Logic and Foundations II

Part 8. Second order arithmetic and non-standard methods

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Real analysis and reverse mathematics (8.5 lectures)
- Part 8. Second order arithmetic and non-standard methods (6.5 lectures)

Part 8. Schedule

- May 21, (0) Introduction to forcing
- May 23, (1) Harrington's conservation result on WKL_0
- May 28, (2) H.Friedman's conservation result on WKL_0
- May 30, (3)
- June 04, (4)
- June 06, (5)
- June 11, (6)

§8.1. Forcing and Harrington's Theorem

Let (M, S) be a countable non-standard model of RCA_0

Definition 1.5

$G(\subseteq M)$ is called an **($\mathfrak{M}$-)generic path**, if for every dense set $D \in \mathrm{Def}(\mathfrak{M})$, there exists a tree $T \in D$ such that G is an infinite path through T.

Lemma 1.6

Every infinite binary tree $T(\in \mathbb{P})$ has a generic path G.

Lemma 1.7

If G is a generic path, then $(M, S \cup \{G\}) \models \Sigma^0_1$-induction.

Fix a generic path G for $T \in \mathbb{P}$, and let

$$S^T = \{X \subseteq M \mid X \text{ is definable in } (M, S \cup \{G\}) \text{ by a } \Delta^0_1 \text{ formula}\}.$$

Lemma 1.8

$(M, S^T) \models \mathsf{RCA}_0 + T$ has an infinite path.

Lemma 1.9

For any countable model (M, S) of RCA_0, there exists a countable set S_∞ such that $S \subseteq S_\infty \subseteq \mathcal{P}(M)$ and $(M, S_\infty) \models \mathsf{WKL}_0$.

Proof Construct $S_0 \subseteq S_1 \subseteq \cdots$ as follows: $S_0 = S$, and

$$S_{(n,m)+1} = S^T_{(n,m)}, \text{ where } T \text{ is the } m\text{-th infinite tree in } S_n (\subseteq S_{(n,m)}).$$

Here, $(n, m) = \frac{(n+m)(n+m+1)}{2} + n$, and so $(n, m) \geq n$. Finally, let $S_\infty = \bigcup_{i\in\omega} S_i$.
It is clear from the definition that this is the desired set. □

Theorem 1.10 (Harrington)

For any Π^1_1 sentence σ, $\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{RCA}_0 \vdash \sigma$.

Proof Suppose σ is a Π^1_1 sentence that is not provable in RCA_0. By Gödel's completeness theorem, there exists a countable model $(M, S) \models \mathsf{RCA}_0 + \neg\sigma$. Now, $\neg\sigma$ can be expressed as $\exists X \varphi(X)$ with $\varphi \in \Pi^1_0$. Then there exists $A \in S$ such that $(M, S) \models \mathsf{RCA}_0 + \varphi(A)$. By constructing S_∞ by Lemma 1.9, we have $(M, S_\infty) \models \mathsf{WKL}_0 + \varphi(A)$. Note that since $\varphi(X)$ is arithmetical, the truth value of $\varphi(A)$ depends only on M and A. Therefore, $(M, S_\infty) \models \mathsf{WKL}_0 + \neg\sigma$, which implies $\mathsf{WKL}_0 \nvdash \sigma$. □

§8.2. Semi-Regular Cuts and Friedman's Theorem

The goal of this section is to prove a theorem of H. Friedman that "WKL_0 is Π^0_2 conservative over PRA." First of all, we introduce a formal system of finitistic arithmetic PRA, which stands for Primitive Recursive Arithmetic, to handle all primitive recursive functions on the natural numbers. Its language consists of symbols for the primitive recursive functions, and its axioms are their defining equations, along with Σ_0 induction. A model of PRA is of the form $(M, \mathtt{f}_0^M, \mathtt{f}_1^M, \cdots)$, also denoted as (M, F) or just M.

Now, we fix a nonstandard model (M, F) of PRA (i.e., $M \neq \omega$). Also, let $p \in F$ be a primitive recursive function that lists the prime numbers in the ascending order, i.e., $p(0) = 2, p(1) = 3, p(2) = 5, \cdots$.

Definition 2.1

A set $X(\subseteq M)$ has a **code** $c \in M$ or is coded by c, if

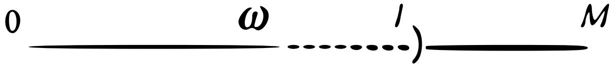
$$X = \{n \in M : M \models \exists d < c\ (c = p(n) \cdot d)\}.$$

Such a set X is called **M-finite**, and the number of elements in X is denoted by $|X|$ or $|c|$.

Note $|x|$ is a primitive recursive function on M, i.e., $|x| \in F$. Also, if $X(\neq \varnothing)$ has a code c, the largest element of X can be denoted by $\max(X)$ or $\max(c) \in F$.

Definition 2.2

A proper initial segment I of M is called a **cut** of M, denoted $I \subseteq_e M$[1], if it is closed under the successor function (i.e., $a \in I \Rightarrow a+1 \in I$).
Furthermore, a cut $I \subseteq_e M$ is called a **semi-regular cut**, if $X \cap I$ is bounded within I for any M-finite set X with $|X| \in I$.



Note. If X is an M-finite set and $X \cap I$ is bounded in I, then $X \cap I$ is also M-finite, and so the largest element of $X \cap I$ exists.

The analogy between the semi-regular cuts of nonstandard models of arithmetic and the regular cardinals in models of set theory was discovered by Paris and his colleagues in the UK in the mid-1970s. Recall that a regular cardinal is a cardinal such that the range of any function from a smaller cardinal to it is always bounded.

[1] In Chapter 5, all initial segments were denoted by $\subseteq_e$.

Let (M, F) be a nonstandard model of PRA.

Theorem 2.3 (Kirby-Paris)

If $I \subseteq_e M$ is a semi-regular cut, then $(I, F\lceil I) \models$ PRA, where $F\lceil I$ is the set of functions obtained by restricting the domain of each function f in F to I.

Proof First, we show that I is closed under primitive recursive functions. For each $n \in \omega$, define the unary primitive recursive function g_n as follows:

$$\mathrm{g}_0(x) = x + 1,$$

$$\mathrm{g}_{n+1}(x) = \overbrace{\mathrm{g}_n \mathrm{g}_n \cdots \mathrm{g}_n}^{x+2}(x)^2$$

For any primitive recursive function symbol $\mathtt{f}$, there exists some $n \in \omega$ such that

$$\text{PRA} \vdash \mathtt{f}(x_1, x_2, \cdots, x_k) < \mathrm{g}_n(\max\{x_1, x_2, \cdots, x_k\})$$

(where for $k = 0$, the value of $\max$ is set as 0). Let's briefly demonstrate this fact.

[2] To see $\mathrm{g}_{n+1}(x)$ is primitive recursive, we first introduce a two-variable function $\mathrm{g}'_{n+1}(x, y)$ as follows: $\mathrm{g}'_{n+1}(x, 0) = 0, \mathrm{g}'_{n+1}(x, y + 1) = \mathrm{g}_n(\mathrm{g}'_{n+1}(x, y))$. Then $\mathrm{g}_{n+1}(x) = \mathrm{g}'_{n+1}(x, x + 2)$ is primitive recursive. Compare with the Ackermann function in part 1.

For the three initial functions of primitive recursion, we have $\mathrm{Z}() = 0 < \mathsf{g}_0(0)$, $\mathrm{S}(x) = x + 1 < 2x + 2 = \mathsf{g}_1(x)$, and $\mathrm{P}^n_i(x_1, \ldots, x_n) = x_i < \mathsf{g}_0(\max\{x_1, x_2, \ldots, x_k\})$. For function composition, we consider one-variable functions for simplicity. If $h_1(x) < \mathsf{g}_n(x)$ and $h_2(x) < \mathsf{g}_n(x)$, then their composite function $h_1(h_2(x)) < \mathsf{g}_n(\mathsf{g}_n(x)) \leq \mathsf{g}_{n+1}(x)$. For primitive recursion $f(x, y+1) = h(x, y, f(x, y))$ defined by $f(x, 0) < \mathsf{g}_n(x)$ and $h(x, y, z) < \mathsf{g}_n(\max\{x, y, z\})$, we have $f(x, y) < \mathsf{g}_n^{y+2}(\max\{x, y\}) \leq \mathsf{g}_{n+1}(\max\{x, y\})$. Hence, every primitive recursive function is bounded by some g_n.

To confirm that I is closed under all primitive recursive functions, it suffices to show closure for each g_n. By definition 2.2, I is closed under the successor function, so the case $n = 0$ holds. Now, by way of contradiction, assume it is closed under g_n, but not g_{n+1}. Then choose $a \in I$ such that $\mathsf{g}^M_{n+1}(a) \notin I$, and define

$$X = \{\mathsf{g}^M_n(a), \mathsf{g}^M_n\mathsf{g}^M_n(a), \cdots, \overbrace{\mathsf{g}^M_n\mathsf{g}^M_n\cdots\mathsf{g}^M_n}^{a+2}(a)\}.$$

Since X is an M-finite set with $|X| = a + 2 \in I$, so $X \cap I$ is bounded and has a maximum element b. However, since I is closed under g_n, we have $\mathsf{g}^M_n(b) \in X \cap I$, contradicting the maximality of b.

From the above, $(I, F\lceil I)$ can be considered a substructure of (M, F), and thus the truth values of Σ_0 formulas are the same in both structures. Finally, to show $(I, F\lceil I) \models \Sigma_0$-ind., let $\varphi(x)$ be a Σ_0 formula and assume $(I, F\lceil I) \models \varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1))$. Choose any $c \in I$ and let $\psi(x) = \varphi(x) \vee c < x$. Then, $\psi(x)$ is also Σ_0 and it is easy to see $(M, F) \models \psi(0) \wedge \forall x(\psi(x) \to \psi(x+1))$. Since $(M, F) \models \Sigma_0$-induction, $(M, F) \models \forall x \psi(x)$ and so $(M, F) \models \psi(c)$, which means $(M, F) \models \varphi(c)$, and thus $(I, F\lceil I) \models \varphi(c)$. Since $c \in I$ is arbitrary, we obtain $(I, F\lceil I) \models \forall x \varphi(x)$. Therefore, $(I, F\lceil I) \models \Sigma_0$-induction, and so $(I, F\lceil I) \models$ PRA. □

Definition 2.4

Let $I \subseteq_e M$ and let S be the set of all M-finite sets. A set $B \subseteq I$ is called an ***M*-coded set** if there exists a $X \in S$ such that $B = X \cap I$. Then, B is also coded by a code c of X. We denote the set of all M-coded subsets of I by $S\lceil I$.

Note. We can consider $(I, S\lceil I)$ as a structure of second-order arithmetic, with basic operations $+^I, \cdot^I$, etc., which are obtained by restricting the corresponding operations (primitive recursive functions) on M to I.

Lemma 2.5

If $I \subseteq_e M$ is a semi-regular cut, then $(I, S\lceil I) \models \mathsf{WKL}_0$.

Proof It is clear that $(I, S\lceil I)$ satisfies the basic axioms of arithmetic. Therefore, what we need to show is that it satisfies (Δ^0_1-CA), weak König's lemma (WKL), and Σ^0_1 induction. Let's start with Σ^0_1 induction. It suffices to show (bounded Σ^0_1-CA).

First, we consider how to transform a formula θ in $(I, S\lceil I)$ into a formula θ^* in (M, F). For each set parameter $B \in S\lceil I$ in θ, let c_B be a code of B, that is, a code of X such that $B = X \cap I$. θ^* is obtained from θ by replacing every subformula "$t \in B$" with "$\exists d < c_B\ (c_B = p(t) \cdot d)$". Then, if θ is Σ^0_0, also is θ^*. It is easy to see that for any $a \in I$,

$$(I, S\lceil I) \models \theta(a) \ \Leftrightarrow\ (M, F) \models \theta^*(a).$$

Next, consider a Σ^0_1 formula $\varphi(x) = \exists y \theta(x, y)$ (where θ is Σ^0_0 in $(I, S\lceil I)$). The goal is to show that for arbitrary $c \in I$, $\{x < c \mid \varphi(x)\} \in S\lceil I$. Take any $d \in M - I$ and define

$$Z = \{(a, b) : a <_M c,\ b <_M d \text{ and } (M, F) \models \theta^*(a, b) \wedge \forall x < b \neg\theta^*(a, x)\}.$$

That is, for $(a, b) \in Z$, b is the smallest element in M such that $\theta^*(a, b)$ holds. It is evident that Z is M-finite with $|Z| \leq_M c$. From the semi-regularity of I, $Z \cap (I \times I)$ is bounded, and so there exists $d' \in I$, such that for all $a <_M c$,

$$\exists b \in I\ (a, b) \in Z \ \Leftrightarrow\ \exists b <_M d'\ (a, b) \in Z.$$

Since (M, F) satisfies Σ_0 induction (the least number principle), for all $a <_M c$,

$$\exists b \in I\ (M, F) \models \theta^*(a, b) \ \Leftrightarrow\ \exists b \in I\ (a, b) \in Z \text{ } (\because \Rightarrow \text{ by the least number principle})$$
$$\Leftrightarrow\ \exists b <_M d'\ (a, b) \in Z \ \Leftrightarrow\ \exists b <_M d'\ (M, F) \models \theta^*(a, b) \text{ } (\because \Leftarrow \text{ by the same principle})$$

Therefore, for all $a <_M c$,

$$(I, S\lceil I) \models \varphi(a) \Leftrightarrow \exists b \in I \ (I, S\lceil I) \models \theta(a, b)$$

$$\Leftrightarrow \exists b \in I \ (M, F) \models \theta^*(a, b)$$

$$\Leftrightarrow \exists b <_M d' \ (M, F) \models \theta^*(a, b)$$

$$\Leftrightarrow (M, F) \models \exists y < d' \ \theta^*(a, y)$$

Since $\exists y < d' \ \theta^*(a, y)$ is a Σ_0 formula, we can show by Σ_0 induction that $X = \{a < c : (M, F) \models \exists y < d' \ \theta^*(a, y)\}$ has a code $\Pi_{a \in X} p(a)$. That is, $\{a < c : (I, S\lceil I) \models \varphi(a)\}$ is an M-coded set X, and hence it belongs to $S\lceil I$.

Since $(\Delta^0_1$-CA) + (WKL) is equivalent to $(\Sigma^0_1$-SP) [3], it suffices to show that $(I, S\lceil I) \models (\Sigma^0_1$-SP). Let $\varphi_i(x) = \exists y \theta_i(x, y)$, $\theta_i(x, y) \in \Sigma^0_0$ $(i = 0, 1)$, and assume $(I, S\lceil I) \models \neg\exists x(\varphi_0(x) \land \varphi_1(x))$. Similar to the above, let θ^*_i be the Σ_0 formula obtained by replacing the set parameters of θ_i with their definitions. Now, fix any $d \in M - I$, and define

$$Y = \{a <_M d \mid \exists b <_M d \ (M, F) \models \theta^*_0(a, b) \land \forall x < b \neg\theta^*_1(a, x)\}$$

That is, Y is the set of element a such that, when b increases from below, $\theta^*_0(a, b)$ holds before $\theta^*_1(a, b)$. Obviously, Y is M-finite, so $Y \cap I \in S\lceil I$. Then, it is easy to see

$$(I, S\lceil I) \models \forall a[(\varphi_0(a) \to a \in Y \cap I) \land (\varphi_1(a) \to a \notin Y \cap I)].$$

Hence, $(I, S\lceil I) \models (\Sigma^0_1$-SP). Therefore, $(I, S\lceil I) \models$ WKL$_0$ has been proved. □

[3] See Lemma 3.6 in part 7

The following lemma is crucial to Friedman's proof.

Lemma 2.6

Let (M, F) be a countable nonstandard model of PRA. Take $c, d \in M$ such that for all primitive recursive functions $\mathtt{f}$, $\mathtt{f}^M(c, c, \cdots, c) <_M d$. Then, there exists a semi-regular cut $I \subseteq_e M$ such that $c \in I$ and $d \notin I$.

Note If we take $c \in M - \omega$ and consider the smallest cut J that contains c and is closed under all primitive recursive functions, J will not be a semi-regular cut. The reason is as follows. Let $\{\mathrm{g}_n\}$ be the sequence of primitive recursive functions constructed in the proof of Theorem 2.3, and let $B(x, y, z) \Leftrightarrow \mathrm{g}_x(y) \leq z$. (The precise definition of the primitive recursive predicate $B(x, y, z)$ is given in the proof below.) Since $J = \{a \in M : \exists n \in \omega\ a <_M \mathrm{g}_n^M(c)\}$, we have $J \models \neg\exists z B(c, c, z)$. If J were a semi-regular cut, then by the lemma above, $J \models \Sigma^0_1$ induction, so there would be a smallest $a \in J$ such that $J \models \neg\exists z B(a, c, z)$. Then $J \models \exists z B(a-1, c, z)$, that is, $\mathrm{g}_{a-1}(c) \in J$, and so there exists $n \in \omega$ such that $\mathrm{g}_{a-1}(c) < \mathrm{g}_n(c)$, which is impossible since $a - 1 \notin \omega$. Therefore, J is not a semi-regular cut. On the other hand, since J is a model of PRA, it has been shown that PRA $\not\vdash \Sigma^0_1$ induction.

Proof First, define the primitive recursive predicate $B(x,y,z)$ as follows:

- $B(0,y,z) \;\Leftrightarrow\; y<z$,
- $B(x+1,y,z) \;\Leftrightarrow\;$ for any M-finite set $X\subset[y,z)$ with $|X|\le y$, there exists $[y',z')\subset[y,z)$ such that $B(x,y',z')$ and $[y',z')\cap X=\emptyset$

Here, $[y,z)=\{w : y\le w<z\}$.

Now, when $B(x,y,z)$ holds, we say "the interval $[y,z)$ is x-large." Then, the interval $[y,z)$ is $(x+1)$-large iff for any subset $X\subset[y,z)$ with $|X|\le y$, there exists a subinterval $[y',z')\subset[y,z)$ that is x-large and disjoint from X.

We observe that the definition of $B(x+1,y,z)$ is Σ_0, since a subset $X\subset[y,z)$ with cardinality at most y can be encoded by a number at most $p(z)^y$. So this makes $B(x,y,z)$ a primitive recursive predicate.

For the sequence $\{\mathrm{g}_n\}$ of primitive recursive functions constructed in the proof of Theorem 2.3, it can be shown that for each $n \in \omega$,

$$\mathsf{PRA} \vdash \mathrm{g}_n(y) \leq z \ \rightarrow B(n, y, z).$$

Indeed, this is clear when $n = 0$. Assuming it holds for n, let's show it for $n + 1$. Suppose $\mathrm{g}_{n+1}(y) \leq z$. Since $\mathrm{g}_{n+1}(y) = \mathrm{g}_n^{y+2}(y)$, for any subset $X \subset [y, z)$ with $|X| \leq y$, there exists some $c < y + 2$ such that the interval $[\mathrm{g}_n^c(y), \mathrm{g}_n^{c+1}(y))$ does not contain any element of X. Let $y' = \mathrm{g}_n^c(y)$ and $z' = \mathrm{g}_n^{c+1}(y)$. Then $\mathrm{g}_n(y') = z'$. So by the inductive hypothesis, $B(n, y', z')$ holds, which fulfills the definition of $B(x + 1, y, z)$.

Next, take $c, d \in M$ as in the statement of the lemma. Then for any $n \in \omega$, $\mathrm{g}_n^M(c) <_M d$, and so $B(n, c, d)$. By the overspill principle, there exists $b \in M - \omega$ such that $\forall a \leq_M b \ B(a, c, d)$.[4]

[4]Using Σ_0 induction in PRA, one can take the smallest x such that $\neg B(x, c, d)$ and set $b = x - 1$.

Now, since (M, F) is a countable model of PRA, there are only countably many M-finite sets. So, we can construct a sequence of M-finite sets $\{X_n\}$, such that each M-finite set appears infinitely often in the sequence. Using this, we define the decreasing sequence of intervals $\{[c_n, d_n)\}$ as follows:

$$[c_0, d_0) = [c, d),$$

$$[c_{n+1}, d_{n+1}) = \begin{cases} [c_n, d_n) & \text{if } |X_n| \geq_M c_n, \\ [c', d') & \text{otherwise, take any } [c', d') \subset [c_n, d_n) \text{ such that} \\ & B(b-n, c', d') \text{ and } [c', d') \cap X = \emptyset. \end{cases}$$

For any $a \in M$, obviously $\{a\}$ is M-finite, so for sufficiently large n, $[c_n, d_n) \cap \{a\} = \emptyset$, that is, $a \notin [c_n, d_n)$. Therefore, $\bigcap_n [c_n, d_n) = \varnothing$.

Now, let $I = \{a \in M : \exists n\ a <_M c_n\} = \{a \in M : \forall n\ a <_M d_n\}$. We show that I becomes a semi-regular cut. If X is M-finite and $|X| \in I$, by the definition of $\{X_n\}$, there are infinitely many n such that $X = X_n$. Then, there exists n such that $X = X_n$ and $|X| <_M c_n$. Thus, $[c_{n+1}, d_{n+1}) \cap X = \varnothing$. Therefore, $X \cap I$ is bounded by c_{n+1} in I. Hence, I is a semi-regular cut. □

Theorem 2.7 (Friedman)

For any Π_2 sentence σ, $\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{PRA} \vdash \sigma$.

Proof. To show the contraposition, take a Π_2 sentence $\sigma = \forall y \exists z \theta(y, z)$ with $\theta \in \Sigma_0$ that is not provable in PRA. Then, $\mathsf{PRA} \cup \{\neg\exists z \theta(c, z)\} \cup \{\mathtt{f}(c, c, \cdots, c) < d : \mathtt{f}$ is a symbol of a primitive recursive function$\}$ is consistent, and hence by the completeness theorem, it has a countable model (M, F, c, d). Now, by Lemma 2.6, there exists a semi-regular cut $I \subseteq_e M$ such that $c \in I$ and $d \notin I$. Since $\neg\exists z \theta(c, z)$ is a Π_1 sentence and $M \models \neg\exists z \theta(c, z)$, it follows that $I \models \neg\exists z \theta(c, z)$, i.e., $I \models \neg\sigma$. On the other hand, by Lemma 2.5, we have $(I, S\lceil I) \models \mathsf{WKL}_0$. Thus, $(I, S\lceil I) \models \mathsf{WKL}_0 + \neg\sigma$, and so $\mathsf{WKL}_0 + \neg\sigma$ is consistent, hence σ cannot be proved in WKL_0 either. □

As we saw in part 7, a wide range of mathematics can be developed within WKL_0. Nevertheless, Friedman's theorem shows that WKL_0 is Π_2-conservative over PRA, which can be viewed as a partial realization of Hilbert's "finitistic reductionism" or an essence of the "Hilbert Program."

Hilbert's Program

The main goal of Hilbert's program was to provide secure foundations for all mathematics, to counteract the intuitionism, led by Brouwer who had been attacking non-constructive methods in mathematics. Hilbert proposed the method of "proof theory" or "meta-mathematics", by which mathematical arguments are treated as symbolic manipulations, and thus can be analyzed themselves mathematically.

Let T be a large system (e.g., set theory ZFC) that can develop most of mathematics. Let t be a small system (e.g., PRA) capable of performing symbolic manipulatios of T. Then, Hilbert considered that a Π^0_1 sentence which does not assert existence (e.g., Fermat's Last Theorem: $\forall n > 2 \forall x, y, z > 0(x^n + y^n \neq z^n)$) would be provable in t if it is provable in T. Therefore, the validity of a Π^0_1 sentence could be recognized by any non-constructive methods.

In the following, we assume that both T and t include at least PRA. Then,

Hilbert's (reductionism) program HP

HP: for any Π^0_1 sentence φ, if $T \vdash \varphi$ then $t \vdash \varphi$.

Theorem 2.8

For any Π^0_1 sentence φ, if $T \vdash \varphi$, then $t + \mathrm{Con}(T) \vdash \varphi$. Here, $\mathrm{Con}(T)$ is a Π^0_1 sentence representing the consistency of T.

Proof. Let $\varphi \equiv \forall n \theta(n)$ (where $\theta(n)$ is Σ^0_0 or primitive recursive), and assume $T \vdash \varphi$. So, since $\mathrm{Bew}_T(\ulcorner \varphi \urcorner)$ is a true Σ^0_1 sentence, by the Σ^0_1-completeness of t, $t \vdash \mathrm{Bew}_T(\overline{\ulcorner \varphi \urcorner})$. On the other hand, from the proof of Lemma 4.5.1 D3, $t \vdash \neg\theta(n) \to \mathrm{Bew}_t(\overline{\ulcorner \neg\theta(\bar{n}) \urcorner})$, i.e., $t \vdash \neg\theta(n) \to \mathrm{Bew}_t(\overline{\ulcorner \neg\varphi \urcorner})$. Since $\mathrm{Bew}_t(\overline{\ulcorner \neg\varphi \urcorner}) \to \mathrm{Bew}_T(\overline{\ulcorner \neg\varphi \urcorner})$, it follows that $t \vdash \neg\theta(n) \to \neg\mathrm{Con}(T)$. Therefore, $t + \mathrm{Con}(T) \vdash \theta(n)$, and thus $t + \mathrm{Con}(T) \vdash \varphi$. □

By this theorem, if $t \vdash \mathrm{Con}(T)$, then HP holds. However, by Gödel's second incompleteness theorem, $\mathrm{Con}(T)$ is be not provable in T, so of course not in t.

However, for $T = \mathsf{WKL}_0$ and $t = \mathsf{PRA}$, HP is shown to hold by Friedman's theorem. Observing the richness of mathematics developed in WKL_0, one can view that "Hilbert's program" has been partially realized. Those skeptical about the meaning of HP still likely agree on the importance of rewriting a proof of a Π^0_1 sentence involving non-constructive arguments like weak König's lemma into a constructive proof without them.

Thank you for your attention!