Logic and Computation II

Part 4. Modal logic

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Logic and Computation

- **Part 1. Introduction to Theory of Computation**
- **Part 2. Propositional Logic and Computational Complexity**
- **Part 3. First Order Logic and Decision Problems**
- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
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Part 4. Schedule (tentative)

- March 4, (1) Kripke models and normal logics
- March 6, (2) Kripke completeness
- March 11, (3) Standard translation and bisimulation
- March 13, (4) Decidability results
- ...

Recap

- $M = (W, R, v)$ is a **Kripke model** if (W, R) is a directed graph (**Kripke frame**) and v is a function from the set P of atomic propositions to the power set of W. v can be identified with a function $v' : W \times P \to \{\mathrm{T}, \mathrm{F}\}$ such that $v'(s, p) = \mathrm{T} \Leftrightarrow s \in v(p)$.
- $M, s \models \varphi$, equivalently $V(s, \varphi) = \mathrm{T}$, which means φ holds at a state $s \in W$, is defined as follows:

$$M, s \models p \Leftrightarrow v(s, p) = \mathrm{T},$$
$$M, s \models \neg\varphi \Leftrightarrow M, s \models \varphi \text{ does not hold},$$
$$M, s \models \varphi \to \psi \Leftrightarrow M, s \models \varphi \text{ implies } M, s \models \psi,$$
$$M, s \models \Box\varphi \Leftrightarrow M, t \models \varphi \text{ for all } t \in sR.$$

- φ is valid in a model $M = (W, R, v)$, denote $M \models \varphi$, if $M, s \models \varphi$ for any $s \in W$.
- φ is valid in a frame F, denote $F \models \varphi$, if it is valid in any model (F, v).
- φ is valid, denote $\models \varphi$, if $M \models \varphi$ for any model M.

- A logic (a set of propositions) L is **normal** if it satisfies the following conditions:
  (1) contains all tautologies (or axioms P1, P2, P3),
  (2) contains the **normal axiom**: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
  (3) is closed under the modus ponens rule (MP),
  (4) is closed under the **necessitation rule** (Nec): $\varphi \in L \Rightarrow \Box\varphi \in L$.
- The smallest normal logic is called K, named after Kripke.
- For a normal modal logic L, the **canonical frame** $F_L = (W_L, R_L)$ is defined as:
  (1) W_L is the set of maximal consistent sets in L,
  (2) $(s,t) \in R_L \Leftrightarrow$ for all $\Box\varphi \in s$, we have $\varphi \in t$.
- The **canonical model** $M_L = (F_L, v_L)$ consists of a canonical frame F_L, and v_L s.t.
  $s \in v_L(p)$ (i.e., $M_L, s \models p$) $\Leftrightarrow p \in s$.
- **Truth Lemma.** For any proposition φ, $M_L, s \models \varphi \Leftrightarrow \varphi \in s$.
- **Completeness Theorem.** Any normal modal logic L is **strongly complete** with respect to its canonical model M_L, i.e., if $\forall s(M_L, s \models \Gamma \Rightarrow M_L, s \models \varphi)$ then $\Gamma \vdash_L \varphi$. In particular, K is also strong complete w.r.t. the whole class of models, and it coincides with the set of all valid propositions.

§4.3. Canonical Normal Modal Logics

- When considering the completeness theorem for logics other than $L = \mathsf{K}$, the choice of model classes becomes important. We will consider normal modal logics L whose models are characterized by their frames $\mathcal{F}$.
- However, we should note that even if M_L is a model of L, there may exist another valuation v' such that (F_L, v') is not a model of L, namely $F_L \not\models L$.
- If $F_L \models L$ holds, we say that L is **canonical**.
- For a class of frames $\mathcal{F}$, $\mathrm{L}(\mathcal{F})$ denotes the set of all propositions that are valid in every frame of $\mathcal{F}$. A logic L is called **Kripke complete** if there exists a class $\mathcal{F}$ such that $L = \mathrm{L}(\mathcal{F})$.
- Furthermore, if L is Kripke complete, then letting $\mathcal{F}(L)$ be the collection of all frames that validate every proposition of L, we obtain $\mathrm{L}(\mathcal{F}(L)) = L$.
- Canonical logics are necessarily Kripke complete.

- Major canonical logics are obtained from K by adding new axiom schemata A (possibly multiple). In other words, they form the smallest normal modal logic containing A.
- Typical additional axioms include the following:

$$\mathrm{D} : \Box\varphi \to \Diamond\varphi \text{ (or } \neg\Box\bot), \quad \mathrm{T} : \Box\varphi \to \varphi,$$
$$4 : \Box\varphi \to \Box\Box\varphi, \quad .2 : \Diamond\Box\varphi \to \Box\Diamond\varphi,$$
$$5 : \Diamond\varphi \to \Box\Diamond\varphi, \quad \mathrm{B} : \Diamond\Box\varphi \to \varphi \text{ (or } \varphi \to \Box\Diamond\varphi).$$

- Based on these axioms, we define major systems of normal modal logic as follows

$$\mathsf{T} := \mathsf{K} + \mathrm{T}, \quad \mathsf{B} := \mathsf{K} + \mathrm{B}, \quad \mathsf{D} := \mathsf{K} + \mathrm{D},$$
$$\mathsf{K4} := \mathsf{K} + 4, \quad \mathsf{S4} := \mathsf{T} + 4, \quad \mathsf{S4.2} := \mathsf{S4} + .2,$$
$$\mathsf{S5} := \mathsf{T} + 5 = \mathsf{S4} + \mathrm{B}.$$

  T is a system defined by Feys, B reflects the ideas of Brouwer, and D represents deontic logic. S4 and S5 are systems introduced by Lewis.

- It is easy to see that $\mathsf{K} \subset \mathsf{K4} \subset \mathsf{S4} \subset \mathsf{S4.2} \subset \mathsf{S5}$. On the other hand, we have $\mathsf{K} \subset \mathsf{D} \subset \mathsf{T} \subset \mathsf{S4}$, but D and T cannot be compared with K4. Similarly, $\mathsf{K} \subset \mathsf{B} \subset \mathsf{S5}$, but B cannot be compared with the others.

All the above normal modal logics are Kripke complete and are characterized by the following classes of frames:

Theorem 4.12

(1) $F \models \mathsf{T} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} : sRs$,

(2) $F \models \mathsf{B} \Leftrightarrow F \in \mathcal{F}_{\mathrm{sym}} : sRt \Rightarrow tRs$,

(3) $F \models \mathsf{D} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ser}}(\mathrm{serial}) : \forall s \exists t sRt$,

(4) $F \models \mathsf{K4} \Leftrightarrow F \in \mathcal{F}_{\mathrm{tran}} : sRt \wedge tRu \Rightarrow sRu$,

(5) $F \models \mathsf{S4} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{tran}}$,

(6) $F \models \mathsf{S4.2} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{tran}} \cap \mathcal{F}_{\mathrm{dir}}$,
where $\mathcal{F}_{\mathrm{dir}}$ (directed): $sRt \wedge sRt' \Rightarrow \exists u(tRu \wedge t'Ru)$,

(7) $F \models \mathsf{S5} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{Euc}} = \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{sym}} \cap \mathcal{F}_{\mathrm{tran}}$, where $\mathcal{F}_{\mathrm{Euc}}$ (Euclidean): $sRt \wedge sRt' \Rightarrow tRt'$.

Proof.

(1) ($\Leftarrow$) Suppose the frame $F = (W, R)$ belongs to $\mathcal{F}_{\mathrm{ref}}$. For any model $M = (F, v)$ and any $s \in W$, we want to show that $M, s \models \Box\varphi \to \varphi$. Assume $M, s \models \Box\varphi$. By the interpretation of $\Box\varphi$, for all $t \in sR$, we have $M, t \models \varphi$. Since R is reflexive, $s \in sR$ holds, so $M, s \models \varphi$. Hence, $M, s \models \Box\varphi \to \varphi$ is proved.

($\Rightarrow$) Conversely, suppose the frame $F = (W, R)$ does not belong to $\mathcal{F}_{\mathrm{ref}}$. Then, there exists some $s \in W$ such that $s \notin sR$. Define a valuation v such that for some atomic proposition p, $v(p) = W - \{s\}$, while $v(q)$ is arbitrary for other atomic propositions q. In the model $M = (F, v)$, we have $M, s \not\models p$ and $M, t \models p$ for all $t \neq s$. Since all $t \in sR$ satisfy $t \neq s$, it follows that $M, t \models p$, hence $M, s \models \Box p$. However, $M, s \not\models p$, so $M, s \not\models \Box p \to p$. Therefore, $M \not\models \Box p \to p$ and $F \not\models \Box p \to p$.

The rest are left as exercises. □

Problem 2

In particular, show the following:

(2) $F \models \mathsf{B} \Leftrightarrow F \in \mathcal{F}_{\mathrm{sym}}$.

(5) $F \models \mathsf{S4} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{tran}}$.

- A typical example of non-canonical logic is Gödel-Löb's system GL.
- Gödel proved his first incompleteness theorem by showing that the sentence meaning its own unprovability is unprovable. Then, Henkin asked whether or not the sentence meaning its own provability is provable. His question is expressed as follows:
  $H : \Box(\Box\varphi \leftrightarrow \varphi) \ \rightarrow \ \Box\varphi.$ ($\Box$ = provability)
- Then, Löb gave a positive answer by showing the following stronger proposition.
  $L : \Box(\Box\varphi \rightarrow \varphi) \ \rightarrow \ \Box\varphi.$
  He also showed that this proposition is essential to the second incompleteness theorem.
- A normal logic GL is defined as follows:
  $\mathsf{GL} := \mathsf{K} + L$.

Problem 3

(1) Show that $\mathsf{GL} \supset \mathsf{K4}$.
(2) $F \models \mathsf{GL}$ if and only if F is transitive and contains no infinite R-chains $s_1 R s_2 R s_3 R \ldots$.

Theorem 4.13

GL is not canonical.

Proof. Transform a formula φ in GL into a formula φ° in Peano arithmetic PA as follows:

(1) Replace atomic propositions p with arithmetic statements p°.

(2) Preserve Boolean operations: $(\varphi \odot \psi)^\circ := \varphi^\circ \odot \psi^\circ$.

(3) Define $(\Box\varphi)^\circ$ by $\mathsf{Bew}_{\mathsf{PA}}(\ulcorner\varphi^\circ\urcorner)$. If $\mathsf{GL} \vdash \varphi$, then $\mathsf{PA} \vdash \varphi^\circ$. Let $s_0 := \{\varphi : \mathbb{N} \models \varphi^\circ\}$. Since s_0 is complete, $s_0 \in W_{\mathsf{GL}}$. For any φ, $\Box\varphi \in s_0 \Rightarrow \mathbb{N} \models \mathsf{Bew}_{\mathsf{PA}}(\ulcorner\varphi^\circ\urcorner) \Rightarrow \mathsf{PA} \vdash \varphi^\circ \Rightarrow \mathbb{N} \models \varphi^\circ \Rightarrow \varphi \in s_0$, so $s_0 R s_0$, which forms an infinite chain, contradicting $F_{\mathsf{GL}} \models \mathsf{GL}$. Thus, GL is not canonical. □

An example of a Kripke-incomplete logic is $\mathsf{GH} := \mathsf{K} + H$, which is closely related to GL. It is clear that $\mathsf{GL} \supset \mathsf{GH}$, but the axiom 4 is what separates the two. Nevertheless, we have $\mathcal{F}(\mathsf{GL}) = \mathcal{F}(\mathsf{GH})$, which means that GH is not Kripke complete.

Recap: First-Order Logic

- First-order logic is developed in propositional connectives, quantifiers $\forall x$ and $\exists x$, equality $=$ and other mathematical symbols. The set of mathematical symbols to use is called a **language**.
- A **structure** in a language $\mathcal{L}$ (simply, an $\mathcal{L}$-structure) is defined as a non-empty set A equipped with an interpretation of the symbols in $\mathcal{L}$.
- A **term** is a symbol string to denote an element of a structure. A **formula** is to describe a property of a structure. A formula with no free variables is called a sentence.
- "A sentence φ is true in $\mathcal{A}$", denote $\mathcal{A} \models \varphi$, is defined by Tarski's clauses. $\mathcal{A}$ is a **model** of a theory T (a set of sentences), denote $\mathcal{A} \models T$, if $\forall \varphi \in T\ (\mathcal{A} \models \varphi)$. We say that φ holds in T, denote $T \models \varphi$, if $\forall \mathcal{A}(\mathcal{A} \models T \to \mathcal{A} \models \varphi)$.
- Compactness theorem. If all finite subsets of a theory T have a model, T has a model.
- Löwenheim-Skolem's downward theorem. For a structure $\mathcal{A}$ in a countable language $\mathcal{L}$, there exists a countable $\mathcal{A}' \subset \mathcal{A}$ s.t. $\mathcal{A}' \models \varphi \Leftrightarrow \mathcal{A} \models \varphi$ for any $\mathcal{L}_{A'}$-sentence φ. $\mathcal{A}'$ is called an elementary substructure of $\mathcal{A}$, $\mathcal{A}' \prec \mathcal{A}$.

§4.4. Translation to First-Order Logic

Let us examine how Kripke models $M = (W, R, v)$ can be treated within first-order logic. For this purpose, we interpret an atomic proposition p_i as a subset $v(p_i) = P_i \subset W$ (where $i < n$ or $i \in \mathbb{N}$), and consider the relational structure $M' = (W, R, P_0, P_1, P_2, \dots)$. The modal relation "$M, x \models \varphi$" can be translated into a first-order formula $ST_x(\varphi)$ on M'.

Definition 4.14 (Standard Translation)

The **standard translation** $ST_x(\varphi)$ of a modal proposition φ is defined inductively as follows:

$$\begin{aligned} ST_x(p_i) &:= P_i(x), \\ ST_x(\neg\varphi) &:= \neg ST_x(\varphi), \\ ST_x(\varphi \to \psi) &:= ST_x(\varphi) \to ST_x(\psi), \\ ST_x(\Box\varphi) &:= \forall y(R(x,y) \to ST_y(\varphi)). \end{aligned}$$

Theorem 4.15

For a Kripke model M and its corresponding first-order structure M', the following hold:
(1) $M, s \models \varphi \Leftrightarrow M' \models ST_s(\varphi)$,
(2) $M \models \varphi \Leftrightarrow M' \models \forall x\ ST_x(\varphi)$.

Proof. By induction on the construction of φ. □

Let us now look at some applications.

Corollary 4.16 (Compactness)

For any set of modal propositions Φ, if every finite subset of Φ is satisfiable, then Φ itself is satisfiable. That is, there exists a model M and a state s such that $M, s \models \Phi$.

Proof. For a set of modal propositions Φ, define $ST_x(\Phi) := \{ST_x(\varphi) : \varphi \in \Phi\}$. By Theorem 4.15, if a modal proposition φ is satisfiable in M, s, then its standard translation $ST_s(\varphi)$ has a model M'. Therefore, if every finite subset of Φ is satisfiable, then every finite subset of $ST_s(\Phi)$ also has a model. By the compactness theorem of first-order logic, $ST_s(\Phi)$ is satisfiable in some M', which implies that Φ is also satisfiable in M, s. □

Corollary 4.17 (Weak LS Downward Property)

If a set of modal propositions Φ has an infinite model, then it also has a countably infinite model. (Note: LS = Löwenheim-Skolem)

Proof. Since $ST_x(\Phi)$ has the weak LS downward property, we can transform its model into a Kripke model. □

Note.

- $M \models \varphi$ is expressed as a first-order formula $\forall x ST_x(\varphi)$, but $F \models \varphi$ may not. The latter can be represented by a monadic second-order formula $\forall \vec{P} \forall x ST_x(\varphi)$.
- In fact, the frame that makes Löb's axiom L valid cannot be characterized by a first-order formula. This is because if arbitrarily long finite sequences of R exist, then by compactness, an infinite sequence of R also exists.
- In first-order logic, Ehrenfeucht-Fraïssé game connects the concepts of elementary equivalence to isomorphism. In modal logic, this corresponds to the idea of "bisimulation".

Definition 4.18 (Bisimulation)

Let $M = (W, R, v)$, $M' = (W', R', v')$ be Kripke models. $Z \subset W \times W'$ is a **bisimulation** between M and M' if the following holds. (0) $Z \neq \emptyset$.
(1) sZs', then for any $p \in P$, $M, s \models p \Leftrightarrow M', s' \models p$.
(2) If sZs' and sRt, then there is $t' \in W'$ such that $s'R't'$ and tZt' (the forth condition).
(3) If sZs' and $s'R't'$, then there is $t \in W$ such that sRt and tZt' (the back condition).
If there exists a bisimulation Z between M and M' s.t. sZs', we write $M, s \underline{\leftrightarrow} M', s'$.

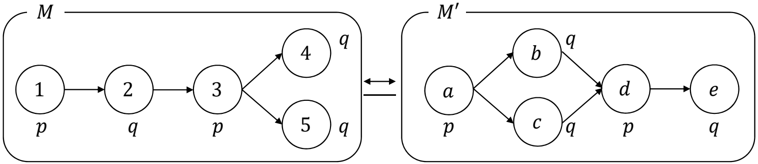

Figure: $Z = \{(1, a), (2, b), (2, c), (3, d), (4, e), (5, e)\}$

Definition 4.19 (Modal equivalence)

Let M and M' be Kripke models. M, s and M', s' are **modally equivalent**, denote $M, s \equiv M', s'$, if for all modal propositions φ, $M, s \models \varphi \Leftrightarrow M', s' \models \varphi$.

Theorem 4.20 (Bisimulation invariant theorem)

If $M, s \underline{\leftrightarrow} M', s'$, then $M, s \equiv M', s'$.

Proof.

- We assume there exists a bisimulation Z such that $M, s \underline{\leftrightarrow} M', s'$, and our goal is to show that for all φ, $M, t \models \varphi \Leftrightarrow M', t' \models \varphi$ if tZt'.
- We prove this by induction on the construction of modal proposition φ.
- The case $\varphi = \Box\psi$ is essential. Suppose $M, s \models \Box\psi$, and we show $M', s' \models \Box\psi$. So, we will show that for any $t' \in s'R'$, $M', t' \models \psi$.
- $M, s \underline{\leftrightarrow} M', s'$ gives sZs', so by the backward condition, there is t such that sRt and tZt'. By $M, s \models \Box\psi$ and sRt, we have $M, t \models \psi$. Since tZt', $M, t \underline{\leftrightarrow} M', t'$, it follows that $M', t' \models \psi$ from the induction hypothesis. □

The converse of the above theorem does not hold in general. However, there are some special classes of Kripke models where the converse of the theorem also holds, which is called the **Hennessy-Milner property**.

We say that $M = (W, R, v)$ is a finite branching model if sR is a finite set for any $s \in W$.

Theorem 4.21

The class of finite branching models has the Hennessy-Milner property.

Proof. Assume M, M' are finite branching and $M, s \equiv M', s'$. Let Z be the set of pairs (w, w') such that $M, w \equiv M', w'$. It is obvious that $Z \neq \varnothing$. Condition (1) of Definition 4.18 can be obtained from $M, s \equiv M', s'$. To prove (2) of Definition 4.18, suppose sZs' and sRt. Since $M, s \models \neg\Box\bot$ by sRt, we have $M', s' \models \neg\Box\bot$ and so there is t' such that $s'R't'$. Since there are only a finite number of such t' due to the finite branch property, and so we list them as $t'_1, t'_2, \ldots, t'_n$.
Suppose to the contrary that for all $i \leq n$, not tZt'_i. Then for each $i \leq n$, there is ψ_i such that $M, t \models \psi_i$ and $M', t'_i \models \neg\psi_i$. So $M, t \models \bigwedge_i \psi_i$ and $M', s' \models \Box \bigvee_i \neg\psi_i$. Then, $M, s \models \Box \bigvee_i \neg\psi_i$ by sZs'. So, $M, t \models \bigvee_i \neg\psi_i$ by sRt, which contradicts $M, t \models \bigwedge_i \psi_i$. Therefore, for some $i \leq n$, we have tZt'_i. This complete the proof for (2). Similarly, we can prove for (3). □

Thank you for your attention!