ The automaton has a winning strategy in $G(M, t)$."
- Assume the determinacy of this game (either player has a winning strategy),
  "M does not accept $t \Leftrightarrow$ The path finder has a winning strategy in $G(M, t)$."
- For the moment, we also assume the following (which we will prove in next week).
  "The parity game is positionally determined."

Now we present the main lemma.

Lemma 6.21

For any PTA M, there is a PTA M' that accepts the complement of $L(M)$.

Proof.

- Let $M = (Q, \Omega, \delta, Q_0, \pi)$ be a PTA and L^c the complement of $L(M)$. First, we will define a parity game $G(M, t)$ such that
  "an input tree t belongs to $L^c \Leftrightarrow$ player II has a winning strategy."
- $G(M, t) = (V_{\mathsf{I}}, V_{\mathsf{II}}, E, \pi)$ is defined as follows:
  $V_1 = \{0,1\}^* \times Q, \quad V_2 = \{(d, (q, q_0, q_1)) \in \{0,1\}^* \times Q^3 : \delta(q, t(d), q_0, q_1)\}$,
  $E = \{(d, (q, q_0, q_1)), (d\hat{\ }i, q_i)) \in V_2 \times V_1 : i = 0, 1\} \cup \{((d, q), (d, (q, q_0, q_1))) \in V_1 \times V_2\}$.
- The game starts with I by choosing an element from $\{\epsilon\} \times Q_0$.
- The priority function of the game essentially follows π of PTA M, i.e., the priority for $(d, (q, q_0, q_1)) \in V_2$ and $(d, q) \in V_1$ are both $\pi(q)$. Then, the same $\pi(q)$ always appears twice consecutively, but it does not matter with the parity condition. Player I wins when the smallest priority appearing infinitely often is even.

- Let S_{II} be the set of total functions from Q^3 to $\{0,1\}$. Then, II's memoryless strategy can be viewed as $\sigma : \{0,1\}^* \to S_{\text{II}}$. Hence, it can also be viewed as a S_{II}-labelled tree.
- So, given an input tree t and a memoryless strategy σ for II (not necessarily a winning strategy), we have a $\Omega \times S_{\text{II}}$-labelled tree. Then, we treat an infinite path through this tree $(a_0, s_0)(a_1, s_1)(a_2, s_2)\cdots$ such that $a_i = t(d_0 d_1 \cdots d_{i-1}), s_i = \sigma(d_0 d_1 \cdots d_{i-1})$ (where $d_i \in \{0,1\}$, $i \geq n$) as an ω-word $\alpha = (a_0, s_0, d_0)(a_1, s_1, d_1)(a_2, s_2, d_2)\cdots$ on $\Omega' = \Omega \times S_{\text{II}} \times \{0,1\}$.
  Let $L(t,\sigma)$ denote the set of all such ω-words.
- We can define an NPA A which accepts an ω-word $\alpha = (a_0, s_0, d_0)(a_1, s_1, d_1)\cdots$ iff a sequence $q_0 q_1 q_2 \cdots$ can be chosen consistently with α to satisfy the parity condition. Actually, we set $A = (Q, \Omega', \delta', Q_0, \pi)$, where Q, Q_0, π are the same as the PTA M, and $\Omega' = \Omega \times S_{\text{II}} \times \{0,1\}$, and

$$\delta' = \{(q, (a, s, i), q_i) : \text{there exists } (q, a, q_0, q_1) \in \delta \text{ s.t. } s(q, q_0, q_1) = i\}.$$

  Note that this definition depends on Ω', but does not directly on II's strategy σ.

Claim 1

II's memoryless strategy σ is the winning strategy $\Leftrightarrow L(t,\sigma) \cap L(A) = \varnothing$.

($\Rightarrow$) By way of contradiction, let $\alpha \in L(t,\sigma) \cap L(A)$.

- For $\alpha \in L(a)$, there exists a run $q_0 q_1 q_2 \cdots$ of A on input α satisfying the parity condition.
- On the other hand, for II's strategy σ, if player I chooses $(q, a, q_0, q_1) \in \delta$ following δ', then they produce a play $q_0 q_1 q_2 \cdots$ in which I wins. So, σ is not a winning strategy for II.

($\Leftarrow$) By way of contradiction, suppose strategy σ is not a winning strategy for II.

- If player I chooses $(q, a, q_0, q_1) \in \delta$ appropriately, there exists $\alpha = (a_0, s_0, d_0)(a_1, s_1, d_1) \cdots$ such that its corresponding $q_0, q_1, q_2 \cdots$ satisfies the parity condition.
- Thus $\alpha \in L(t,\sigma) \cap L(A)$.

Now, since $L(A)$ is an ω-regular language, there exists a DPA $A' = (P, \Omega', \eta, q_0, \pi')$ that accepts the complement of $L(A)$ on Ω'.

- Then we construct a desired PTA M' from a DPA A'. That is, $M' = (P, \Omega, \eta', P_0, \pi')$,

$$\eta' = \{(p, a, p_0, p_1) : \exists s \in S_{\mathrm{II}}\ ((p, (a, s, 0), p_0) \in \eta \wedge (p, (a, s, 1), p_1) \in \eta)\}.$$

Claim 2

$t \in L(M') \Leftrightarrow t \notin L(M)$.

$(\Rightarrow)$ Suppose $t \in L(M')$, and fix an accepting run-tree r.

- For each node $d \in \{0,1\}^*$ in r, there exists $s_d \in S_{\mathrm{II}}$ satisfying η'. Then, we merge them to define a memoryless strategy $\sigma : \{0,1\}^* \to S_{\mathrm{II}}$.
- Next, for an ω-word α in $L(t, \sigma)$, consider a run of DPA A'. This is nothing but the sequence of labels of the tree r for the $\{0,1\}^\omega$ components of α, and so it satisfies the parity condition. Thus, $\alpha \in L(A')$, which means $\alpha \notin L(A)$.
- Hence, $L(t, \sigma) \cap L(A) = \varnothing$. By Claim 1, σ is a memoryless winning strategy for II in $G(M, t)$. Therefore, $t \notin L(M)$.

- ($\Leftarrow$) Suppose $t \notin L(M)$.
- Then player II has a memoryless winning strategy σ in $G(M,t)$, which can be viewed as a S_{II}-labeled tree. From claim 1, $L(t,\sigma) \cap L(A) = \varnothing$, so $L(t,\sigma) \subset L(A')$.
- A P-labeled sequence of DPA A' for a finite subsequence of ω-word α in $L(t,\sigma)$ is uniquely determined. Based on them, there exists a P-labeled tree r which is a run-tree of M' for t.
- Since each P-labeled path of the tree r satisfies the parity condition, r satisfies the acceptance condition of M' and so M' accepts the input tree t. □

The outline of the proof is shown in the following diagram.

PTA M does not accept t. $\Longleftrightarrow$ II has a winning strategy σ for t in the game $G(M, t)$.

$\Updownarrow$

All the paths through the $\Omega \times S_{\mathrm{II}}$-labeled tree $T^{t,\sigma}$ negates the parity condition.

$\Updownarrow$

The ω-language $L(t, \sigma)$ on $\Omega' = \Omega \times S_{\mathrm{II}} \times \{0,1\}$ consists of ω-words negating the parity condition.

$\Updownarrow$

PTA M' accepts t. $\Longleftrightarrow$ $L(t, \sigma) \cap L(A) = \emptyset$.

Let A be an NPA which accepts all ω-words on Ω' satisfying the parity cond.

Let A' be a DPA which accepts the complement of $L(A)$. Let M' be a PTA constructed form A'.

Using a parity game similar to $G(M,t)$ above, it is easy to show the following.

Lemma 6.22 (PTA emptiness problem)

It is decidable whether the accepted language of PTA is empty or not.

Proof. Given PTA $M=(Q,\Omega,\delta,Q_0,\pi)$, consider the following parity game $G(M)=(V_1,V_2,E,Q_0,\pi')$.

- $V_1=Q, \quad V_2=\delta,$
- $E=\{(q,(q,a,q_0,q_1))\in V_1\times V_2\}\cup\{((q,a,q_0,q_1),q_i)\in V_2\times V_1 : i=0,1\},$
- $\pi'(q)=\pi(q), \quad \pi'((q,a,q_0,q_1))=\pi(q).$

This is like removing the position information $d\in\{0,1\}^*$ from the above $G(M,t)$. Therefore,

Player I has a winning strategy in $G(M)$ starting from a state in $Q_0 \Leftrightarrow L(M)\neq\varnothing$

And if player I has a winning strategy in $G(M)$, he has a memoryless winning strategy. Since V_1,V_2 are finite sets, it is decidable in finite steps that player I has a winning strategy. □

S2S and MTA

- Now we will show the equivalence of S2S and MTA.
- First, to translate an S2S formula $\varphi(\vec{x}, \vec{X})$ into a tree language, we need something like the characteristic sequence we defined to translate S1S.
- For simplicity, we replace the first-order variable x with second-order variable X representing the singleton set, and consider the translation of the formula $\varphi(\vec{X})$ with no free occurrences of first-order variables.
- Let $\vec{T} = (T_1, \ldots, T_n)$ be an n-tuple of subsets of $\{0,1\}^*$. Letting $\Omega = \{0,1\}^n$, we express $\vec{T}$ by an Ω-labeled tree $t : \{0,1\}^* \to \{0,1\}^n$ such that for each $i = 1, \ldots, n$,

$$T_i = \{d \in \{0,1\}^* : i\text{-th element of } t(d) \text{ is } 1\}$$

Then, such a t is called the **characteristic representation tree** (representation tree, in short) of $\vec{T}$.

Lemma 6.23

Given an S2S formula $\varphi(\vec{X})$, there exists an MTA M_φ on $\Omega = \{0,1\}^n$ such that,

$$L(M_\varphi) = \{\text{The representation tree of } \vec{T} : \varphi(\vec{T}) \text{ holds}\}.$$

Proof. The atomic formula of S2S has a form

$$S_{b_1} S_{b_2} \dots S_{b_k} x \in X \ (\text{where } b_i = 0, 1).$$

Then (d, T) satisfies the above relation iff the word $db_k \dots b_2 b_1$ belongs to T. So, it is easy to construct a PTA M that accepts the set of the representation trees of such (d, T)'s. Furthermore, since the class of languages accepted by MTA's is closed under Boolean operations and projections, any S2S formula has an equivalent MTA. □

Conversely, let $\{P_a : a \in \Omega\}$ $(P_a = t^{-1}(a))$ be the partition of $\{0,1\}^*$ determined by the Ω- labeled tree t. If an S2S formula φ holds in the structure

$$(\{0,1\}^* \cup \mathcal{P}(\{0,1\}^*), S_0(x), S_1(x), \in, P_a)_{a\in\Omega},$$

φ is said to holds in t. Then,

Lemma 6.24

Given an MTA M on Ω, there exists an S2S formula φ_M containing $P_a (a \in \Omega)$ as a set constant such that

$$t \in L(M) \Leftrightarrow \varphi_M \text{ holds in } t.$$

Proof. The idea of constructing the S2S formula φ_M from MTA M is almost the same as the proof of the lemma for S1S. First, the basic predicates of S1S can be used in S2S. For example, "$x = y$", "$X \subseteq Y$", "$X = Y$" etc. can be used. In addition, we define

- "$x = \epsilon$" : $\neg\exists y(S_0 y = x \vee S_1 y = x)$.
- "$\mathrm{Path}(X)$" : $\exists x \in X(x = \epsilon) \wedge \forall x \in X(x \neq \epsilon \to \exists y \in X(S_0 y = x \vee S_1 y = s))\wedge$ $\forall x \in X \exists! y(S_0 x = y \vee S_1 x = y)$.

Now, let $M = (Q, \Omega, \delta, Q_0, \mathcal{F})$ be a *complete* (no dead ends in state transitions) MTA . Then, if the input tree is represented by $\{P_a : a \in \Omega\}$, the run-tree $\vec{Y} = \{Y_q\}$ (Y_q is the set of vertices with label q) is expressed as follows.

$$
\begin{aligned}
\mathrm{run}(\vec{Y}) \quad = \quad & \bigvee_{q \in Q_0} \epsilon \in Y_q \\
& \wedge \forall x \bigvee_{(q,a,q_0,q_1) \in \delta} (x \in Y_q \wedge P_a(x) \wedge x0 \in Y_{q_0} \wedge x1 \in Y_{q_1}) \\
& \wedge \forall x \bigwedge_{p \neq q} \neg(x \in Y_p \wedge x \in Y_q)
\end{aligned}
$$

Furthermore, the Muller acceptance condition is expressed as

$$
\begin{aligned}
\varphi_M \quad = \quad & \exists \vec{Y} (\mathrm{run}(\vec{Y}) \\
& \wedge \forall X (\mathrm{Path}(X) \to \bigvee_{F \in \mathcal{F}} (\bigwedge_{q \in F} Y_q \cap X \text{ is infinite} \wedge \bigwedge_{q \notin F} Y_q \cap X \text{ is finite})
\end{aligned}
$$

Obviously, this satisfies the lemma. □

Corollary 6.25

S2S is decidable.

Proof. Let σ be an S2S sentence. Its truth can be determined by checking whether or not the emptiness problem of the MTA language equivalent to $\sigma \wedge (X = X)$. This problem is decidable by the lemma in Page 12 of this slides. □

Homework

Let $\Omega = \{a, b\}$.
(1) Construct a PTA M_1 that accepts Ω-labeled trees in which a appears finitely.
(2) Construct a a PTA M_2 that accepts Ω-labeled trees in which a appears infinitely many times only in one path.

Homework

By SωS, we denote the monadic second-order theory of $\mathcal{T}_\omega = (\mathbb{N}^*, \{S_i(x)\}_{i\in\mathbb{N}}, \subset, \preccurlyeq)$, where $S_i(w) = w\hat{\ }i$ $(i \in \mathbb{N})$, $\subset$ is the prefix and $\preccurlyeq$ is the lexicographic order.
Now let $f : \mathbb{N}^* \to \{0,1\}^*$ be

$$f(n_0 n_1 \dots n_{k-1}) = 0^{n_0}10^{n_1}1 \cdots 10^{n_{k-1}}1, \quad \text{and } f(\epsilon) = \epsilon.$$

Letting D be the range of f, we have $\mathcal{D} = (D, \{S_i^D(x)\}_{i\in\mathbb{N}}, \subset^D, \preccurlyeq^D) \cong \mathcal{T}_\omega$.
Then show that $\mathcal{D}$ is S2S-definable (Note: $\subset$ and $\preccurlyeq$ cannot be defined in $(\mathbb{N}^*, \{S_i(x)\}_{i\in\mathbb{N}})$). From this, derive that SωS is decidable.

Thank you for your attention!