Logic and Foundations II

Part 8. Second order arithmetic and non-standard methods

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Real analysis and reverse mathematics (8.5 lectures)
- Part 8. Second order arithmetic and non-standard methods (6.5 lectures)

Part 8. Schedule

- May 21, (0) Introduction to forcing
- May 23, (1) Harrington's conservation result on WKL_0
- May 28, (2) H.Friedman's conservation result on WKL_0
- May 30, (3) Friedman's result (continued) and a self-embedding theorem I
- June 04, (4) A self-embedding theorem II
- June 06, (5) A self-embedding theorem III and STY theorem I
- June 11, (6) STY theorem II

Theorem 3.1 (Self-Embedding Theorem)

Let $\mathfrak{M} = (M, S)$ be a countable model of WKL_0 with $M \neq \omega$. Then, there exists a proper initial segment I of M such that $\mathfrak{M}\lceil I = (I, S\lceil I)$ is isomorphic to $\mathfrak{M}$. Here, $S\lceil I = \{X \cap I \mid X \in S\}$.

We first prove the following lemma, which will be frequently used later.

Lemma 3.2 (Compactness in WKL_0)

(1) For any Π^0_1 formula $\varphi(X)$, there exists a Π^0_1 formula $\hat{\varphi}$ such that WKL_0 proves:

$$\hat{\varphi} \leftrightarrow \exists X\, \varphi(X).$$

(2) For any Π^0_1 formula $\varphi(k, X)$, WKL_0 proves:

$$\forall n\, \exists X\, \forall k < n\, \varphi(k, X) \to \exists X\, \forall k\, \varphi(k, X).$$

We define **G-Σ^0_1 formulas** or simply **G formulas** by generalizing Σ^0_1 formulas as follows. The G formulas are obtained from Σ^0_1 formulas by using $\wedge, \vee$, bounded universal quantifier $\forall x < y$ and unbounded existential quantifier $\exists x$, and set quantifiers $\forall X, \exists X$.
In WKL_0, we can prove that a G formula is equivalent to a Σ^0_1 formula.

Definition 3.3 (G formulas in RCA_0)

A sequence $G_0 \subset G_1 \subset G_2 \subset \cdots$ of sets of $\mathcal{L}^2_{\mathrm{OR}}$-formulas is defined inductively modulo 4 as follows: for each $e \in \mathbb{N}$,

$$\begin{aligned}
G_0 &= \{\text{finite disjunctions } (\vee) \text{ of atomic formulas or their negations}\},\\
G_{4e+1} &= \{\exists x\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e} \text{ formulas}\} \cup G_{4e},\\
G_{4e+2} &= \{\forall x < y\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+1} \text{ formulas}\} \cup G_{4e+1},\\
G_{4e+3} &= \{\exists X\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e+2} \text{ formulas}\} \cup G_{4e+2},\\
G_{4e+4} &= \{\forall X\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+3} \text{ formulas}\} \cup G_{4e+3}.
\end{aligned}$$

Finally, we set $\mathbf{G} = \bigcup_{e \in \mathbb{N}} G_e$. The formulas in G are called **G formulas**.

In the following, we will define Sat for G formulas.

From now on, a structure $\mathfrak{M} = (M, S)$ is denoted by V. Then, for each $p \in M$, let $M_p = \{a \in M \mid \mathfrak{M} \models a < p\}$, $S_p = \{X \cap M_p \mid X \in S\}$ and $V_p = (M_p, S_p)$.

For any formula φ in $\mathcal{L}^2_{\mathrm{OR}}$, let φ^{V_p} be a formula obtained by restricting the ranges of variables to $V_p = (M_p, S_p)$. More precisely, in φ^{V_p}, quantification over numbers is bounded by p, and quantification over sets is also considered as ranging binary sequences of length p, which can be coded by numbers $< 2^p$. So, φ^{V_p} can be regarded as a Δ^0_1 formula in V. Thus, by using $\mathrm{Sat}_{\Sigma^0_1}$, we define the **satisfaction predicate** $\mathrm{Sat}^p(z, \xi)$ as follows:

$$\mathrm{Sat}^{\mathbf{p}}(\ulcorner\varphi\urcorner, \xi) \equiv \mathrm{Sat}_{\Sigma^0_1}(\ulcorner\varphi^{V_p}\urcorner, \xi \restriction V_p), \quad \text{i.e., } \varphi(\xi)^{V_p}.$$

Here, ξ is a finite function that assigns elements of $M_p \cup S_p$ to free variables in φ, and $\xi \restriction V_p$ is the assignment obtained from ξ by restricting its values to V_p.

We also remark that a variable z in $\mathrm{Sat}^p(z, \xi)$ can potentially express a non-standard number. In V, we can verify that Sat^p satisfies Tarski's truth definition clauses (cf. Theorem IV.2.26 in [P. Hájek and P. Pudlák, *Metamathematics of First-oder Arithmetic*, Springer, 1993.]).

Next, we define the **satisfaction relation for G formulas** as follows:

Definition 3.4

For each $z \in G$, define the satisfaction relation $\mathrm{Sat}(z, \xi)$ as follows:

$$\mathrm{Sat}(z, \xi) \leftrightarrow \exists p\, \mathrm{Sat}^p(z, \xi \restriction V_p).$$

For simplicity, we abbreviate $\mathrm{Sat}^p(z, \xi \restriction V_p)$ as $\mathrm{Sat}^p(z, \xi)$.
In the following, we identify a formula with its code.

Lemma 3.5

In a model V of WKL_0, $\mathrm{Sat}(z, \xi)$ satisfies Tarski's truth definition clauses for G formulas.

Proof idea. In fact, if z is Σ^0_1, $\mathrm{Sat}(z, \xi) \Leftrightarrow \exists p\, \mathrm{Sat}^p(z, \xi) \Leftrightarrow \exists p\, z(\xi)^{V_p} \Leftrightarrow z(\xi)$.
The critical case is $z = \forall X\, z'$ (where z' is a G formula).

$$\begin{aligned}\mathrm{Sat}\,(\forall X\, z', \xi) &\Leftrightarrow \exists p\, \mathrm{Sat}^p\,(\forall X\, z', \xi) \Leftrightarrow \exists p\, \forall U\, \mathrm{Sat}^p\,(z', \xi \cup \{(X, U)\}) \\ &\Leftrightarrow \forall U\, \exists p\, \mathrm{Sat}^p\,(z', \xi \cup \{(X, U)\}) \qquad (\Leftarrow \text{ by compactness (Lemma 3.2(2))}) \\ &\Leftrightarrow \forall U\, \mathrm{Sat}\,(z', \xi \cup \{(X, U)\}),\end{aligned}$$

where $\xi \cup \{(X, U)\}$ is an extension of ξ with X assigned to U.

Lemma 3.6

In a model $V = (M, S)$ of WKL_0, we fix any $e \in M$ and an M-finite assignment map ξ. Then, there exists a $p \in M$ such that for all G_e formulas z whose free variables all belong to the domain of ξ, then $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^p(z, \xi)$ holds.

Proof. Since the domain of the assignment map ξ is M-finite, the set of G_e formulas whose free variables are in the domain of ξ is essentially M-finite (disregarding repetitions of the same formulas within a disjunction or conjunction). This fact can be demonstrated by Σ^0_1 induction on e.

Therefore, for M-finitely many G_e formulas z, if $\mathrm{Sat}(z, \xi)$ holds, let p_z be p such that $\mathrm{Sat}^p(z, \xi)$, or otherwise let $p_z = 0$. Then, if we put $q = \max\{p_z\}$, [1] then we have $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^q(z, \xi)$. □

[1] Strictly speaking, strong Σ^0_1 collection principle ($\mathrm{S}\Sigma_1$) is used here. (Refer to Problem 1 following Lemma 1.8 in Chapter 7.)

Definition 3.7 (reflection)

In a model V of WKL_0, for any e, p, and for two assignment maps ξ, ξ' with the same domain, the relation $\mathrm{Ref}^p_e(\xi, \xi')$ is defined as follows:

$$\mathrm{Sat}(z, \xi) \Rightarrow \mathrm{Sat}^p(z, \xi'), \text{for each } G_e \text{ formula } z \text{ with free variables in the domain of } \xi.$$

Lemma 3.8

In a model V of WKL_0, supposing $\mathrm{Ref}^p_e(\xi, \xi')$ with M-finite ξ, ξ', the following holds:

(1) If $e = 4d + 1$, $\forall a \, \exists a' < p \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$, where y is a variable not in the domain of ξ.

(2) If $e = 4d + 2$, for each numerical variable x belonging to ξ, $\forall a' < \xi'(x) \, \exists a < \xi(x) \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$, with y not in ξ.

(3) If $e = 4d + 3$, $\forall U \, \exists U' \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$, where Y is a variable not belonging to the domain of ξ.

(4) If $e = 4d + 4$, $\forall U' \, \exists U \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$, with Y not in ξ.

Proof Let $V = (M, S)$ be a model of WKL_0, and let ξ, ξ' be M-finite assignments with the same domain such that $\mathrm{Ref}^p_e(\xi, \xi')$ is satisfied.

(1) For $e = 4d + 1$. Show $\forall a \, \exists a' < p \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$.
Fix any $a \in M$. Let Z be the set of all codes of G_{e-1} formulas z satisfying $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ and in a non-redundant form (i.e., no same formula is repeated in disjunctions or conjunctions), whose free variables are either y or belong to the domain of ξ. According to the argument in the proof of Lemma 3.6, this set Z is M-finite within V. Thus, by (bounded Σ^0_1-CA) (Lemma 7.1.8), Z exists.

Now, consider a G_e-formula $z' = \exists y \bigwedge_{z \in Z} z$. Since $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ for each $z \in Z$, it follows from Lemma 3.5 that $\mathrm{Sat}(\bigwedge_{z \in Z} z, \xi \cup \{(y, a)\})$ and so $\mathrm{Sat}(z', \xi)$.

Therefore, by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds. Thus, there exists $a' < p$ such that $\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$ holds for each $z \in Z$, fulfilling the requirement.

(2) For $e = 4d + 2$. Show $\forall a' < \xi'(x) \, \exists a < \xi(x) \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$.
Fix any $a' < \xi'(x)$. To prove by contradiction, assume that for any $a < \xi(x)$ there exists a G_{e-1} formula z such that $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ and $\neg\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$.
Let Z be the set of all $z \in G_{e-1}$ satisfying $\neg\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$ and in a non-redundant form, whose free variables are either y or belong to the domain of ξ.

(2) (continued) Like in case (1), Z exists by (bounded Σ^0_1-CA). Consider a G_e formula $z' = \forall y < x \bigvee_{z \in Z} z$. By the other assumption, for each $a < \xi(x)$, there exists $z \in Z$ such that $\mathrm{Sat}(z, \xi \cup \{(y,a)\})$, so $\mathrm{Sat}(z', \xi)$ holds.

Therefore, by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds. Thus for each $a' < \xi'(x)$, there exists $z \in Z$ such that $\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$, which contradicts the definition of Z.

(3) For $e = 4d + 3$. $\forall U\, \exists U'\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y,U)\}, \xi' \cup \{(Y,U')\})$ can be shown like (1).

(4) For $e = 4d + 4$. Show $\forall U'\, \exists U\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y,U)\}, \xi' \cup \{(Y,U')\})$.
Fix any U'. Let Z be the set of $z \in G_{e-1}$ satisfying $\neg\mathrm{Sat}^p(z, \xi' \cup \{(Y,U')\})$ and in a non-redundant form, whose free variables are either y or belong to the domain of ξ. Consider a G_e formula $z' = \forall Y \bigvee_{z \in Z} z$. By contradiction, assume for each U, there exists $z \in Z$ such that $\mathrm{Sat}(z, \xi \cup \{(Y,U)\})$. Thus, $\mathrm{Sat}(z', \xi)$ holds, and by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds, which contradicts the definition of Z.

Thus, the proof is complete. □

Theorem 3.1 (Self-Embedding Theorem)

Let $\mathfrak{M} = (M, S)$ be a countable model of WKL_0 with $M \neq \omega$. Then, there exists a proper initial segment I of M such that $\mathfrak{M}\lceil I = (I, S\lceil I)$ is isomorphic to $\mathfrak{M}$.

Proof Let $V = (M, S)$ be a countable nonstandard model of WKL_0, and fix $q \in M$. Since V_q is M-finite within V, we can also make an M-finite mapping ξ_0 that assigns each number and set in V_q to distinct variables.
Now, take any nonstandard number $e \in M$. By Lemma 3.6, for any G_e-formula z whose free variables belong to the domain of ξ_0, there exists p such that $\mathrm{Sat}(z, \xi_0) \Leftrightarrow \mathrm{Sat}^p(z, \xi_0)$ holds.

In the following, by repeatedly using Lemma 3.8 (the back-and-forth method), we construct two ω-sequences of assignment mappings $\xi_0 \subseteq \xi_1 \subseteq \cdots \subseteq \xi_k \subseteq \ldots$ and $\xi'_0 \,(= \xi_0) \subseteq \xi'_1 \subseteq \cdots \subseteq \xi'_k \subseteq \ldots$ $(k \in \omega)$, where $\mathrm{Ref}^p_{e-k}(\xi_k, \xi'_k)$ holds for all $k \in \omega$, and $\bigcup_k \mathrm{range}(\xi_k) = V$ and $\bigcup_k \mathrm{range}(\xi'_k)$ forms the desired initial segment of the model V.

To begin with, we enumerate the elements of V as $M = \{a_i \mid i \in \omega\}$, $S = \{U_i \mid i \in \omega\}$. We inductively construct ξ_k, ξ'_k with the same domain $(k \in \omega)$ by cases:

(i) For $e - k = 4d + 1$. Let a be the element a_i in $M - \mathrm{range}(\xi_k)$ with the smallest index i, and let $a' < p$ be obtained by Lemma 3.8(1). Then, let y be a new numerical variable not in the domain of ξ_k, and set $\xi_{k+1} = \xi_k \cup \{(y, a)\}$, $\xi'_{k+1} = \xi_k \cup \{(y, a')\}$.

(ii) For $e-k=4d+2$. Let $\xi'_k(x_0)$ be the largest in the order in M among all $\xi'_k(x)$'s. Then, let a' be the element a_i in $M-\mathrm{range}(\xi'_k)$ and satisfying $a_i<\xi'_k(x_0)$ with the smallest index i, and let $a<\xi(x_0)$ be obtained by Lemma 3.8(2). Then, let y be a new numerical variable, and set $\xi_{k+1}=\xi_k\cup\{(y,a)\}$, $\xi'_{k+1}=\xi_k\cup\{(y,a')\}$.

(iii) For $e-k=4d+3$. Let U be $U_i\in S$ with the smallest index i, that is different from any set in $\mathrm{range}(\xi_k)$ with regards to the numbers in $\mathrm{range}(\xi_k)$. Also, let U' be obtained by Lemma 3.8(3). Then, let Y be a new set variable, and set $\xi_{k+1}=\xi_k\cup\{(Y,U)\}$, $\xi'_{k+1}=\xi_k\cup\{(Y,U')\}$.

(iv) For $e-k=4d+4$. Let U' be $U_i\in S$, with the smallest index i, that is different from any set in $\mathrm{range}(\xi'_k)$ with regards to the numbers in $\mathrm{range}(\xi'_k)$. Also, let U be obtained by Lemma 3.8(4). Then, let Y be a new set variable, and set $\xi_{k+1}=\xi_k\cup\{(Y,U)\}$, $\xi'_{k+1}=\xi'_k\cup\{(Y,U')\}$

From the above construction, it is easy to see that $\mathrm{Ref}^p_{e-k}(\xi_k,\xi'_k)$ holds for each $k\in\omega$.

From (i) and (iii), it is obvious that $\bigcup_k \mathrm{range}(\xi_k)=(M,S)$. Also, from (ii), we can easily see that the set I consisting of a belonging to $\bigcup_k \mathrm{range}(\xi'_k)$ forms an initial segment of M. Then, from (iv) it follows that $\bigcup_k \mathrm{range}(\xi'_k)=(I,S\lceil I)$.

Next, we prove by induction that both ξ_k, ξ'_k are injective for all $k \in \omega$. It is clear from the definition that $\xi_0 = \xi'_0$ is injective.

In (i), we first extend the injective mapping ξ_k to an injective ξ_{k+1}, and then extend the injective ξ'_k to a mapping ξ'_{k+1} that satisfies $\mathrm{Ref}^p_{e-k-1}(\xi_{k+1}, \xi'_{k+1})$. The injectivity of ξ_{k+1} is clear from the construction. Since the injectivity is expressed by a G_2 formula, ξ'_{k+1} is also injective.

Similarly for (ii), (iii) and (iv).

Thus, $\bigcup_k \xi_k$ and $\bigcup_k \xi'_k$ are also injective.
Let $f = (\bigcup_k \xi'_k) \circ (\bigcup_k \xi_k)^{-1}$, which becomes a bijection from V to $V\lceil I$. It is evident that f acts as the identity map on V_q.
Furthermore, since $\mathrm{Ref}^p_0(\xi_k, \xi'_k)$ holds for each $k \in \omega$, it is clear that f is an isomorphism. Thus, the proof of the theorem is complete. □

Let's briefly describe how the Self-Embedding Theorem 3.1 can be applied to nonstandard analysis.

- According to Gödel's completeness theorem and compactness theorem,

$$\mathsf{WKL}_0 \vdash \varphi \Leftrightarrow \text{for any non-}\omega\text{ model } \mathfrak{M} \text{ of } \mathsf{WKL}_0, \mathfrak{M} \models \varphi.$$

- Since any infinite structure has an elementarily equivalent countable structure by the Löwenheim-Skolem Theorem,

$$\mathsf{WKL}_0 \vdash \varphi \Leftrightarrow \text{for any countable non-}\omega\text{ model } \mathfrak{M} \text{ of } \mathsf{WKL}_0, \mathfrak{M} \models \varphi.$$

- Choose a countable non-ω model $\mathfrak{M} = (M, S)$ of WKL_0. Theorem 3.1 states that $\mathfrak{M}$ has an initial segment isomorphic to itself. But by swapping their roles of $\mathfrak{M}$ and an isomorphic initial segment, $\mathfrak{M}$ is seen to have an isomorphic end-extension ${}^*\mathfrak{M} = ({}^*M, {}^*S)$, which allows us to carry out some nonstandard analysis arguments.
- For example, in $\mathfrak{M} = (M, S)$, a real number a is indeed a set in S. Thus, a is an initial segment ${}^*a\lceil M$ of some set ${}^*a \in {}^*S$. Since ${}^*a$ may be taken bounded in ${}^*\mathfrak{M}$, it can be coded by an element of ${}^*M$. Therefore, a real number in $\mathfrak{M}$ can be treated like a rational number in ${}^*\mathfrak{M}$.

Application (The Maximum Principle)

$\mathsf{WKL}_0 \vdash$ Any continuous function $f : [0,1] \to [0,1]$ has a maximum value.

Proof.

$$\mathfrak{M} = (M, S) \qquad\qquad {}^*\mathfrak{M} = ({}^*M, {}^*S)$$

$$\begin{array}{ccc} f : [0,1] \cap \mathbb{Q} \to [0,1] & \Longrightarrow & {}^*f : \{q_i\}_{i<a} \to 2^b \\ \| \qquad\qquad \| & & (a, b \in^* M - M, f =^* f \cap M) \\ \{q_i\}_{i \in M} \qquad 2^M & & \Downarrow \\ {}^*m \cap M \text{ is } \sup f & \Longleftarrow & {}^*m = \max\{{}^*f(q_i)\}_{i<a} \end{array}$$

Other Applications

$\mathsf{WKL}_0 \vdash$ The Cauchy-Peano Theorem (Tanaka, 1997)

$\mathsf{WKL}_0 \vdash$ The existence of Haar measure for a compact group (Tanaka-Yamazaki, 2000)

$\mathsf{WKL}_0 \vdash$ The Jordan curve theorem (Sakamoto-Yokoyama, 2007)

§4. STY Theorem

In §1, we proved Harrington's theorem that WKL_0 is conservative over RCA_0 with respect to Π^1_1 sentences. The proof utilized the tree forcing argument.

The STY theorem, standing for Simpson-T.-Yamazaki, extends Harrington's conservation result to the class of senteces in the form $\forall X \exists! Y \varphi(X,Y)$ (where $\varphi(X,Y)$ is arithmetic) [2]

In the original proof of the STY theorem, the forcing argument over so-called universal trees is devised to enable the construction of models with stronger properties. However, due to its technical complexity, we here adopt a new method of **symmetric models** composed of generic set sequences, also introduced by Simpson (2000).

[2] A formula in this form is called "Tanaka" and a formula obtained from Tanaka formulas applying $\vee, \wedge, \forall x, \exists y$ and $\forall X$ is called "G-Tanaka." Shore (JSL 2023) further extended the conservation to the G-Tanaka formulas.

Theorem 4.1 (STY theorem)

For any sentence σ in the form $\forall X \exists! Y \varphi(X,Y)$ (where $\varphi(X,Y)$ is arithmetic),

$$\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{RCA}_0 \vdash \sigma,$$

where $\exists! Y \varphi(X,Y)$ means $\exists Y \varphi(X,Y) \wedge \forall Y_1 \forall Y_2 (\varphi(X,Y_1) \wedge \varphi(X,Y_2) \to Y_1 = Y_2)$.

A key to the proof of this theorem is the following lemma.

Lemma 4.2

Let $\mathfrak{M} = (M,S)$ be a countable nonstandard model of RCA_0 with $A \in S$. Then, there exist sets S_1 and S_2 satisfying the following conditions:

1. $S_1 \cap S_2 = \mathrm{Rec}^{\mathfrak{M}}(A) = \{X \subseteq M \mid \mathfrak{M} \models X \leq_T A\}$
2. $(M,S_i) \models \mathsf{WKL}_0$, for $i = 1,2$.
3. (M,S_1) and (M,S_2) satisfies the same sentences in $\mathcal{L}_2(M \cup \{A\})$.

In the above lemma, it is not necessary that S contains $S_1 \cup S_2$. Also, since elements of S other than A are not essentially used, it is sufficient for the lemma that $(M, \{A\})$ is a countable model of Σ^0_1 induction. We first assume the lemma to prove the main theorem.

Proof of Theorem 4.1 Suppose $\mathsf{WKL}_0 \vdash \forall X \exists! Y \varphi(X, Y)$ with an arithmetic formula $\varphi(X, Y)$. For contradiction, assume $\mathsf{RCA}_0 \nvdash \forall X \exists! Y \varphi(X, Y)$. By the completeness theorem, there exists a countable model $\mathfrak{M} = (M, S)$ of RCA_0 such that

$$(M, S) \models \neg \forall X \exists! Y \varphi(X, Y).$$

Consequently, there exists some $A \in S$ such that either

(i) $(M, S) \models \exists Y_1 \exists Y_2 (\varphi(A, Y_1) \wedge \varphi(A, Y_2) \wedge Y_1 \neq Y_2)$, or

(ii) $(M, S) \models \forall Y \neg \varphi(A, Y)$.

Case (i) There exist $B_1, B_2 \in S$ such that $(M, S) \models \varphi(A, B_1) \wedge \varphi(A, B_2) \wedge B_1 \neq B_2$. By Lemma 1.9 (Harrington's lemma), there exists $S' \supseteq S$ such that $(M, S') \models \mathsf{WKL}_0$. Since (M, S) and (M, S') agree on first-order parts, they validate the same arithmetic formulas. Hence, $(M, S') \models \varphi(A, B_1) \wedge \varphi(A, B_2) \wedge B_1 \neq B_2$. However, since $\mathsf{WKL}_0 \vdash \forall X \exists! Y \varphi(X, Y)$, we have $(M, S') \models \forall X \exists! Y \varphi(X, Y)$, a contradiction.

Case (ii) By Lemma 4.2, there exist sets S_1 and S_2 such that

(a) $S_1 \cap S_2 = \mathsf{Rec}^{\mathfrak{M}}(A)$,

(b) $(M, S_i) \models \mathsf{WKL}_0$,

(c) (M, S_1) and (M, S_2) satisfy the same sentences of $\mathcal{L}_2(M \cup \{A\})$.

From (b) and $\mathsf{WKL}_0 \vdash \forall X \exists ! Y \varphi(X, Y)$, there exists a unique $B_i \in S_i$ such that $(M, S_i) \models \varphi(A, B_i)$ for each $i = 1, 2$. By (c), for any $n \in M$,

$$\begin{aligned} n \in B_1 &\Leftrightarrow (M, S_1) \models \exists Y (\varphi(A, Y) \wedge n \in Y) \\ &\Leftrightarrow (M, S_2) \models \exists Y (\varphi(A, Y) \wedge n \in Y) \\ &\Leftrightarrow n \in B_2 \end{aligned}$$

Therefore, $B_1 = B_2$ and thus $B_1 \in S_1 \cap S_2$. From (a), $B_1 \in \mathsf{Rec}^{\mathfrak{M}}(A)$. Since (M, S) is a model of RCA_0 and $B_1 \in S$, $(M, S) \models \exists Y \varphi(A, Y)$, a contradiction. □

In the following, we will introduce several new concepts such as a generic sequence, to proceed with the proof of Lemma 4.2.

First, let us consider $\mathfrak{M} = (M, S)$ as a countable nonstandard model of WKL_0 [3]. Take any $A \in S$ and consider the formulas involving it. If $\varphi(X, A)$ is a Π^0_1 formula with a unique free variable X and a parameter A, the set $\{X \in S \mid \mathfrak{M} \models \varphi(X, A)\}$ is called a $\Pi^{0,A}_1$ **class** in $\mathfrak{M}$. Note that a set $P \subseteq S$ is a $\Pi^{0,A}_1$ class iff there exists a binary tree $T \subseteq 2^{<M}$ recursive in A such that $P = [T]$. Here, $[T]$ represents the set of all infinite paths through a tree T.

From now on, the display of parameter A is omitted due to complexity in description. By $\langle P_e \mid e \in M\rangle$, we denote a computable enumeration of all Π^0_1 classes. Formally, using the Π^0_1 satisfaction predicate $\mathrm{Sat}_{\Pi^0_1}(x, X)$, we define it as: for any $e \in M, X \in S$,

$$X \in P_e \Leftrightarrow \mathfrak{M} \models \mathrm{Sat}_{\Pi^0_1}(e, X).$$

We also write $P_e(X)$ for $X \in P_e$.

[3] Note that in the claim of Lemma 4.2, $\mathfrak{M} = (M, S)$ was a countable nonstandard model of RCA_0.

Definition 4.3

For an M-finite subset $p \subseteq M \times M^{<M}$ (denoted as $p \subseteq_{\mathrm{fin}} M \times M^{<M}$) [4], a sequence of sets $\langle X_n \mid n \in M \rangle$ **meets** p, if for every $(e, \langle n_1, \cdots, n_k \rangle) \in p$,

$$X_{n_1} \oplus \cdots \oplus X_{n_k} \in P_e,$$

where $X_{n_1} \oplus \cdots \oplus X_{n_k} = \{(x,1) \mid x \in X_{n_1}\} \cup \{(x,2) \mid x \in X_{n_2}\} \cdots \cup \{(x,k) \mid x \in X_{n_k}\}$. The condition $X_{n_1} \oplus \cdots \oplus X_{n_k} \in P_e$ is also expressed as $P_e(X_{n_1}, \cdots, X_{n_k})$.

Definition 4.4

Define a p.o. set $(\mathbb{P}^{\mathfrak{M}}, \leq)$ as follows:

$$\mathbb{P}^{\mathfrak{M}} = \{p \subseteq_{\mathrm{fin}} M \times M^{<M} \mid \text{there exists } \langle X_n \mid n \in M \rangle \in S^M \text{ that meets } p\},$$

and the order $p \leq q$ on $\mathbb{P}^{\mathfrak{M}}$ is defined as $p \supseteq q$. [5]

[4] $M^{<M}$, i.e., $\mathrm{Seq}^{\mathfrak{M}}$, includes all M-finite sequences from M.
[5] The reason why the order is the reverse inclusion is that when $q \subseteq p$, p has more conditions, hence fewer sequences meet it.

In WKL_0, the condition that "there exists $\langle X_n \mid n \in M \rangle \in S^M$ that meets p" can be rephrased as the existence of an infinite path in an infinite tree, since the part "(something) meets p" is a Π^0_1 condition. Thus by compactness, the whole condition can be expressed by a Π^0_1 formula.

Furthermore, $p(\subseteq_{\text{fin}} M \times M^{<M})$ can be considered an element of M, so $\mathbb{P}^{\mathfrak{M}}$ can be regarded as a Π^0_1 subset of M.

Henceforth, unless otherwise stated, $\mathbb{P}^{\mathfrak{M}}$ will simply be referred to as $\mathbb{P}$.

A sequence $\langle G_n \mid n \in M \rangle$ is said to be a **generic sequence** if for any dense subset $D \in \mathrm{Def}(\mathfrak{M})$ of $\mathbb{P}$, there exists a $p \in D$ that $\langle G_n \mid n \in M \rangle$ meets. [6]

[6] Even if some G_n does not belong to S, the definition remains valid as long as their existence does not violate the Σ^0_1 induction.

Thank you for your attention!