Logic and Computation II

Part 4. Modal logic

Kazuyuki Tanaka

BIMSA

March 4, 2025

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Computation

- **Part 1. Introduction to Theory of Computation**
- **Part 2. Propositional Logic and Computational Complexity**
- **Part 3. First Order Logic and Decision Problems**
- **Part 4. Modal logic**
- **Part 5. Modal μ-calculus**
- **Part 6. Automata on infinite objects**
- **Part 7. Recursion-theoretic hierarchies**

Part 4. Schedule (tentative)

- March 4, (1) Kripke models and normal logics
- March 6, (2) Kripke completeness
- March 11, (3) Standard translation and bisimulation
- March 13, (4) Decidability results
- ...

Classical Logic vs. Modal Logic

- In contrast to the objective world of classical logic, which deals with truth and falsity, modal logic is a branch of logic that also considers subjective expressions commonly used in everyday language, such as "might be," "is necessary," and "knows."
- In the modern setting, modal logic is obtained from classical logic by adding new operators $\Box$ and $\Diamond$, which normally express "necessity" and "possibility," respectively. Namely, $\Box\varphi$ ($\Diamond\varphi$) means that φ holds in all (some) possible worlds.
- $\Box$ and $\Diamond$ exhibit properties similar to those of $\forall$ and $\exists$, respectively. Generally, while modal (propositional) logic is less expressive than first-order logic, this limitation allows us to determine the truth value of a given proposition. Thus, Modal logic can be viewed as intermediate between propositional logic and first-order logic, effectively making it a kind of "0.5-order logic."

Historical Notes

- C.I. Lewis (1918, 1932) is often regarded as the founder of modern modal logic. He was not satisfied with Russell's treatment of (material) implication $\to$. Then he proposed five axiomatic systems of strict implication, S1, S2, ..., S5 (from weaker to stronger), which became the prototype for formal systems in modal logic.
- The attempt to extend classical logic by adding a new operator $\Box$ to handle modality began with Gödel (1933), who applied the provability predicate used in his proof of the incompleteness theorem to translate Lewis's system S4, though systems S3 and below are difficult to translate in this way.
- There are various structures that can serve as models for modal logic, including algebraic and topological models. Kripke (1959) defined a structure, called a Kripke model, as a directed graph (or transition system) with each vertex representing a state (or possible world), which has predominantly been used for modal logic with $\Box$, $\Diamond$.
- There has been a growing trend to use modal logic as a language to describe the properties of transition systems. This area of research, known as model checking, has gained significant attention.

Recap

- Propositional logic is the study of logical connections between propositions.
- If any truth-value function V satisfying all propositions in Γ also satisfies φ, then φ is said to be a **tautological consequence** of Γ, written as $\Gamma \models \varphi$.
- We consider an axiomatic system that derives all valid propositions only using $\neg, \rightarrow$. We can omit $\vee$ and $\wedge$ by setting $\varphi \vee \psi := \neg\varphi \rightarrow \psi$, $\varphi \wedge \psi := \neg(\varphi \rightarrow \neg\psi)$.
- A **proof** of φ in Γ is a sequence of propositions $\varphi_0, \varphi_1, \cdots, \varphi_n (= \varphi)$ satisfying the following conditions: for $k \leq n$, (1) φ_k belongs to $\{\mathrm{P1}, \mathrm{P2}, \mathrm{P3}\} \cup \Gamma$, where

  P1. $\varphi \rightarrow (\psi \rightarrow \varphi)$ P2. $\big(\varphi \rightarrow (\psi \rightarrow \theta)\big) \rightarrow \big((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta)\big)$

  P3. $(\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$,

  or (2) there exist $i, j < k$ such that $\varphi_j = \varphi_i \rightarrow \varphi_k$ (MP).
- If a proof of φ in Γ exists, φ is called a **theorem** in Γ, written as $\Gamma \vdash \varphi$.

Completeness theorem for propositional logic

$$\Gamma \vdash \varphi \quad \Leftrightarrow \quad \Gamma \models \varphi.$$

§4.1. Simple modal logic and Kripke models

In today's lecture, we consider propositional logic with a single $\Box$-operator.

Definition 4.1 (Modal Propositional Language)

Let $P := \{p, q, r, \dots\}$ be atomic propositions. We often use p as a meta-variable for atomic propositions. A proposition φ is constructed by the following syntax:

$$\varphi := p \mid \neg\varphi \mid (\varphi \to \varphi) \mid \Box\varphi$$

Here, $\Box\varphi$ means "a proposition φ necessarily holds." The symbol $\Box$ is read as "box."

Other operators are defined as follows:

$$\bot := \neg(p \to p), \qquad \varphi \wedge \psi := \neg(\varphi \to \neg\psi),$$
$$\varphi \vee \psi := \neg\varphi \to \psi, \qquad \Diamond\varphi := \neg\Box\neg\varphi.$$

Thus, $\Diamond\varphi$ means "it is possible that φ holds." The symbol $\Diamond$ is referred to as "diamond."

Kripke (1959) defined an interpretation of modal logic in a transition system (called a **Kripke model**), that is, a directed graph (also called a **Kripke frame**) with each vertex representing a possible world. In other words, a modal proposition describes some property of a transition system.

Definition 4.2 (Kripke Frames)

A **Kripke frame** is a directed graph (W, R) that allows for self-loops, where W is a non-empty set and R is a binary relation on W. The elements of W are called **states** or **possible worlds**, and R is called a **transition relation**.

Notation. For a binary relation R on W and any s, t in W, Rt denotes the set $\{s' : (s', t) \in R\}$, and sR denotes $\{t' : (s, t') \in R\}$. Therefore, $s \in Rt$ and $t \in sR$ are equivalent, and in this case, we also write sRt.

Definition 4.3 (Kripke Models)

A structure $M = (W, R, v)$ is called a **Kripke model** if (W, R) is a Kripke frame and v is a function from the set P of atomic propositions to the power set of W. The function v can be identified with a function $v' : W \times P \to \{\mathrm{T}, \mathrm{F}\}$ such that $v'(s, p) = \mathrm{T} \Leftrightarrow s \in v(p)$, which means an atomic proposition $p \in P$ holds at a state $s \in W$.

By $M, s \models \varphi$, we denote a proposition φ holds at a state s in M. This is defined as follows:

$$
\begin{gathered}
M, s \models p \Leftrightarrow v(s, p) = \mathrm{T}, \\
M, s \models \neg\varphi \Leftrightarrow M, s \models \varphi \text{ does not hold}, \\
M, s \models \varphi \to \psi \Leftrightarrow M, s \models \varphi \text{ implies } M, s \models \psi, \\
M, s \models \Box\varphi \Leftrightarrow M, t \models \varphi \text{ for all } t \in sR.
\end{gathered}
$$

We extend v to a function V on the general propositions φ by

$$V(s, \varphi) = \mathrm{T} \Leftrightarrow M, s \models \varphi.$$

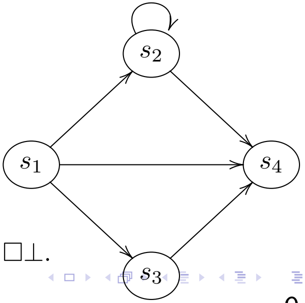
In the above definition, the only difference from classical propositional logic is the treatment of $\Box\varphi$.
$\Box\varphi$ holds at a state s if and only if φ holds at all states t that are reachable from s.
Moreover, for $\wedge, \vee, \Diamond$, we can derive the following from their definitions:

$$M, s \models \varphi \wedge \psi \Leftrightarrow M, s \models \varphi \text{ and } M, s \models \psi,$$
$$M, s \models \varphi \vee \psi \Leftrightarrow M, s \models \varphi \text{ or } M, s \models \psi,$$
$$M, s \models \Diamond\varphi \Leftrightarrow \text{there exists } t \in sR \text{ such that } M, t \models \varphi.$$

Example 1

Consider the frame $F = (W, R)$ given by the following diagram: Let $P = \{p, q\}$ and define the truth assignment as follows:

$$V(p) = \{s_2, s_3\}, \quad V(q) = \{s_4\}.$$



Show that the following statements hold:

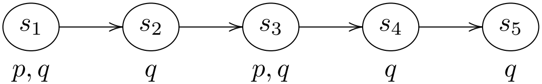
$$M, s_3 \models \Box q, \quad M, s_2 \models \neg\Box q, \quad M, s_1 \models \Diamond\Box q, \quad M, s_4 \models \Box\bot.$$

Definition 4.4 (Validity)

- A proposition φ is said to be valid in a Kripke model $M = (W, R, v)$ if for any $s \in W$, we have $M, s \models \varphi$. In this case, we write $M \models \varphi$.
- A proposition φ is said to be valid in a Kripke frame F if it is valid in any Kripke model (F, v). In this case, we write $F \models \varphi$.
- A proposition φ is said to be valid, denote $\models \varphi$, if $F \models \varphi$ for any frame F.

Problem 1

Consider the frame $F = (W, R)$ as shown below. Let $P = \{p, q\}$, and define the truth assignment $V(p) = \{s_1, s_3\}, V(q) = W$.



(1) Find all states s where $\Diamond\Box p$ holds.
(2) Find all states s where $\Diamond(\Box p \to p)$ holds.

§4.2. Modal logic K and its completeness theorem

- A formal deductive system of modal logic can be obtained from that of propositional logic by adding **normal axioms** and the **necessitation rule** (Nec).
- A set of propositions (simply referred to as a **logic**) that satisfies the following conditions is called a **normal logic**.

Definition 4.5

A logic L is said to be **normal** if it satisfies the following conditions:

(1) contains all tautologies (or axioms P1, P2, P3),

(2) contains the **normal axiom**: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,

(3) is closed under the modus ponens rule (MP),

(4) is closed under the **necessitation rule** (Nec): $\varphi \in L \Rightarrow \Box\varphi \in L$.

- The smallest normal logic is called K, named after Kripke.

Lemma 4.6 (K's Soundness Theorem)

Any proposition in K is valid.

Proof. Since K are the same as the theorems derived finitely from the conditions of Definition 4.5, the following proof proceeds by induction on the length of derivations.

- Given any Kripke model $M = (W, R, v)$ and any state $s \in W$.
- Since the truth values of propositional connections are defined independently at each state, conditions (1) and (3) of Definition 4.5 easily follows from the corresponding conditions of propositional logic.
- To show that (2) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ holds at a state s, suppose $\Box(\varphi \to \psi)$ and $\Box\varphi$ are both true at s. From the interpretation of $\Box$, for any $t \in sR$, we have $M, t \models \varphi \to \psi$ and $M, t \models \varphi$. By (3) at t, it follows that $M, t \models \psi$, which holds for all $t \in sR$, so $M, s \models \Box\psi$, establishing (2).
- Finally, for (4), if φ holds at all states $t \in W$ of M, then it also holds at all $t \in sR$, hence $M, s \models \Box\psi$. □

- Conversely, we can also show that every valid proposition belongs to K.
- This argument generalizes to the completeness theorem for any normal logic L. For that purpose, we first define derivability in L.

Definition 4.7

Let L be a normal modal logic. A proposition φ is derivable in L from a theory (set of propositions) Γ ($\Gamma \vdash_L \varphi$) if there exist $\theta_1, \theta_2, \ldots, \theta_k$ in Γ such that

$$\theta_1 \to (\theta_2 \to (\cdots \to (\theta_k \to \varphi) \ldots)) \in L.$$

- The set of derivable propositions is closed under MP but not necessarily under Nec, so it is not necessarily normal. A theory Γ is **consistent** in L if it does not derive $\bot$.

Lemma 4.8

Any consistent theory is included in a maximal consistent (i.e., complete) set.

This can be proved in the same way as classic proposition logic.

In accordance with $\Gamma \vdash_L \varphi$, we also want to consider $\Gamma \models_L \varphi$. For this, we first define the canonical frame and model for L.

Definition 4.9

For a normal modal logic L, the **canonical frame** $F_L = (W_L, R_L)$ is defined as:
(1) W_L is the set of maximal consistent sets in L,
(2) $(s, t) \in R_L \Leftrightarrow$ for all $\Box\varphi \in s$, we have $\varphi \in t$.
The **canonical model** $M_L = (F_L, v_L)$ is defined as:
(1) $F_L = (W_L, R_L)$ is a canonical frame,
(2) $s \in v_L(p)$ (i.e., $M_L, s \models p$) $\Leftrightarrow p \in s$.

Lemma 4.10 (Truth Lemma)

For any proposition φ, $M_L, s \models \varphi \Leftrightarrow \varphi \in s$.

Proof. We proceed by induction on the construction of φ. The essential case is $\varphi \equiv \Box\psi$.
($\Leftarrow$) Assume $\Box\psi \in s$. To show that $M_L, s \models \Box\psi$, take any $t \in sR_L$. By the definition of R_L, we have $\psi \in t$, and by the induction hypothesis, it follows that $M_L, t \models \psi$.

($\Rightarrow$) Assume $\Box\psi \notin s$. Then, we can show that $\{\theta : \Box\theta \in s\} \cup \{\neg\psi\}$ is consistent. By way of contradiction, assume that it were inconsistent. Then, there exist propositions $\theta_1, \theta_2, \ldots, \theta_k$ in $\{\theta : \Box\theta \in s\}$ such that

$$\theta_1 \to (\theta_2 \to (\cdots \to (\theta_k \to \psi) \ldots)) \in L.$$

Since L is closed under the necessitation rule (Nec), we have:

$$\Box(\theta_1 \to (\theta_2 \to (\cdots \to (\theta_k \to \psi) \ldots))) \in L.$$

Applying the normality axiom, we obtain:

$$\Box\theta_1 \to (\Box\theta_2 \to (\cdots \to (\Box\theta_k \to \Box\psi) \ldots)) \in L.$$

Since $\Box\theta_i \in s$ for each i, it follows that $\Box\psi \in s$, contradicting our assumption. Hence, $\{\theta : \Box\theta \in s\} \cup \{\neg\psi\}$ must be consistent.
By Lemma 4.8, let t be a maximal consistent set containing $\{\theta : \Box\theta \in s\} \cup \{\neg\psi\}$, then $(s, t) \in R_L$ and $M_L, t \not\models \psi$. Therefore, $M_L, s \not\models \Box\psi$. $\Box$

If $\Gamma \nvdash_L \varphi$, then there exists a maximal consistent set s containing $L \cup \Gamma \cup \{\neg\varphi\}$. By Lemma 4.10, we conclude that $M_L, s \models \neg\varphi$, so $M_L \not\models \varphi$. Thus, if we define $\Gamma \models_L \varphi$ as $\forall s(M_L, s \models \Gamma \Rightarrow M_L, s \models \varphi)$, we obtain a version of (strong) completeness theorem, though this is not very usable.

For $L = \mathsf{K}$, by lemma 4.6, we obtain the following.

Theorem 4.11 (Completeness Theorem for K)

For K, the (strong) completeness theorem holds for the whole class of models.
In particular, K coincides with the set of all valid propositions.

- When considering the completeness theorem for logics other than $L = \mathrm{K}$, the choice of model classes becomes important. As we will see in the next lecture, we often consider normal modal logics L whose models are characterized by their frames $\mathcal{F}$.
- However, even if M_L is a model of L, there may exist another valuation v' such that (F_L, v') is not a model of L. Thus, it is possible that $F_L \not\models L$.
- If $F_L \models L$ holds, we say that L is **canonical**.
- For a class of frames $\mathcal{F}$, $\mathrm{L}(\mathcal{F})$ denotes the set of all propositions that are valid in every frame of $\mathcal{F}$. A logic L is called **Kripke complete** if there exists a class $\mathcal{F}$ such that $L = \mathrm{L}(\mathcal{F})$.
- Furthermore, if L is Kripke complete, then letting $\mathcal{F}(L)$ be the collection of all frames that validate every proposition of L, we obtain $\mathrm{L}(\mathcal{F}(L)) = L$.
- Canonical logics are necessarily Kripke complete.

Canonical Normal Modal Logics

- Major canonical logics are obtained by adding new axiom schemata A (possibly multiple) to K. In other words, they form the smallest normal modal logic containing K + A.
- Typical additional axioms include the following:

$$\mathrm{D}: \Box\varphi \to \Diamond\varphi \text{ (or } \neg\Box\bot), \quad \mathrm{T}: \Box\varphi \to \varphi,$$
$$4: \Box\varphi \to \Box\Box\varphi, \quad .2: \Diamond\Box\varphi \to \Box\Diamond\varphi,$$
$$5: \Diamond\varphi \to \Box\Diamond\varphi, \quad \mathrm{B}: \Diamond\Box\varphi \to \varphi \text{ (or } \varphi \to \Box\Diamond\varphi).$$

- Based on these axioms, we define major systems of normal modal logic.

$$\mathsf{T} := \mathsf{K} + \mathrm{T}, \quad \mathsf{B} := \mathsf{K} + \mathrm{B}, \quad \mathsf{D} := \mathsf{K} + \mathrm{D},$$
$$\mathsf{K4} := \mathsf{K} + 4, \quad \mathsf{S4} := \mathsf{T} + 4, \quad \mathsf{S4.2} := \mathsf{S4} + .2,$$
$$\mathsf{S5} := \mathsf{T} + 5 = \mathsf{S4} + \mathrm{B}.$$

- It is easy to see that K ⊂ K4 ⊂ S4 ⊂ S4.2 ⊂ S5. On the other hand, we have K ⊂ D ⊂ T ⊂ S4, but D and T cannot be compared with K4. Similarly, K ⊂ B ⊂ S5, but B cannot be compared with the others.

All the above normal modal logics are Kripke complete and are characterized by the following classes of frames:

Theorem 4.12

(1) $F \models \mathsf{T} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} : sRs$,

(2) $F \models \mathsf{B} \Leftrightarrow F \in \mathcal{F}_{\mathrm{sym}} : sRt \Rightarrow tRs$,

(3) $F \models \mathsf{D} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ser}}(\mathrm{serial}) : \forall s \exists t sRt$,

(4) $F \models \mathsf{K4} \Leftrightarrow F \in \mathcal{F}_{\mathrm{tran}} : sRt \wedge tRu \Rightarrow sRu$,

(5) $F \models \mathsf{S4} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{tran}}$,

(6) $F \models \mathsf{S4.2} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{tran}} \cap \mathcal{F}_{\mathrm{dir}}$,
where $\mathcal{F}_{\mathrm{dir}}$ (directed): $sRt \wedge sRt' \Rightarrow \exists u(tRu \wedge t'Ru)$,

(7) $F \models \mathsf{S5} \Leftrightarrow F \in \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{Euc}} = \mathcal{F}_{\mathrm{ref}} \cap \mathcal{F}_{\mathrm{sym}} \cap \mathcal{F}_{\mathrm{tran}}$, where $\mathcal{F}_{\mathrm{Euc}}$ (Euclidean): $sRt \wedge sRt' \Rightarrow tRt'$.

Thank you for your attention!