Logic and Foundations II

Part 8. Second order arithmetic and non-standard methods

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Real analysis and reverse mathematics (8.5 lectures)
- Part 8. Second order arithmetic and non-standard methods (6.5 lectures)

Part 8. Schedule

- May 21, (0) Introduction to forcing
- May 23, (1) Harrington's conservation result on WKL_0
- May 28, (2) H.Friedman's conservation result on WKL_0
- May 30, (3) Friedman's result (continued) and a self-embedding theorem I
- June 04, (4) A self-embedding theorem II
- June 06, (5)
- June 11, (6)

§8.2. Semi-Regular Cuts and Friedman's Theorem

The goal of this section is to prove a theorem of H. Friedman that "WKL_0 is Π^0_2 conservative over PRA." Primitive Recursive Arithmetic PRA consists of defining axioms for the primitive recursive functions, together with Σ_0 induction.

We fix a nonstandard model (M, F) of PRA (i.e., $M \neq \omega$). Also, let $p \in F$ be a primitive recursive function that lists the prime numbers in the ascending order, i.e., $p(0) = 2, p(1) = 3, p(2) = 5, \cdots$.

Definition 2.1

A set $X(\subseteq M)$ has a **code** $c \in M$ if $X = \{n \in M : M \models \exists d < c\ (c = p(n) \cdot d)\}$.
Such a set X is called **M-finite**, and the number of elements in X is denoted by $|X|$ or $|c|$.

Definition 2.2

A proper initial segment I of M is called a **cut** of M, denoted $I \subseteq_e M$, if it is closed under the successor function (i.e., $a \in I \Rightarrow a + 1 \in I$).
Furthermore, a cut $I \subseteq_e M$ is called a **semi-regular cut**, if $X \cap I$ is bounded within I for any M-finite set X with $|X| \in I$.

Let (M, F) be a nonstandard model of PRA.

Theorem 2.3 (Kirby-Paris)

If $I \subseteq_e M$ is a semi-regular cut, then $(I, F\lceil I) \models$ PRA, where $F\lceil I$ is the set of functions obtained by restricting the domain of each function f in F to I.

Proof For each $n \in \omega$, define the unary primitive recursive function g_n as follows:

$$\mathrm{g}_0(x) = x + 1, \quad \mathrm{g}_{n+1}(x) = \overbrace{\mathrm{g}_n \mathrm{g}_n \cdots \mathrm{g}_n}^{x+2}(x)$$

We can show for any primitive recursive function $\mathtt{f}$, there exists some $n \in \omega$ such that

$$\mathsf{PRA} \vdash \mathtt{f}(x_1, x_2, \cdots, x_k) < \mathrm{g}_n(\max\{x_1, x_2, \cdots, x_k\})$$

To confirm that I is closed under all g_n, by way of contradiction, assume it is closed under g_n, but not g_{n+1}. Then we can choose $a \in I$ such that $\mathrm{g}_{n+1}^M(a) \notin I$, and define

$$X = \{\mathrm{g}_n^M(a), \mathrm{g}_n^M \mathrm{g}_n^M(a), \cdots, \overbrace{\mathrm{g}_n^M \mathrm{g}_n^M \cdots \mathrm{g}_n^M}^{a+2}(a)\}.$$

Since X is an M-finite set with $|X| = a + 2 \in I$, so $X \cap I$ is bounded and has a maximum element b. However, since I is closed under g_n, we have $\mathrm{g}_n^M(b) \in X \cap I$, contradicting the maximality of b. So, $(I, F\lceil I)$ is a substructure of (M, F), and also satisfies Σ_0-induction.□

Definition 2.4

Let $I \subseteq_e M$ and let S be the set of all M-finite sets. $B = X \cap I$ is called an **M-coded set** if $X \in S$. The set of all M-coded subsets of I is denoted by $S\lceil I$.

Note. We can consider $(I, S\lceil I)$ as a structure of second-order arithmetic, with basic operations $+^I, \cdot^I$, etc., which are obtained by restricting the corresponding operations (primitive recursive functions) on M to I.

Lemma 2.5

If $I \subseteq_e M$ is a semi-regular cut, then $(I, S\lceil I) \models \mathsf{WKL}_0$.

Proof First, for a Σ^0_0 formula θ in $(I, S\lceil I)$, we construct a Σ_0 formula θ^* in (M, F) by replacing every atomic formula $t \in B$ in θ with $\exists d < c_B\ (c_B = p(t) \cdot d)$, where c_B is a code of X such that $B = X \cap I$. Then, for any $a \in I$, $(I, S\lceil I) \models \theta(a) \ \Leftrightarrow\ (M, F) \models \theta^*(a)$.

To prove $(I, S\lceil I)$ satisfies (bounded Σ^0_1-CA), take a Σ^0_1 $\varphi(x) = \exists y \theta(x, y)$ and any $c \in I$. Then, for all $a <_M c$, $(I, S\lceil I) \models \varphi(a) \Leftrightarrow \exists b \in I\ (I, S\lceil I) \models \theta(a, b) \Leftrightarrow \exists b \in I\ (M, F) \models \theta^*(a, b) \Leftrightarrow (M, F) \models \exists y < d'\ \theta^*(a, y)$ for some $d' \in I$. Thus, $\{x < c \mid \varphi(x)\} \in S\lceil I$, since $X = \{a < c : (M, F) \models \exists y < d'\ \theta^*(a, y)\}$ has a code $\Pi_{a \in X} p(a)$. Hence, $(I, S\lceil I)$ satisfies (bounded Σ^0_1-CA). Similarly, $(I, S\lceil I) \models (\Sigma^0_1\text{-SP})$. Therefore, $(I, S\lceil I) \models \mathsf{WKL}_0$. □

Lemma 2.6

Let (M, F) be a countable nonstandard model of PRA. Take $c, d \in M$ such that for all primitive recursive functions $\mathtt{f}$, $\mathtt{f}^M(c, c, \cdots, c) <_M d$. Then, there exists a semi-regular cut $I \subseteq_e M$ such that $c \in I$ and $d \notin I$.

Proof First, define the primitive recursive predicate $B(x, y, z)$ as follows:

- $B(0, y, z) \Leftrightarrow y < z$,
- $B(x+1, y, z) \Leftrightarrow$ for any M-finite set $X \subset [y, z)$ with $|X| \leq y$, there exists $[y', z') \subset [y, z)$ such that $B(x, y', z')$ and $[y', z') \cap X = \emptyset$

Here, $[y, z) = \{w : y \leq w < z\}$.

Now, when $B(x, y, z)$ holds, we say "the interval $[y, z)$ is x-large." Then, the interval $[y, z)$ is $(x+1)$-large iff for any subset $X \subset [y, z)$ with $|X| \leq y$, there exists a subinterval $[y', z') \subset [y, z)$ that is x-large and disjoint from X.

We observe that the definition of $B(x+1, y, z)$ is Σ_0, since a subset $X \subset [y, z)$ with cardinality at most y can be encoded by a number at most $p(z)^y$. So this makes $B(x, y, z)$ a primitive recursive predicate.

For the sequence $\{\mathrm{g}_n\}$ of primitive recursive functions constructed in the proof of Theorem 2.3, it can be shown that for each $n \in \omega$,

$$\mathsf{PRA} \vdash \mathrm{g}_n(y) \leq z \ \rightarrow B(n, y, z).$$

Indeed, this is clear when $n = 0$. Assuming it holds for n, let's show it for $n + 1$. Suppose $\mathrm{g}_{n+1}(y) \leq z$. Since $\mathrm{g}_{n+1}(y) = \mathrm{g}_n^{y+2}(y)$, for any subset $X \subset [y, z)$ with $|X| \leq y$, there exists some $c < y + 2$ such that the interval $[\mathrm{g}_n^c(y), \mathrm{g}_n^{c+1}(y))$ does not contain any element of X. Let $y' = \mathrm{g}_n^c(y)$ and $z' = \mathrm{g}_n^{c+1}(y)$. Then $\mathrm{g}_n(y') = z'$. So by the inductive hypothesis, $B(n, y', z')$ holds, which fulfills the definition of $B(x + 1, y, z)$.

Next, take $c, d \in M$ as in the statement of the lemma. Then for any $n \in \omega$, $\mathrm{g}_n^M(c) <_M d$, and so $B(n, c, d)$. By the overspill principle, there exists $b \in M - \omega$ such that $\forall a \leq_M b \ B(a, c, d)$.

Now, since (M, F) is a countable model of PRA, there are only countably many M-finite sets. So, we can construct a sequence of M-finite sets $\{X_n\}$, such that each M-finite set appears infinitely often in the sequence. Using this, we define the decreasing sequence of intervals $\{[c_n, d_n)\}$ as follows:

$$[c_0, d_0) = [c, d),$$

$$[c_{n+1}, d_{n+1}) = \begin{cases} [c_n, d_n) & \text{if } |X_n| \geq_M c_n, \\ [c', d') & \text{otherwise, take any } [c', d') \subset [c_n, d_n) \text{ such that} \\ & B(b-n, c', d') \text{ and } [c', d') \cap X_n = \varnothing. \end{cases}$$

For any $a \in M$, obviously $\{a\}$ is M-finite, so for sufficiently large n, $[c_n, d_n) \cap \{a\} = \emptyset$, that is, $a \notin [c_n, d_n)$. Therefore, $\bigcap_n [c_n, d_n) = \varnothing$.

Now, let $I = \{a \in M : \exists n\ a <_M c_n\} = \{a \in M : \forall n\ a <_M d_n\}$. We show that I becomes a semi-regular cut. If X is M-finite and $|X| \in I$, by the definition of $\{X_n\}$, there are infinitely many n such that $X = X_n$. Then, there exists n such that $X = X_n$ and $|X| <_M c_n$. Thus, $[c_{n+1}, d_{n+1}) \cap X = \varnothing$. Therefore, $X \cap I$ is bounded by c_{n+1} in I. Hence, I is a semi-regular cut. □

Theorem 2.7 (Friedman)

For any Π_2 sentence σ, $\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{PRA} \vdash \sigma$.

Proof. To show the contraposition, take a Π_2 sentence $\sigma = \forall y \exists z \theta(y, z)$ with $\theta \in \Sigma_0$ that is not provable in PRA. Then, $\mathsf{PRA} \cup \{\neg \exists z \theta(c, z)\} \cup \{\mathtt{f}(c, c, \cdots, c) < d : \mathtt{f}$ is a symbol of a primitive recursive function$\}$ is consistent, and hence by the completeness theorem, it has a countable model (M, F, c, d). Now, by Lemma 2.6, there exists a semi-regular cut $I \subseteq_e M$ such that $c \in I$ and $d \notin I$. Since $\neg \exists z \theta(c, z)$ is a Π_1 sentence and $M \models \neg \exists z \theta(c, z)$, it follows that $I \models \neg \exists z \theta(c, z)$, i.e., $I \models \neg \sigma$. On the other hand, by Lemma 2.5, we have $(I, S\lceil I) \models \mathsf{WKL}_0$. Thus, $(I, S\lceil I) \models \mathsf{WKL}_0 + \neg \sigma$, and so $\mathsf{WKL}_0 + \neg \sigma$ is consistent, hence σ cannot be proved in WKL_0 either. □

As we saw in part 7, a wide range of mathematics can be developed within WKL_0. Nevertheless, Friedman's theorem shows that WKL_0 is Π_2-conservative over PRA, which can be viewed as a partial realization of Hilbert's program or his "finitistic reductionism."

Hilbert's Program

The main goal of Hilbert's program was to provide secure foundations for all mathematics, to counteract the intuitionism, led by Brouwer who had been attacking non-constructive methods in mathematics. Hilbert proposed the method of "proof theory" or "meta-mathematics", by which mathematical arguments are treated as symbolic manipulations, and thus can be analyzed themselves mathematically.

Let T be a large system (e.g., set theory ZFC) that can develop most of mathematics. Let t be a small system (e.g., PRA) capable of performing symbolic manipulatios of T. Then, Hilbert considered that a Π^0_1 sentence which does not assert existence (e.g., Fermat's Last Theorem: $\forall n > 2 \forall x, y, z > 0(x^n + y^n \neq z^n)$) would be provable in t if it is provable in T. Therefore, the validity of a Π^0_1 sentence may be argued with non-constructive methods.

Hilbert's (reductionism) program HP

HP: for any Π^0_1 sentence φ, if $T \vdash \varphi$ then $t \vdash \varphi$.

Theorem 2.8

Suppose both T and t include PRA. Then, for any Π^0_1 sentence φ, if $T \vdash \varphi$, then $t + \mathrm{Con}(T) \vdash \varphi$. Here, $\mathrm{Con}(T)$ is a Π^0_1 sentence expressing the consistency of T.

Proof. Let $\varphi \equiv \forall n \theta(n)$ (where $\theta(n)$ is Σ^0_0 or primitive recursive), and assume $T \vdash \varphi$. So, since $\mathsf{Bew}_T(\overline{\ulcorner \varphi \urcorner})$ is a true Σ^0_1 sentence, by the Σ^0_1-completeness of t, $t \vdash \overline{\mathsf{Bew}_T(\ulcorner \varphi \urcorner})$. On the other hand, from the proof of Lemma 4.5.1 D3, $t \vdash \neg\theta(n) \to \mathsf{Bew}_t(\ulcorner \neg\theta(\bar{n}) \urcorner)$, i.e., $t \vdash \neg\theta(n) \to \mathsf{Bew}_t(\overline{\ulcorner \neg\varphi \urcorner})$. Since $\mathsf{Bew}_t(\overline{\ulcorner \neg\varphi \urcorner}) \to \mathsf{Bew}_T(\overline{\ulcorner \neg\varphi \urcorner})$, it follows that $t \vdash \neg\theta(n) \to \neg\mathsf{Con}(T)$. Therefore, $t + \mathrm{Con}(T) \vdash \theta(n)$, and thus $t + \mathrm{Con}(T) \vdash \varphi$. □

By this theorem, if $t \vdash \mathrm{Con}(T)$, then HP holds. However, by Gödel's second incompleteness theorem, $\mathrm{Con}(T)$ is not provable in T, hence also not in t.

However, for $T = \mathsf{WKL}_0$ and $t = \mathsf{PRA}$, HP is shown to hold by Friedman's theorem. Observing the richness of mathematics developed in WKL_0, one can view that "Hilbert's program" has been partially realized. Those skeptical about the meaning of HP still likely agree on the importance of rewriting a proof of a Π^0_1 sentence involving non-constructive arguments like weak König's lemma into a constructive proof without them.

§8.3. A self-embedding theorem of WKL_0

In this section, we introduce a self-embedding theorem of WKL_0, by which we can devise methods of nonstandard analysis in WKL_0.

Gödel stated in 1973 that "nonstandard analysis is the future of analysis." However, Henson and Keisler have shown in 1986 that nonstandard arguments in n-th order arithmetic require $(n+1)$-th order arithmetic. Therefore, conducting complete nonstandard analysis for second-order arithmetic Z_2 is impossible within the framework of second-order arithmetic alone. Nevertheless, as demonstrated in my paper[1], certain amount of nonstandard analysis can still be developed within WKL_0.

The main tool of our nonstandard method is a self-embedding theorem of WKL_0 (Theorem 3.1), which extends Friedman's self-embedding theorem (§5.3) to WKL_0. This section primarily discusses the proof of this theorem.

[1] K. Tanaka, The self-embedding theorem of WKL_0 and a non-standard method, *Annals of Pure and Applied Logic* 84 (1997), pp.41–49.

Theorem 3.1 (Self-Embedding Theorem)

Let $\mathfrak{M} = (M, S)$ be a countable model of WKL_0 with $M \neq \omega$. Then, there exists a proper initial segment I of M such that $\mathfrak{M}\lceil I = (I, S\lceil I)$ is isomorphic to $\mathfrak{M}$. Here, $S\lceil I = \{X \cap I \mid X \in S\}$.

Before proving this theorem, we need some preparations. We first prove the following lemma, which will be frequently used later.

Lemma 3.2 (Compactness in WKL_0)

(1) For any Π^0_1 formula $\varphi(X)$, there exists a Π^0_1 formula $\hat{\varphi}$ such that WKL_0 proves:

$$\hat{\varphi} \leftrightarrow \exists X\, \varphi(X).$$

(2) For any Π^0_1 formula $\varphi(k, X)$, WKL_0 proves:

$$\forall n\, \exists X\, \forall k < n\, \varphi(k, X) \to \exists X\, \forall k\, \varphi(k, X).$$

From now on, we adopt the notation $[T]$ for the set of all infinite paths of a tree T. Do not confuse it with $[p]$, which represents a basic open set in the order topology.

Proof. (1) We identify a set X with its characteristic function, which is also represented as an infinite binary sequence. Then, a Π^0_1 formula $\varphi(X)$ can be expressed as $\forall x\, \theta(X{\restriction}x)$, where θ is Σ^0_0 and $X{\restriction}x$ is a code for a finite binary sequence. We set $T = \{t \mid \forall s \subseteq t\ \theta(s)\}$. Then T is a tree, and $X \in [T]$ iff $\varphi(X)$ holds. Thus, $\exists X\, \varphi(X)$ is equivalent to $[T] \neq \varnothing$, which is expressed as a Π^0_1 formula "T is infinite ($\forall n \exists t \in \{0,1\}^n t \in T$)".

(2) Express a Π^0_1 formula $\varphi(k, X)$ as $\forall x\ \theta(k, X{\restriction}x)$ (where θ is Σ^0_0), and define a tree $T = \{t \mid \forall k \leq \mathrm{leng}(t) \forall x \leq \mathrm{leng}(t)\ \theta(k, t\lceil x)\}$. Here, $\mathrm{leng}(t)$ denotes the length of the finite binary sequence t. If $\forall n\, \exists X\, \forall k < n\, \varphi(k, X)$ holds, then $\forall n\, \exists X\, \forall k < n\, \forall x < n\ \theta(k, X\lceil x)$, so $t = X\lceil n \in T$ for all n, thus T is infinite. Hence, in WKL_0, there exists an infinite path $X \in [T]$ satisfying $\forall k\, \varphi(k, X)$. □

Here is another demonstration for (2). If we express $\varphi(k, X)$ as $X \in [T_k]$, then $\exists X\, \forall k < n\, \varphi(k, X)$ can be expressed as $\bigcap_{k<n}[T_k] \neq \varnothing$. Since this is true for any n, we have $\bigcap_{k<\infty}[T_k] \neq \varnothing$ by the compactness of the Cantor space since $[T_k]$'s are closed sets.

Both (1) and (2) are referred to as "compactness (of binary trees) in WKL_0".

We define **G-Σ^0_1 formulas** or simply **G formulas** by generalizing Σ^0_1 formulas as follows. The G formulas are obtained from Σ^0_1 formulas by using $\land, \lor$, bounded universal quantifier $\forall x < y$ and unbounded existential quantifier $\exists x$, and set quantifiers $\forall X, \exists X$.

In WKL_0, we can prove that a G formula is equivalent to a Σ^0_1 formula. To prove it, it suffices to show that the class of Σ^0_1 formulas is closed under set quantifiers $\forall X, \exists X$, because the closure condition under $\forall x < y$ is nothing but the collection principle $\mathsf{B}\Sigma^0_1$ derivable from Σ^0_1 induction, and the other closure conditions are almost obvious.

The closure condition under $\forall X$ can be obtained from Lemma 3.2(1) by taking the negation on both sides. The closure condition under $\exists X$ can be demonstrated by noting that $\exists X \exists x\ \theta(x, X\lceil x)$ (where θ is Σ^0_0) can be rewritten as $\exists t \exists x\ \theta(x, t)$.

Now, we redefine the G-formulas explicitly in RCA_0 in the next slide.

Definition 3.3 (G-formulas)

A sequence $G_0 \subset G_1 \subset G_2 \subset \cdots$ of sets of $\mathcal{L}^2_{\mathrm{OR}}$-formulas is defined inductively modulo 4 as follows: for each $e \in \mathbb{N}$,

$$\begin{aligned} G_0 &= \{\text{finite disjunctions } (\vee) \text{ of atomic formulas or their negations}\}, \\ G_{4e+1} &= \{\exists x\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e} \text{ formulas}\} \cup G_{4e}, \\ G_{4e+2} &= \{\forall x < y\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+1} \text{ formulas}\} \cup G_{4e+1}, \\ G_{4e+3} &= \{\exists X\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e+2} \text{ formulas}\} \cup G_{4e+2}, \\ G_{4e+4} &= \{\forall X\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+3} \text{ formulas}\} \cup G_{4e+3}. \end{aligned}$$

Finally, we set $\mathbf{G} = \bigcup_{e \in \mathbb{N}} G_e$. The formulas in G are called **G-formulas**.

By Lemma 5.5.3, there is no formula that defines the truth values of all formulas. But, Lemma 5.3.4 shows that if we restrict the formulas to a class like Σ_n, then there exists a formula Sat_{Σ_n} to define the truth values of formulas in the class. This is also the case for Σ^0_n in second order arithmetic. In the following, we will define Sat for G-formulas.

From now on, a structure $\mathfrak{M} = (M, S)$ is denoted by V. Then, for each $p \in M$, set $M_p = \{a \in M \mid \mathfrak{M} \models a < p\}$, $S_p = \{X \cap M_p \mid X \in S\}$ and denote $V_p = (M_p, S_p)$.

Since M_p may not be closed under operations such as addition, V_p may not be a substructure of V. However, just by restricting the ranges of variables to these sets, the **satisfaction predicate** $\mathrm{Sat}^p(z, \xi)$ for V_p can be naturally defined within $V = (M, S)$. Here, z represents the code of a formula φ, and ξ is a finite function that assigns elements of $M_p \cup S_p$ to free variables appearing in φ. Thus, supposing that a formula $\varphi(\vec{x}, \vec{X})$ has no free variables other than $\vec{x}, \vec{X}$, and $\xi(\vec{x}) = \vec{a}, \xi(\vec{X}) = \vec{U}$, we have in V,

$$\mathrm{Sat}^{\mathbf{p}}(\ulcorner\varphi\urcorner, \xi) \equiv \varphi(\tilde{\mathbf{a}}, \tilde{\mathbf{U}})^{\mathbf{V_p}}, \text{ roughly } V_p \models \varphi(\vec{a}, \vec{U}).$$

Here, in $\varphi(\vec{a}, \vec{U})^{V_p}$, quantification over numbers is bounded by p, and quantification over sets is also considered as ranging binary sequences of length p, which can be coded by numbers $< 2^p$. Thus, $\mathrm{Sat}^p(z, \xi)$ can be defined as a Δ^0_1 formula in V (cf. Lemma 5.3.4).

We also remark that z in $\mathrm{Sat}^p(z, \xi)$ is a variable which can potentially express a non-standard number. In V, it can be easily verified that Sat^p satisfies Tarski's truth definition clauses for all standard formulas (cf. Theorem IV.2.26 in [P. Hájek and P. Pudlák, *Metamathematics of First-oder Arithmetic*, Springer, 1993.]).

Next, we define the **satisfaction relation for G-formulas** as follows:

Definition 3.4

For each $z \in G$, define the satisfaction relation $\mathrm{Sat}(z, \xi)$ as follows:

$$\mathrm{Sat}(z, \xi) \leftrightarrow \exists p \, \mathrm{Sat}^p(z, \xi \restriction V_p).$$

Here, $\xi \restriction V_p$ is the assignment obtained by restricting the values of ξ to V_p.

For simplicity, we abbreviate $\mathrm{Sat}^p(z, \xi \restriction V_p)$ as $\mathrm{Sat}^p(z, \xi)$. It is provable in RCA_0 that for the code z of a Σ^0_1 formula, if $\mathrm{Sat}^p(z, \xi)$ holds, then $\mathrm{Sat}^{p'}(z, \xi)$ also holds for any $p' \geq p$.

Moreover, we will show in WKL_0 that it also the case for the codes z of G.

In the following, we identify a formula with its code.

Lemma 3.5

In a model V of WKL_0, $\mathrm{Sat}(z,\xi)$ satisfies Tarski's truth definition clauses for G formulas.

Proof. We prove the statement by induction on the complexity of the formula z.
If z is an atomic formula or its negation, $\mathrm{Sat}(z,\xi) \Leftrightarrow \exists p\,\mathrm{Sat}^p(z,\xi) \Leftrightarrow \exists p\, z(\xi)^{V_p} \Leftrightarrow z(\xi)$.
If $z = \bigvee_{i<n} z_i$ (where each z_i is a G formula),

$$\mathrm{Sat}\left(\bigvee_{i<n} z_i,\xi\right) \Leftrightarrow \exists p\,\mathrm{Sat}^p\left(\bigvee_{i<n} z_i,\xi\right) \Leftrightarrow \exists p \bigvee_{i<n} \mathrm{Sat}^p(z_i,\xi)$$
$$\Leftrightarrow \bigvee_{i<n} \exists p\,\mathrm{Sat}^p(z_i,\xi) \Leftrightarrow \bigvee_{i<n} \mathrm{Sat}(z_i,\xi).$$

If z is $\exists x\, z'$ or $\exists X\, z'$ (where z' is a G formula), the proof follows analogously.

When $z = \bigwedge_{i<n} z_i$ (where each z_i is a G formula),

$$\mathrm{Sat}\left(\bigwedge_{i<n} z_i, \xi\right) \Leftrightarrow \exists p\, \mathrm{Sat}^p\left(\bigwedge_{i<n} z_i, \xi\right) \Leftrightarrow \exists p \bigwedge_{i<n} \mathrm{Sat}^p(z_i, \xi)$$
$$\Leftrightarrow \bigwedge_{i<n} \exists p\, \mathrm{Sat}^p(z_i, \xi) \quad (\Leftarrow \text{ by } \Sigma_1 \text{ collection principle})$$
$$\Leftrightarrow \bigwedge_{i<n} \mathrm{Sat}(z_i, \xi).$$

If z is $\forall x < y\, z'$ (where z' is a G formula), the proof is analogous.
If $z = \forall X\, z'$ (where z' is a G formula),

$$\mathrm{Sat}(\forall X\, z', \xi) \Leftrightarrow \exists p\, \mathrm{Sat}^p(\forall X\, z', \xi) \Leftrightarrow \exists p\, \forall U\, \mathrm{Sat}^p(z', \xi \cup \{(X, U)\})$$
$$\Leftrightarrow \forall U\, \exists p\, \mathrm{Sat}^p(z', \xi \cup \{(X, U)\}) \quad (\Leftarrow \text{ by compactness (Lemma 3.2(2))})$$
$$\Leftrightarrow \forall U\, \mathrm{Sat}(z', \xi \cup \{(X, U)\}),$$

where $\xi \cup \{(X, U)\}$ is an extension of ξ with X assigned to U. $\square$

Lemma 3.6

In a model $V = (M, S)$ of WKL_0, we take any $e \in M$ and an M-finite assignment map ξ. Then, there exists a $p \in M$ such that for all G_e formulas z whose free variables all belong to the domain of ξ, then $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^p(z, \xi)$ holds.

Proof. Since the domain of the assignment map ξ is M-finite, the set of G_e formulas whose free variables belong to its domain of ξ is essentially M-finite (disregarding repetitions of the same formula within disjunctions or conjunctions). This fact can be demonstrated by Σ^0_1 induction on e.

Therefore, for M-finitely many G_e formulas z where $\mathrm{Sat}(z, \xi)$ holds, let p_z be such that $\mathrm{Sat}^p(z, \xi)$, or set $p_z = 0$ otherwise. Then, if $q = \max\{p_z\}$, $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^q(z, \xi)$ holds. [2]
□

To be continued.

[2]Strictly speaking, strong Σ^0_1 collection principle ($\mathrm{S}\Sigma_1$) is used here. (Refer to Problem 1 following Lemma 1.8 in Chapter 7.)

Thank you for your attention!