Logic and Computation II
Part 5. Modal μ-calculus

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April 8, 2025

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Logic and Computation II

- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
- **Part 6. Automata on infinite objects**
- **Part 7. Recursion-theoretic hierarchies**

Part 5. Schedule (tentative)

- March 27, (1) Introduction to modal μ-calculus and monadic second-order logic
- April 1, (2) Basics of modal μ-calculus
- April 3, (3) The adequacy theorem
- April 8, (4) CTL

Recap: The Evaluation Game

A Kripke model M is fixed. For a state s and a formula φ, we consider the **evaluation game** $\mathcal{E}(M, s, \varphi)$. In this game, two players $\exists$ (female) and $\forall$ (male) decompose the formula while moving a token from s over the frame (much like a board game).

We first consider multi-modal logic with a model $M = (W, (R_i)_{i \in I}, v)$. A **position** in the game $\mathcal{E}(M, s_0, \varphi)$ is a pair (s, ψ) where $s \in W$ and ψ is a subformula of φ. The rules are as follows: (s_0, φ) is the initial position.

- At position $(s, \psi \vee \psi')$, it is $\exists$'s turn to choose either (s, ψ) or (s, ψ').
- At position $(s, \psi \wedge \psi')$, $\forall$ chooses a next position (s, ψ) or (s, ψ').
- At position $(s, \Box_i \psi)$, $\forall$ chooses a next position (t, ψ) such that there is an i-labeled edge from s to t. (If no such t exists, $\forall$ loses.)
- At $(s, \Diamond_i \psi)$, $\exists$ chooses such a position (t, ψ). (If no such t exists, $\exists$ loses.)

When a terminal position (s, p) or $(s, \neg p)$ is reached, $\exists$ **wins** if $s \in v(p)$ or $s \notin v(p)$, resp.

Adequacy Theorem

$\exists$ has a winning strategy in $\mathcal{E}(M, s, \varphi) \iff M, s \models \varphi$

Extending the game to modal μ-calculus

For simplicity, we assume that any two fixpoint variables appearing in a formula are distinct. We add the following rules for new positions:

- A position $(s, \eta X.\theta)$ automatically changes to (s, θ).
- Later, when the play reaches (t, X), the game goes to $(t, \eta X.\theta)$.

This means that the game does not need to terminate in a finite way. If the play terminates in a finite number of moves, the winning condition is the same as in the modal logic case.

If the play is infinite, then the winner is decided by the form of the *outermost* (largest) fixed point formula appearing infinitely often:

- If the outmost fixpoint is of the form $\mu X.\varphi$, then $\forall$ wins.
- If it is of the form $\nu X.\varphi$, then $\exists$ wins.

To prove the adequacy theorem, it suffices to show:

1. If $M, s \models \varphi$, then $\exists$ has a (*memoryless*) winning strategy in the game $\mathcal{E}(M, s, \varphi)$.
2. If $M, s \not\models \varphi$, then $\forall$ has a (*memoryless*) winning strategy in the game $\mathcal{E}(M, s, \varphi)$.

In fact, $\exists$'s strategy always chooses positions (t, ψ) s.t. $M, t \models_{V'} \psi$, and similarly for $\forall$, where V' is a certain extension of V, which is obtained while fixpoint operators are unfolded during the game.

Constructing temporary valuations

We construct a temporary valuation V'.
List up the subformulas of φ with fixpoint operators in prefix and order them by size as:

$$\eta_1 X_1.\psi_1,\ \eta_2 X_2.\psi_2,\ \ldots,\ \eta_n X_n.\psi_n,$$

where η_i is μ or ν. So, if $\eta_i X_i.\psi_i$ is a subformula of $\eta_j X_j.\psi_j$, then $j \leq i$. Hence, X_j may appear free in $\eta_i X_i.\psi_i$ for $j < i$.
Now we set

$$V_0 := V, \quad V_{i+1} := V_i \cup \left\{X_{i+1} \mapsto ||\eta_{i+1} X_{i+1}.\psi_{i+1}||^M_{V_i}\right\}.$$

Then the truth values of the subformulas (with free variables) of φ are all determined by $V' = V_n$, and so a strategy can be defined by choosing a position (t, ψ) s.t. $M, t \models_{V'} \psi$. However, this doesn't guarantee a winning strategy, especially when the game proceeds infinitely. For such cases, we need to ensure that the largest subformula appearing infinitely often is headed by a ν-operator.

Ordinal-assisted strategy

From the list of fixpoint subformulas of φ, we extract the subformulas headed by μ:

$$\mu Y_1.\theta_1,\ \mu Y_2.\theta_2,\ \ldots,\ \mu Y_m.\theta_m$$

For an ordinal sequence $r = (\alpha_1, \ldots, \alpha_m)$, we define V_n^r as follows:

$$V_0^r := V, \quad V_{i+1}^r := V_i^r \cup \left\{ X_{i+1} \mapsto \begin{cases} ||\eta_{i+1} X_{i+1}.\psi_{i+1}||^M_{V_i^r} & \text{if } \eta_{i+1} = \nu \\ ||\mu^{\alpha_j} Y_j.\theta_j||^M_{V_i^r} & \text{if } \eta_{i+1} X_{i+1} = \mu Y_j \end{cases} \right\}$$

where $||\mu^\alpha X.\theta(X)||^M_V := \Psi^\alpha$ for $\Psi(S) = ||\theta(X)||^M_{V(X):=S}$.

Lemma. Let ψ is a subformula of φ. If $M, t \models_{V_n} \psi$, then there exists a minimal ordinal sequence r (in lexicographic order) such that $M, t \models_{V_n^r} \psi$. Denote such an r by $r^\mu(t, \psi)$.

Note. Let $\bar{r}$ be $(\bar{\alpha}_1, \ldots, \bar{\alpha}_m)$ where $\bar{\alpha}_j$ is the least closure ordinal of $\mu Y_j.\theta_j$, that is, $V_n^{\bar{r}}(Y_j) = ||\mu^{\bar{\alpha}_j} Y_j.\theta_j||^M_{V^{\bar{r}}i}$. Then, $r^\mu(t, \psi) < \bar{r}$ for all t, ψ.

Key properties of r^μ

For a valid move over true positions in the game,

- If ψ is headed by ν, then r^μ remains unchanged.
- If ψ is constructed by Boolean or modal operators, r^μ stays the same or decreases.
- If $\psi \equiv \mu Y.\theta(Y)$, then r^μ **strictly decreases**. More strictly, after a position $(t, \mu Y.\theta(Y))$, a move from (t, Y) to $(t, \theta(Y))$ decreases r^μ, because for $t \in ||\mu^\alpha Y.\theta(Y)||^M_{V^r_n}$, there exists some $\beta < \alpha$ such that $t \in ||\theta(\mu^\beta Y.\theta(Y))||^M_{V^r_n}$.

Winning strategy for $\exists$: Always choose (t, ψ) such that:

$$M, t \models_{V_n^{r^\mu(t,\psi)}} \psi.$$

Similarly, if $M, s \not\models_V \varphi$, define $r^\nu(t, \psi)$, and $\forall$'s strategy is to choose (t, ψ) such that:

$$M, t \not\models_{V_n^{r^\nu(t,\psi)}} \psi.$$

Thus, the adequacy theorem is proved.

Miscellaneous

Tips. ν for infinity (always) μ for finiteness (eventually)

- $\nu X.p \wedge \Box_a X$ $\cdots\cdots\cdots$ p always holds along every a-path.
- $\nu X.p \wedge \Box_a \Box_a X$ $\cdots\cdots\cdots$ p holds at every even position along every a-path.
- $\nu X.q \vee (p \wedge \Box_a X)$ $\cdots\cdots\cdots$ p holds until q holds along every a-path.
- $\mu X.p \vee \Diamond_a X$ $\cdots\cdots\cdots$ p eventually holds on some a-path.
- $\mu X.p \vee \Box_a X$ $\cdots\cdots\cdots$ p eventually holds on every a-path.
- $\mu X.q \vee (p \wedge \Box_a X)$ $\cdots$ along every a-path, p holds until q holds and q eventually holds.
- $\mu X.\nu Y.(p \wedge \Box_a X) \vee (\neg p \wedge \Box_a Y)$ $\cdots\cdots$ p holds only finitely often on every a-path.
- $\nu X.\mu Y.(p \wedge \Diamond_a X) \vee \Diamond_a Y)$ $\cdots\cdots$ p holds infinitely often on some a-path.
- $\nu X.\mu Y.\Diamond_a X \vee \Diamond_b Y$ $\cdots\cdots\cdots$ there exists a $\{a, b\}$-path with infinitely many a.

Problems

Prob 2. For two $\mathcal{L}_\mu$-formulas φ, ψ, we write $\varphi \equiv \psi$ if for any model M, $[\![\varphi]\!]^M = [\![\psi]\!]^M$.
Show the following equivalences.
(1) $\mu X.\varphi(X) \equiv \varphi(\mu X.\varphi(X))$. (2) $\mu X.\varphi(X) \equiv \neg\nu X.\neg\varphi(\neg X)$.

Prob 3. Find an example of a model M and a formula $\theta(X)$ such that $[\![\mu^\omega X.\theta(X)]\!]$ is not equivalent to the fixpoint $[\![\mu X.\theta(X)]\!]$. For such an example, consider which ordinal α makes $[\![\mu^\alpha X.\theta(X)]\!]$ and $[\![\mu X.\theta(X)]\!]$ equivalent.

§5.4. Applications: CTL

As applications of μ-calculus, we introduce CTL (Computation Tree Logic), a representative temporal logic, and two related systems. These logics are widely used in computer science.

Although CTL may have several modal operators, we only focus on two binary operators:

$$\mathsf{A}(\varphi \mathsf{U} \psi), \quad \mathsf{E}(\varphi \mathsf{U} \psi)$$

These mean:

- $\mathsf{A}(\varphi \mathsf{U} \psi)$: On *all* infinite paths, ψ holds eventually and φ holds until then.
- $\mathsf{E}(\varphi \mathsf{U} \psi)$: On *some* path, ψ holds eventually and φ holds until then.

Here, a path is a sequence of states generated by computation. U stands for "Until". We also include the usual modal operator $\Box$.

Syntax of CTL

Definition (CTL Formula)

Let p be an atomic proposition. A CTL formula is inductively defined as:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box\varphi \mid \mathsf{A}(\varphi \mathsf{U} \varphi) \mid \mathsf{E}(\varphi \mathsf{U} \varphi)$$

Other logical connectives are defined as usual.
Unless stated otherwise, the frame (W, R) used in CTL is assumed to be **serial**: $\forall s\, \exists t\, (sRt)$.

Semantics of CTL

Definition (Satisfaction Relation)

Let $M = (W, R, v)$ be a relational model. The satisfaction relation $M, s \models \varphi$ is defined inductively as follows:

- For p, $\neg\varphi$, $\varphi \to \psi$ and $\Box\varphi$, the usual clauses apply.
- $M, s_0 \models \mathsf{A}(\varphi \mathsf{U} \psi)$: For every infinite path $s_0 R s_1 R s_2 R \dots$, there exists i such that $M, s_i \models \psi$ and $M, s_j \models \varphi$ for all $j < i$.
- $M, s_0 \models \mathsf{E}(\varphi \mathsf{U} \psi)$: There exists a finite path $s_0 R s_1 R \dots R s_i$ such that $M, s_i \models \psi$ and $M, s_j \models \varphi$ for all $j < i$.

Derived Operators and μ-Calculus Translation

Various modal operators can be defined from the above two. For example,

$$\mathsf{AG}\varphi := \neg\mathsf{E}(\top\mathsf{U}\neg\varphi), \quad \text{where } \top := p \to p.$$

Here, "A" stands for "for All paths" and "G" stands for "Globally in the future."

Importantly, CTL formulas can be translated into modal μ-calculus formulas.
Let $(\varphi)^\natural$ be the translation of a CTL formula φ into a modal μ-calculus formula:

$$(\mathsf{A}(\varphi\mathsf{U}\psi))^\natural := \mu X.\psi^\natural \vee (\varphi^\natural \wedge \Box X)$$

$$(\mathsf{E}(\varphi\mathsf{U}\psi))^\natural := \mu X.\psi^\natural \vee (\varphi^\natural \wedge \Diamond X)$$

All other connectives remain unchanged. Note: only one variable X is used in this translation.

Theorem: Translation to Modal μ-Calculus

Theorem 1 (Translation Theorem)

For any serial relational structure M and any CTL formula φ:

$$M, s \models \varphi \iff M, s \models (\varphi)^\natural$$

A CTL formula φ is said to be **valid** if $M, s \models \varphi$ for every serial model $M = (W, R, v)$ and every state s, and we write:

$$\models_{\mathsf{CTL}} \varphi$$

Validity in CTL can be reduced to the validity of the single-variable modal μ-calculus.

Proof System for CTL

Definition (Proof System)

Axioms:

(1) Classical tautologies.

(2) Modal axiom: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

(3) Seriality: $\Box\varphi \to \Diamond\varphi$.

(4) $\mathsf{A}(\varphi \mathsf{U} \psi) \leftrightarrow \psi \vee (\varphi \wedge \Box \mathsf{A}(\varphi \mathsf{U} \psi))$.

(5) $\mathsf{E}(\varphi \mathsf{U} \psi) \leftrightarrow \psi \vee (\varphi \wedge \Diamond \mathsf{E}(\varphi \mathsf{U} \psi))$.

Inference Rules:

(6) Modus ponens: from φ and $\varphi \to \psi$, infer ψ.

(7) Necessitation: from φ, infer $\Box\varphi$.

(8) If $\psi \vee (\varphi \wedge \Box\theta) \to \theta$, then $\mathsf{A}(\varphi \mathsf{U} \psi) \to \theta$.

(9) If $\psi \vee (\varphi \wedge \Diamond\theta) \to \theta$, then $\mathsf{E}(\varphi \mathsf{U} \psi) \to \theta$.

We write $\vdash_{\mathsf{CTL}} \varphi$ to indicate that φ is provable in this system.

Completeness of CTL

Theorem 2 (Completeness)

$$\vdash_{\mathsf{CTL}} \varphi \iff \models_{\mathsf{CTL}} \varphi$$

Proof. ($\Rightarrow$) Axioms (1)–(7) are straightforward.
We show (8): Assume $\mathsf{A}(\varphi \mathsf{U} \psi) \to \theta$ is not valid. So, $\mathsf{A}(\varphi \mathsf{U} \psi) \to \theta$ is false at some state s in a model M, i.e.,

$$M, s \models \mathsf{A}(\varphi \mathsf{U} \psi) \wedge \neg\theta.$$

We look for a state t along a path from s such that

$$M, t \models (\psi \vee (\varphi \wedge \Box\theta)) \wedge \neg\theta. \quad \ldots(\star)$$

If ψ is true at s, then $(\star)$ holds at s. If not, then by $\mathsf{A}(\varphi \mathsf{U} \psi)$ φ holds at s.
If also $\Box\theta$ holds at $t = s$, then $(\star)$ again holds.
If not, then there exists a successor s' such that $M, s' \models \neg\theta$.
At s', $\mathsf{A}(\varphi \mathsf{U} \psi)$ still holds, and we repeat the reasoning. Proceeding along a path thus constructed, we eventually reach a state t where ψ holds, and at that state $(\star)$ holds.
Hence, $(\psi \vee (\varphi \wedge \Box\theta)) \to \theta$ is not valid. A similar argument proves (9).

Completeness of CTL (a part of the proof)

($\Leftarrow$): Suppose $\nvdash_{\mathsf{CTL}} \varphi$. We aim to construct a finite serial model M such that $M, s \not\models \varphi$.

Let $Sub(\varphi)$ be the set including all the subformulas of φ, closed under the following:

- If $\mathsf{A}(\theta \mathsf{U} \psi) \in Sub(\varphi)$, then it also includes $\Box\mathsf{A}(\theta \mathsf{U} \psi)$
- If $\mathsf{E}(\theta \mathsf{U} \psi) \in Sub(\varphi)$, it includes $\Diamond\mathsf{E}(\theta \mathsf{U} \psi)$

Then put

$$Sub^+(\varphi) := Sub(\varphi) \cup \{\mathsf{A}(\top \mathsf{U} \top), \Box\mathsf{A}(\top \mathsf{U} \top), \top, \Box\top, \bot\}$$

Define a model $M = (W, R, v)$ as follows:

- $W := \{(\Gamma, \Delta) \mid \Gamma \cup \Delta = Sub^+(\varphi)$ is a partition, and $\Gamma \nvdash_{\mathsf{CTL}} \bigvee \Delta\}$
- Transition: $(\Gamma, \Delta) R (\Pi, \Sigma)$ if for every formula $\Box\theta \in \Gamma$, we have $\theta \in \Pi$
- Valuation: $(\Gamma, \Delta) \in v(p)$ iff $p \in \Gamma$

Note that if $\Gamma \cup \Delta \subset Sub^+(\varphi)$ and $\Gamma \nvdash_{\mathsf{CTL}} \bigvee \Delta$, then there exists a partition $\Gamma^+ \cup \Delta^+ = Sub^+(\varphi)$ such that $\Gamma \subset \Gamma^+$, $\Delta \subset \Delta^+$, and still $\Gamma^+ \nvdash_{\mathsf{CTL}} \bigvee \Delta^+$.
$\because$ If $\Gamma \nvdash_{\mathsf{CTL}} \bigvee \Delta$ then for any θ, $\Gamma \cup \{\theta\} \nvdash_{\mathsf{CTL}} \bigvee \Delta$ or $\Gamma \nvdash_{\mathsf{CTL}} \bigvee \Delta \cup \{\theta\}$.

Lemma 3

For any $s = (\Gamma, \Delta) \in W$ *and any formula* θ *that is not of the form* $\mathsf{A}(\cdot\mathsf{U}\cdot)$ *or* $\mathsf{E}(\cdot\mathsf{U}\cdot)$*:*

- *If* $\theta \in \Gamma$*, then* $M, s \models \theta$
- *If* $\theta \in \Delta$*, then* $M, s \not\models \theta$

Proof. By induction on the construction of formula θ.
(1) Case: $\theta = p$ (atomic proposition). By the definition of $v(p)$, $\theta \in \Gamma$ iff $M, s \models \theta$.

(2) Case: $\theta = \varphi \to \psi$. First suppose $\theta \in \Gamma$. Then, $\varphi \in \Gamma$ and $\psi \in \Delta$ implies $\Gamma \vdash_{\mathsf{CTL}} \bigvee \Delta$, a contradiction. Thus, $\varphi \notin \Gamma$ or $\psi \notin \Delta$, that is, $\varphi \in \Delta$ or $\psi \in \Gamma$. By induction hypothesis, $M, s \not\models \varphi$ or $M, s \models \psi$, that is, $M, s \models \varphi \to \psi$. Next, suppose $\theta \in \Delta$. Then, we have $\varphi \notin \Gamma$ and $\psi \notin \Delta$, that is, $\varphi \in \Delta$ and $\psi \in \Gamma$, hence $M, s \not\models \varphi \to \psi$.

(3) Case: $\theta = \Box\varphi$. First suppose $\theta \in \Gamma$. Then, by the definition of transition relation R, $(\Gamma, \Delta)R(\Pi, \Sigma)$ implies $\varphi \in \Pi$, so $M, (\Pi, \Sigma) \models \varphi$. Next suppose $\theta \in \Delta$. We want to show that there exists (Π, Σ) such that $(\Gamma, \Delta)R(\Pi, \Sigma)$ and $\varphi \in \Sigma$. Let $\Gamma/\Box = \{\alpha : \Box\alpha \in \Gamma\}$.
Then, we have $\Gamma/\Box \nvdash_{\mathsf{CTL}} \varphi$. For otherwise, $\Gamma \vdash_{\mathsf{CTL}} \Box\varphi$ $(\in \Delta)$, a contradiction.
Now, we can construct a partition $\Pi \cup \Sigma = Sub^{+}(\varphi)$ such that $\Gamma/\Box \subset \Pi$, $\varphi \in \Delta$, and still $\Pi \nvdash_{\mathsf{CTL}} \bigvee \Sigma$. By $\Gamma/\Box \subset \Pi$, we have $(\Gamma, \Delta)R(\Pi, \Sigma)$.

To extend the truth lemma to formulas involving $\mathsf{A}(\cdot \mathsf{U} \cdot)$ or $\mathsf{E}(\cdot \mathsf{U} \cdot)$, the model must encode path information explicitly. Such constructions are more involved and omitted here.

Remark: Case: $\theta = \mathsf{A}(\varphi \mathsf{U} \psi)$. Suppose $\theta \in \Gamma$. Then, by axiom (4), either $\psi \in \Gamma$ or (both $\varphi \in \Gamma$ and $\Box \mathsf{A}(\varphi \mathsf{U} \psi) \in \Gamma$). If $\psi \in \Gamma$, then by induction hypothesis, $M, s \models \psi$, hence by the definition of $\mathsf{A}(\cdot \mathsf{U} \cdot)$, we have $M, s \models \mathsf{A}(\varphi \mathsf{U} \psi)$. Next assume $\varphi \in \Gamma$ and $\Box \mathsf{A}(\varphi \mathsf{U} \psi) \in \Gamma$. By ind. hyp., $M, s \models \varphi$. Now, take any $s_1 = (\Pi, \Sigma)$ such that $(\Gamma, \Delta) R (\Pi, \Sigma)$. Then, $\theta = \mathsf{A}(\varphi \mathsf{U} \psi) \in \Pi$. So, by the same argument, we have $M, s_1 \models \psi$ or (both $M, s_1 \models \varphi$ and $\Box \mathsf{A}(\varphi \mathsf{U} \psi) \in \Pi$). Thus, we can show that for any path, φ holds always or until ψ holds, which is not sufficient.

Thank you for your attention!