Logic and Computation II

Part 6. Automata on infinite objects

Kazuyuki Tanaka

BIMSA

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北京雁栖湖
应用数学研究院
Beijing Institute
of Mathematical
Sciences and Applications

Logic and Computation II

- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
- **Part 6. Automata on infinite objects**
- **Part 7. Recursion-theoretic hierarchies**

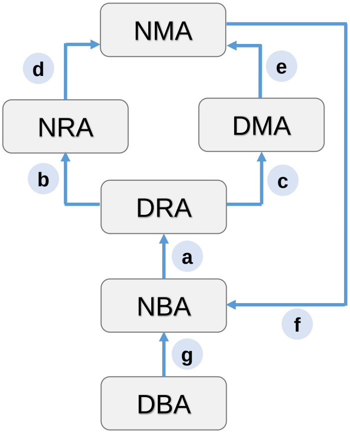
Part 6. Schedule (tentative)

- Apr.15, (1) Second-order arithmetic and analytical hierarchy
- Apr.17, (2) Büchi automata
- Apr.22, (3) Safra's theorem
- Apr.24, (4) The decidability of S1S
- May 6, (5) Tree automata
- May 8, (6) The decidability of S2S
- May 13, (7) Finite model theory
- May 15, (8) Parity games

Recap

- For an infinite run σ, the set of states that appear infinitely in σ is denoted by $\mathrm{Inf}(\sigma)$.
- The ω-language $L(M) \subset \Omega^\omega$ accepted by a Büchi automaton M is defined as
$$L(M) = \{\alpha \in \Omega^\omega \mid \text{there is a run } \sigma \text{ of } M \text{ on } \alpha \text{ such that } \mathrm{Inf}(\sigma) \cap F \neq \varnothing\}.$$
- The ω-language $L(M)$ accepted by a Muller automaton M is defined as
$$L(M) = \{\alpha \in \Omega^\omega \mid \text{there is a run } \sigma \text{ of } M \text{ on } \alpha \text{ such that } \mathrm{Inf}(\sigma) \in \mathcal{F}\}.$$
The Büchi condition $\mathrm{Inf}(\sigma) \cap F \neq \varnothing$ can be expressed in terms of the Muller condition
$$\mathcal{F} = \{A \subseteq Q \mid A \cap F \neq \varnothing\}.$$
- The ω-language $L(M)$ accepted by a Rabin automaton M is defined as
$$L(M) = \{\alpha \in \Omega^\omega \mid \exists\, \sigma \text{ on } \alpha \text{ s.t. for some } i, \mathrm{Inf}(\sigma) \cap G_i \neq \varnothing, \mathrm{Inf}(\sigma) \cap R_i = \varnothing\}.$$
The Rabin condition can be expressed by the Muller condition:
$$\mathcal{F} = \{A \subseteq Q \mid \bigvee_i (A \cap G_i \neq \varnothing \wedge A \cap R_i = \varnothing)\}$$

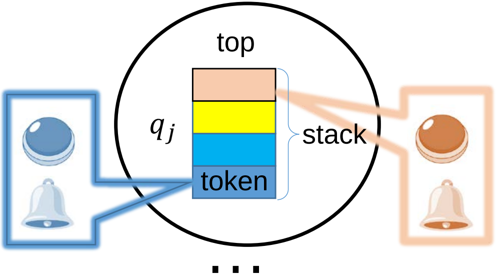
- (b), (e) and (g) are obvious. (c) and (d) have been explained above. Now, we are going to show (f).
  - Let M be an NMA with an accepting set $\mathcal{F}$. Goal: construct an NBA N to simulate M.
  - For input x, N mimics M by nondeterministically guessing a run σ of M on x.
  - At some point, N nondeterministically predicts that all states of M not in $\mathrm{Inf}(\sigma)$ have appeared and also guesses that $\mathrm{Inf}(\sigma)$ is a certain set $A \in \mathcal{F}$.
  - Then check if A is indeed $\mathrm{Inf}(\sigma)$ as follows:
    - Any state of σ (from that point) is in A, and
    - Let s be the state of N representing that every state of A appeared at least once. Then N accepts the input if s appears inf. many times.
- (a):NBA$\to$ DRA is the most difficult to prove.



In the figure, "XXA $\to$ YYA" means "for any XXA M_1, there exists a YYA M_2 such that $L(M_1) = L(M_2)$".

Definition of Stacks

- A **stack** is a pile of colored **tokens**, which is placed on **spots** of a **board**, namely, nodes of the diagram of NBA B. A stack moves from one spot to another along the edges, sometimes changes its contents, and sometimes gets removed.
- A board with some stacks on some spots is a **state** of DRA R.
- The board is connected to different **bells** and **buzzers** for each color.
- The **height** of the stack σ is written as $|\sigma|$.



Buzzer and bell for each color

- The colors in play (on the board) are ordered by their **age**, namely the time they appeared in play. Tokens of the same color in play come into play at the same time and are all of the same age. When adding new tokens, use colors that are not currently in play (reusable) and put them on top of the stack.
- At time t, the stacks are linearly ordered by the reverse lexicographic order $\sigma \ll_t \tau$.
  - σ is a proper extension of τ (τ is obtained by removing the top of σ), or
  - Neither σ nor τ is an extension of the other, and at the lowest position where σ and τ differ in color, the color of σ is older.
- The simulation starts with a board with one white token at each initial state spot of B.
- At each time, the three steps (Move, Cover, Remove) are all executed in this order.
- It should be noted that to make a DRA R deterministic, we must determine the order of all construction steps. However, since a detailed description would make the whole construction less visible, I leave the details to the reader.

Move

- An input symbol a is given. For each $q \in Q$, (a copy of) the stack at spot q is moved to each $p \in \delta(q, a)$.
- If there are multiple stacks to put in p, put the smallest stack with respect to $\ll_t$.
- If a color disappears in this process, sound the buzzer of that color.

Cover

- For each accepting state $q \in F$, put a token of a new color on the top of a stack at spot q so that stacks with the same visible color are covered with tokens of the same new color, and two with different colors are covered with different new colors.
- New colors enter only in this process. Thus, if color c is placed directly above color d in a stack, then all tokens of color c in play are placed directly above tokens of color d.

Remove

- For any invisible color c in play, remove all tokens above tokens of color c, sound the buzzer of the removed color, and ring the bell of the visible color c.
- Note that when a token is removed in this process, all tokens of that color are removed. The order of removal is not important.

After performing these three steps, there are at most n ($=$ the number of the states) colors left in play. Otherwise, there must be at least one invisible color, then repeat the remove step.

Lemma 6.13

The following are equivalent

(1) An NBA B accepts an input x.

(2) In the DRA R thus constructed, there is a color that rings the bell infinitely many times but sounds the buzzer only finitely many times.

Proof.

To show $(2) \Rightarrow (1)$

- Suppose that there exists a color, say yellow, that rings the bell infinitely many times but the buzzer a finitely many times.
- Let $t_0, t_1, \ldots$ be the times when the yellow bell rings after the last buzzer.
- From time t_0, yellow continues to be in play. Otherwise the buzzer will sound.
- To get a yellow bell at each time t_i, all yellow tokens must be invisible just before and some yellow tokens become visible. In other words, no matter how to move a stack with a yellow token on top from t_i to t_{i+1}, it visits some spot of F.
- Therefore, there exists a run where the state of F appears infinitely.

Proof.

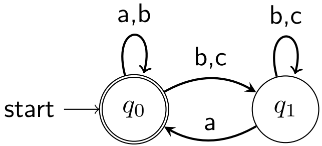
To show $(1) \Rightarrow (2)$

- Conversely, suppose that there is an accepted run ρ of B for x.
- Let σ_t be the stack following ρ at time t.
- Then, set $m = \liminf |\sigma_t|$. In other words, after a certain time t_0, the minimum stack height is m, and it reaches the height m infinitely many times. White (the oldest color) is always in play, so $m \geq 1$, and there are at most n colors in play, so $m \leq n$.
- After time t_0, the color tokens at height $\leq m$ may be replaced by $\ll_t$-smaller ones, which however happens only finite times by the definition of $\ll_t$.
- So, from a certain time t_1, the colors in the stack below m can be assumed to remain unchanged. We assume that the color at the height m is black.
- Since this sequence of actions is an accepted run, a state of F is visited infinitely many times. Although the stack gets a new token each time, eventually the stack height reduces to m again, which rings a black bell.
- Therefore, the black bell rings infinite times and the buzzer sounds only finite times. $\square$

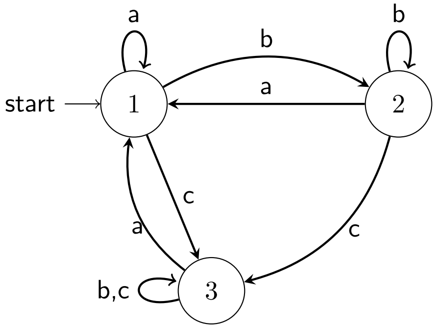
Example 2

Consider $L = \{\{b, c\}^* a \cup b\}^\omega$.

An NBA accepting L



A DRA accepting L



A state of the Rabin automaton R consists of the states of B and the stacks (which may be empty). The number of combinations of stacks is roughly n^n. The treatment of bells and buzzers and auxiliary machineries needs n^n at most. So, the states of R roughly $n^{kn} = 2^{O(n \log n)}$. The acceptance condition consists of n pairs, one for each color.

Therefore, we have

Theorem 6.14 (Safra)

Any NBA with n states can be simulated with a DRA consisting of $2^{O(n \log n)}$ states and n pairs of acceptance conditions. Therefore, it can also be simulated with a DMA with the same number of states.

Corollary 6.15

The class of ω-regular languages is closed with Boolean operations.

Proof.

- We already known that the class of languages accepted by NBA is closed with $\cup$ and $\cap$.
- The closure of complement follows from the above theorem, classes of languages accepted by NBA and DMA are the same.
- In fact, a DMA that accepts the complement of the language of a DMA $M = (Q, \Omega, \delta, q_0, \mathcal{F})$ by replacing the acceptance condition $\mathcal{F}$ of M with $\mathcal{P}(Q) - \mathcal{F}$.

□

Homework

Prove that $L = \{u^\omega : u \in \{0,1\}^+\}$ is not an ω-regular language.

Decidability of S1S

- We showed in Lecture-03-06 that the FO theory of $(\mathbb{N}, +, 0)$ is decidable, but the MSO theory of $(\mathbb{N}, +, 0)$ is undecidable since multiplication is definable there.
- Today, we will show the MSO theory of $(\mathbb{N}, x+1, 0)$, called S1S[i], is decidable, by reducing its decision problem to the emptiness problem of NBA.
- In the following, S1S is treated in the language with $x+1, 0$, a relation symbol ϵ, numerical variables $x, y, \ldots$ and set variables $X, Y, \ldots$.
- We consider that an S1S formula holds iff it is true in the standard structure with the ordinary mathematical sense.
- Recall that pure MSO logic (with the standard structure) is not axiomatizable. Thus, the decidability of S1S also implies its axiomatizability.

[i] The first "S" stands for "second-order", and the next "1S" stands for "One Successor". But note that this is a monadic second-order theory. Indeed, the SO theory of $(\mathbb{N}, +, 0)$ is undecidable.

In S1S, the equality symbol $=$, the inequality symbol $\leq$, and the constant 0 are defined as follows, and they have their usual meanings.

- "$x = y$" : $\forall X(x \in X \leftrightarrow y \in X)$.
- "$X \subseteq Y$" : $\forall x(x \in X \rightarrow x \in Y)$.
- "$X = Y$" : $X \subseteq Y \wedge Y \subseteq X$.
- "$x = 0$" : $\forall y \neg(x = y + 1)$; x has no predecessor.
- "$x = 1$" : $x = 0 + 1$. Since 0 is defined above, 0 is treated like a given symbol. In terms of the original symbols, we can write $\exists y(x = y + 1 \wedge \forall z \neg(y = z + 1))$.
- "$x \leq y$" : $\forall X(x \in X \wedge \forall z(z \in X \rightarrow z + 1 \in X)) \rightarrow y \in X$. That is, any set X that contains x and is closed under successor also contains y.
- "X is finite" : $\exists x \forall y(y \in X \rightarrow y \leq x)$.

Homework

(1) Express the following predicates with S1S formulas.

(a) X is the set of even numbers.

(b) X is finite with even number of elements.

(2) Explain why "X and Y have the same cardinality" cannot be expressed by an S1S formula.

For a set $A \subseteq \mathbb{N}$, the infinite sequence $\alpha \in \{0,1\}^{\mathbb{N}}$ such that $\alpha(i) = 1 \Leftrightarrow i \in A$ is called the **characteristic function** of A.

In the following, a number $a \in \mathbb{N}$ is identified with the singleton set $\{a\}$. Then the characteristic function of a tuple $(a_1, \ldots, a_n, A_1, \ldots, A_m) \in \mathbb{N}^n \times (\mathcal{P}(\mathbb{N}))^m$ can be expressed as an infinite sequence (called a characteristic sequence) over the alphabet $\Omega = \{0,1\}^{m+n}$. This sequence is divided into $m+n$ tracks, where each track is the characteristic function of a_i or A_i.

Example 7 The characteristic sequence of $(3, 5, \{\text{even numbers}\}, \{\text{prime numbers}\})$ is described as follows.

0	0	0	1	0	0	0	0	0	0	0	0	...	(⇐ 3)
0	0	0	0	0	1	0	0	0	0	0	0	...	(⇐ 5)
1	0	1	0	1	0	1	0	1	0	1	0	...	(⇐ even numbers)
0	0	1	1	0	1	0	1	0	0	0	1	...	(⇐ prime numbers)

$\in (\{0,1\}^4)^{\mathbb{N}}$

The following theorem asserts that S1S and NBA have equivalent expressive power. In the proof, the equivalence of NBA, NMA, and DMA is used freely .

Theorem 6.16 (The equivalence of S1S and NBA)

The following holds.

(1) Let $\varphi(\vec{x}, \vec{X})$ be an S1S formula with free numerical variables $\vec{x} = (x_1, \ldots, x_m)$ and free set variables $\vec{X} = (X_1, \ldots, X_n)$. Then there exists an equivalent NBA M_φ on $\Omega = \{0,1\}^{m+n}$ such that

$$L(M_\varphi) = \{\text{the characteristic sequence of } (\vec{a}, \vec{A}) \ : \ \varphi(\vec{a}, \vec{A}) \text{ is true}\},$$

where $\vec{a} = (a_1, \ldots, a_m), \vec{A} = (A_1, \ldots, A_n)$.

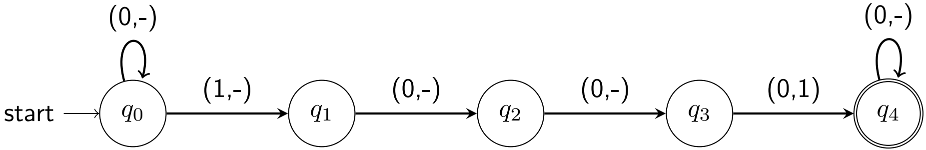
(2) Let M be an NBA on $\Omega = \{0,1\}$. There is an S1S formula $\varphi_M(X)$ such that

$$L(M) = \{\text{the characteristic sequence of } A : \varphi_M(A) \text{ is true}\}.$$

Proof. We show (1) by induction on the construction of the formula φ.
① The atomic formula is in the form of

$$\overbrace{\mathrm{SS}\cdots\mathrm{S}}^{k}x \in X.$$

To accept a characteristic sequence of (a, A), an NBA checks that in the track for x a unique 1 occurs in the a-th position, and in the track for X a 1 is in the $a + k$-th position. For example, the figure below is an NBA (indeed, a DBA) for $k = 3$.



The edge label (b, c) represents the input $\begin{bmatrix} b \\ c \end{bmatrix}$, where b and c are 0 or 1 on the track for x and X, respectively. Also, - indicates either of 0 and 1.

② Next, consider a formula of the form $\varphi_1(\vec{x}, \vec{X}) \wedge \varphi_2(\vec{x}, \vec{X})$. We may assume that there is a DMA $M_i = (Q_i, \Sigma, \delta_i, (q_0)_i, \mathcal{F}_i)$ for each φ_i.
DMA M_3 for $\varphi_1(\vec{x}, \vec{X}) \wedge \varphi_2(\vec{x}, \vec{X}) = (Q_3, \Sigma, \delta_3, (q_0)_3, \mathcal{F}_3)$ is constructed as follows.

$$\begin{aligned} Q_3 &= Q_1 \times Q_2 \\ \delta_3((q_1, q_2), a) &= (\delta_1(q_1, a), \delta_2(q_2, a)) \\ (q_0)_3 &= ((q_0)_1, (q_0)_2) \\ \mathcal{F}_3 &= \{A \subseteq Q_3 \mid \pi_1(A) \in \mathcal{F}_1 \text{ and } \pi_2(A) \in \mathcal{F}_2\} \end{aligned}$$

where π_1 and π_2 are the projections from $Q_1 \times Q_2$ to Q_1 and Q_2, respectively.

③ If $M = (Q, \Sigma, \delta, q_0, \mathcal{F})$ is a DMA for $\varphi(\vec{x}, \vec{X})$, then a DMA for $\neg\varphi$ can be constructed by taking the acceptance condition as $\mathcal{P}(Q) - \mathcal{F}$.

④ Automata for $\vee, \rightarrow$ cases are constructed similarly.

⑤ For $\exists X_1 \varphi(\vec{x}, X_1, \ldots, X_m)$, suppose a DMA M_φ for $\varphi(\vec{x}, X_1, \ldots, X_m)$.
A NMA M of $\exists X_1 \varphi$ takes $a_1, \ldots, a_n, A_2, \ldots, A_m$ together with a nondeterministic guess A_1 as input and mimic M_φ on $a_1, \cdots, a_n, A_1, A_2, \ldots, A_m$.

⑥ An automaton for $\forall X \varphi$ can be constructed as $\neg \exists X \neg \varphi$.

Thus, we can construct an NBA M_φ that accepts the set of characteristic sequences of $\vec{a}, \vec{A}$ satisfying φ.

(Note: The NBA's above may use some working tracks in addition to the input trucks. Especially when φ is a sentence, it is appropriate to arrange for a meaningless track. See the proof of decidability of S1S below.)

We show (2). Let $M = (Q, \{0,1\}, \delta, q_0, \mathcal{F})$ be a DMA. Let X be a set variable for the input binary sequence, and Y_q be the set of times when q is visited. A run $\vec{Y} = \{Y_q\}$ on input X is defined as follows.

$$\begin{aligned}\mathrm{run}(X, \vec{Y}) \quad = \quad & 0 \in Y_{q_0} \\ & \wedge \forall n \bigwedge_q (n \in Y_q \wedge n \notin X \rightarrow \mathrm{S}(n) \in Y_{\delta(q,0)}) \\ & \wedge \forall n \bigwedge_q (n \in Y_q \wedge n \in X \rightarrow \mathrm{S}(n) \in Y_{\delta(q,1)}) \\ & \wedge \forall n \bigwedge_{p \neq q} \neg(n \in Y_p \wedge n \in Y_q)\end{aligned}$$

Furthermore, "a run $\vec{Y}$ is accepted" can be defined:

$$\mathrm{accept}(\vec{Y}) = \bigvee_{F \in \mathcal{F}} \Big(\bigwedge_{q \in F} Y_q \text{ is infinite} \wedge \bigwedge_{q \notin F} Y_q \text{ is finite}\Big)$$

Finally, the desired formula is

$$\varphi_M(X) = \exists \vec{Y} (\mathrm{run}(X, \vec{Y}) \wedge \mathrm{accept}(\vec{Y}))$$

Corollary 6.17

S1S is decidable.

Proof.
Let σ a S1S sentence. Its truth can be determined by the emptiness of an NBA that is equivalent to $\sigma \wedge (X = X)$, which is decidable by the above theorem. $\square$

Note.

- We have treated Pressburger arithmetic on the natural numbers as a regular language. Similarly, we can treat addition of real numbers as infinite decimals in ω-languages.
- A real number may have two distinct decimal notations, but the equality $=$ between them can be recognized by an NBA

Further readings

- Infinite Words. Automata, Semigroups, Logic and Games. Dominique Perrin and Jean-Éric Pin. Pure and Applied Mathematics Vol 141. Elsevier, 2004.
- Automata, Logics, and Infinite Games: A Guide to Current Research. Editors: Erich Grädel, Wolfgang Thomas, Thomas Wilke. Lecture Notes in Computer Science (LNCS, volume 2500), Springer Berlin, Heidelberg, 2002.

Thank you for your attention!