Logic and Foundations II

Part 8. Second order arithmetic and non-standard methods

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Recapitulation and Correction

Theorem 5.6

In RCA_0, ATR_0, $\underset{\sim}{\Delta}^0_1$-Det, $\underset{\sim}{\Sigma}^0_1$-Det, and $\underset{\sim}{\Pi}^0_1$-Det are pairwise equivalent.

Fact: Π^1_1-CA_0 implies ATR_0 and Con(ATR_0).

Theorem 5.7

The determinacy of $\underset{\sim}{\Sigma}^0_1 \wedge \underset{\sim}{\Pi}^0_1$ games and Π^1_1-CA_0 are equivalent over RCA_0.

By generalizing Theorem 5.7, we can also show that the determinacy of games defined by Boolean combinations of $\underset{\sim}{\Sigma}^0_1$ sets can be obtained through iterations of Π^1_1-CA_0.

Moreover, $\underset{\sim}{\Delta}^0_2$-Det can be deduced from transfinite iterations of Π^1_1-CA_0, i.e., Π^1_1-TR_0. For this purpose, we need the effective version of the Hausdorff-Kuratowski theorem on ambiguous Borel sets.

For a well-order $\prec$ on $\mathbb{N}$, define the well-order $\prec^*$ on $\mathbb{N} \times \{0,1\}$ as follows:

$$(x,i) \prec^* (y,j) \quad \text{iff} \quad x \prec y \vee (x = y \wedge i < j).$$

A Π^0_n formula $\varphi(n,i,f)$ is said to **decreasing** along $\prec^*$ if it satisfies:

$$\forall f \in \mathbb{N}^{\mathbb{N}}\ \forall n \forall i \forall m \forall j\ (((m,j) \prec^* (n,i) \wedge \varphi(n,i,f)) \to \varphi(m,j,f)).$$

Definition 5.8 (Effective Difference Hierarchy)

For $n \geq 1$, $A \subseteq \mathbb{N}^{\mathbb{N}}$ belongs to $\mathcal{D}^0_{n+1}$ iff there exists a Π^0_n formula $\varphi(x,i,f)$ decreasing along a well-order $\prec^*$ such that

$$A(f) \Leftrightarrow \exists x(\neg\varphi(x,1,f) \wedge \varphi(x,0,f)).$$

Theorem 5.9 (Effective Difference Hierarchy Theorem)

In ACA$_0$, $\mathcal{D}^0_n = \Delta^0_n$ $(n \geq 2)$.

Theorem 5.10

$\underset{\sim}{\Delta}{}^0_2$-Det is equivalent to Π^1_1-TR$_0$ in RCA$_0$.

Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Real analysis and reverse mathematics (8.5 lectures)
- Part 8. Second order arithmetic and non-standard methods (6.5 lectures)

Part 8. Schedule

- May 21, (0) Introduction to forcing
- May 23, (1) Harrington's conservation result on WKL_0
- May 28, (2) H.Friedman's conservation result on WKL_0
- May 30, (3)
- June 04, (4)
- June 06, (5)
- June 11, (6)

§8.1. Forcing and Harrington's Theorem

In this section, we introduce Harrington's theorem that "WKL_0 is a Π^1_1 conservative extension of RCA_0." The forcing argument of adding infinite paths of an infinite tree as generic paths to a ground model was invented by Jockusch and Soare (Π^0_1 classes and degrees of theories, *Trans. of the A. M. S.* 173 (1972), pp.35–56). Subsequently, Harrington cleverly applied it to non-ω models in second-order arithmetic.

The basic idea of forcing is to generate something that does not exist in the world without causing confusion. First, a set of conditions $\mathbb{P}$ for what to generate is given, and a partial order is defined on $\mathbb{P}$. Ways to interpret these conditions varies depending on applications, and we first proceed without giving particular meanings.

Fix an arbitrary partially ordered set $(\mathbb{P}, <)$, and let $p, q, r, \ldots$ denote elements of $\mathbb{P}$.
A set $G \subseteq \mathbb{P}$ is called an **open set**, if it satisfies the following condition

$$\forall p, q\ (q < p \wedge p \in G \to q \in G).$$

Thus, $(\mathbb{P}, <)$ becomes a topological space. Now, let

$$[p] = \{q \in \mathbb{P} \mid q \leq p\}.$$

Any open set G coincides with $\bigcup_{p \in G}[p]$, and so $\{[p] \mid p \in \mathbb{P}\}$ forms a basis for the topology.

Any set $D \subseteq \mathbb{P}$ is called a **dense** set, if it has a non-empty intersection with every non-empty open set. The condition for D to be dense is equivalent to

$$\forall p \in \mathbb{P}\ \ [p] \cap D \neq \emptyset, \text{ in other words, } \forall p \in \mathbb{P}\ \ \exists d \in D\ \ d \leq p.$$

Definition 1.1

A set $F \subseteq \mathbb{P}$ is called a **filter**, if it satisfies the following conditions:

1) $p \in F \wedge p < q \to q \in F$,
2) $\forall p, q \in F\ \ [p] \cap [q] \cap F \neq \emptyset$.

Definition 1.2

Given a family of sets $\mathcal{D}$, a filter G is called a **$\mathcal{D}$-generic filter** if it intersects every dense set $D \subseteq \mathbb{P}$ belonging to $\mathcal{D}$.

Lemma 1.3

If $\mathcal{D}$ contains at most countably many dense subsets of $\mathbb{P}$, then for any $p \in \mathbb{P}$, there exists a $\mathcal{D}$-generic filter G that contains p.

Proof Enumerate the dense subsets of $\mathbb{P}$ contained in $\mathcal{D}$ as $D_0, D_1, \cdots, D_i, \cdots (i \in \omega)$. For a given $p \in \mathbb{P}$, construct a decreasing sequence $p_0 \geq p_1 \geq \cdots$ from $\mathbb{P}$ as follows: $p_0 = p$, and $p_n \in [p_{n-1}] \cap D_{n-1}$ for each $n > 0$. Then, we set $G = \{q \mid \exists i \; p_i \leq q\}$. Thus, it is obvious that $p \in G$ and G is a $\mathcal{D}$-generic filter. □

Now, we will introduce the forcing conditions used in Harrington's proof. Let $\mathfrak{M} = (M, S)$ be a countable model of RCA_0. Here, M is the first-order part (the domain corresponding to the natural numbers), and S is the second-order part consisting of subsets of M, that is, $S \subseteq \mathcal{P}(M)$. Then, set

$$\mathbb{P} = \{T \in S \mid \mathfrak{M} \models \text{“}T(\subseteq \mathrm{Seq}_2) \text{ is an infinite binary tree”}\},$$

and define a partial order on $\mathbb{P}$ by

$$T_1 \leq T_2 \Leftrightarrow T_1 \subseteq T_2.$$

For each $T \in \mathbb{P}$, we want to generate an infinite path and put it into S. But if we bring in an arbitrary path of T from outside, it might break the condition of $\mathfrak{M}$, e.g., induction axiom. Instead, we approximate an infinite path by $T' \leq T$, and for this purpose, the concept of density is important, namely

$$D \subseteq \mathbb{P} \text{ is dense } \Leftrightarrow \forall T \in \mathbb{P}\ \exists T' \in D\ T' \leq T.$$

$E \subseteq \mathbb{P}$ is said to be **definable in $\mathfrak{M}$** if there exists a formula $\varphi(X)$ (with parameters from $M \cup S$) such that $E = \{T \in \mathbb{P} \mid \mathfrak{M} \models \varphi(T)\}$. The totality of such sets is denoted by $\mathrm{Def}(\mathfrak{M})$. Since we only consider a countable model $\mathfrak{M} = (M, S)$ in a countable language, $\mathrm{Def}(\mathfrak{M})$ is a countable set. By Lemma 1.3, any $T \in \mathbb{P}$ is contained in some $\mathrm{Def}(\mathfrak{M})$-generic filter. Such a filter is simply referred to as an $\mathfrak{M}$-generic filter.

Lemma 1.4

If $F \subseteq \mathbb{P}$ is an $\mathfrak{M}$-generic filter, then there exists a unique infinite path $G = \cap F = \cap_{T \in F} T$ common to all $T \in F$. That is, F is contained in the principal filter generated by G.

Proof For each $k \in M$, let $E_k = \{T \in \mathbb{P} \mid \exists! s \in \{0,1\}^k \ s \in T\}$ be dense and definable in $\mathfrak{M}$. If F is an $\mathfrak{M}$-generic filter, then for each k, there exists some $s_k \in \{0,1\}^k$ such that there is $T_k \in F$ with $T_k \cap \{0,1\}^k = \{s_k\}$. Moreover, if $k < k'$, then s_k is an initial segment of $s_{k'}$, and $s_{k'} \in T_k$. If not, $[T_k] \cap [T_{k'}] = \emptyset$[1], which would contradict the filter condition of F. Thus, let $G = \bigcup_{k \in M} s_k$; then $G = \bigcap_k T_k$ as well. Finally, to show $G = \cap F$, if $G \not\subseteq T \in F$, then there exists some k such that $s_k \notin T$, and $[T] \cap [T_k] = \varnothing$, which contradicts the filter condition of F. □

[1]Here, $[T]$ denotes $\{T' \in \mathbb{P} \mid T' \subset T\}$. In the latter half of part 8, the same notation $[T]$ represents the set of infinite paths of T. Since both are conventional, we would use both as they are.

Definition 1.5

$G(\subseteq M)$ is called an **$\mathfrak{M}$-generic path**, if for every dense set $D \in \mathrm{Def}(\mathfrak{M})$, there exists a tree $T \in D$ such that G is an infinite path through T.

Lemma 1.6

Every $T \in \mathbb{P}$ has an $\mathfrak{M}$-generic path G.

Proof By Lemma 1.3, every T is contained in some $\mathfrak{M}$-generic filter F. Then, by Lemma 1.4, there is a common infinite path G in the trees of F. It is clear from the definition that this G is an $\mathfrak{M}$-generic path. □

From now on, an $\mathfrak{M}$-generic path will simply be referred to as a generic path.

Lemma 1.7

If G is a generic path, then $(M, S \cup \{G\}) \models \Sigma^0_1$-induction.

Proof Let $\varphi(i, X)$ be any Σ^0_1 formula, and choose any $b \in M$, and we will show that $A = \{a \leq_M b \mid \varphi(a, G)\} \in S$ [2]. If $A \in S$, induction on $\varphi(n, G)$ can be shown as follows.

[2] See Lemma 1.8 of part 7 for $\mathrm{RCA}_0 \vdash$ (bounded Σ^0_1-CA). We show (bounded Σ^0_1-CA) $\to \Sigma^0_1$ induction.

Suppose $A \in S$. Then, $B = \{a \mid a \in A \vee a >_M b\} \in S$ since $\mathfrak{M} \models (\Delta^0_1\text{-CA})$. Now, assume $\varphi(0, G)$ and $\forall n(\varphi(n, G) \to \varphi(n+1, G))$. Then, we have $0 \in B$ and $\forall m(m \in B \to m + 1 \in B)$. Since $\mathfrak{M} \models \Sigma^0_1$-induction, by induction on B, we have $B = M$. Therefore, $b \in A$, that is, $\varphi(b, G)$. Since $b \in M$ is arbitrary, we get $\forall n \varphi(n, G)$.

Now we show $A \in S$. Let $\varphi(i, X) \equiv \exists j \theta(i, X \lceil j)$ (where $\theta \in \Sigma^0_0$)[3], and set

$$D_b = \{T \in \mathbb{P} \mid \mathfrak{M} \models \forall a \leq b \ (1)\ \forall t \in T \neg\theta(a, t) \vee \\ (2)\ \exists k \forall t \in T \cap \{0,1\}^k \exists s \subseteq t \theta(a, s)\}.$$

Of course, D_b is definable in $\mathfrak{M}$. Here, note that if $T \in D_b$ and $T' \subseteq T$, then $T' \in D_b$. And as shown below, D_b is dense, so there exists a tree T_0 in D_b that has G as an infinite path. Fix such a T_0. For simplicity, we write $(1)_{T_0}$ for above condition (1) with $T = T_0$, and $(2)_{T_0}$ for condition (2) with $T = T_0$.

[3] $X \upharpoonright j$ represents the code of the initial segment $(f(0), \cdots, f(j-1))$ of the characteristic function f of X. The truth value of the Σ^0_0 formula $\theta(X)$ depends only on a finite part of X, so for sufficiently large j, X can be replaced by $X \upharpoonright j$. See [Simpson, Theorem II.2.7] for details.

Then, for each $a \leq_M b$,

$$\mathfrak{M} \models (1)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \neg\varphi(a, G),$$
$$\mathfrak{M} \models (2)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \varphi(a, G).$$

Since $\mathfrak{M} \models (1)_{T_0} \vee (2)_{T_0}$, we have

$$\mathfrak{M} \models (2)_{T_0} \Leftrightarrow (M, S \cup \{G\}) \models \varphi(a, G)$$

Since (2) is a Σ^0_1 formula, and $\mathfrak{M} \models (\text{bounded}\Sigma^0_1\text{-CA})$ (Lemma 1.8, Chapter 7), $A = \{a \leq_M b \mid \mathfrak{M} \models (2)_{T_0}\} \in S$.

Finally, we show that D_b is dense. Choose any $\tilde{T} \in \mathbb{P}$. For each $\sigma \in \{0, 1\}^{\leq b}$, define a tree T_σ inductively as follows:

$T_\varnothing = \tilde{T}$,
$T_{\sigma^\frown 0} = \{t \in T_\sigma \mid \forall s \subseteq t\ \neg\theta(a, s)\}$, where $a = \text{leng}(\sigma)$,
$T_{\sigma^\frown 1} = T_\sigma$.

Here, $\varnothing$ is the empty sequence, and $\sigma^\frown i$ denotes the sequence σ followed by $i (= 0, 1)$.

Next, let $S_b = \{\sigma \in \{0,1\}^{b+1} \mid T_\sigma \text{ is an infinite tree}\}$. Then, since "$T_\sigma$ is an infinite tree" is expressed by a Π^0_1 formula $\forall n \exists \tau \in \{0,1\}^n\ \tau \in T_\sigma$, by (bounded Σ^0_1-CA), we have $S_b \in S$. Also, since $\overbrace{<1,1,\cdots,1>}^{b+1} \in S_b$, we get $S_b \neq \varnothing$.
Thus, let σ_b be the lexicographically first element in S_b.

Take any $a \leq_M b$. $\sigma_b(a) = 0$, then $(\sigma_b \lceil a)^\frown 0 \subset \sigma_b$, so

$$T_{\sigma_b} \subseteq T_{(\sigma_b \lceil a)^\frown 0} \subseteq \{t \mid \neg\theta(a,t)\},$$

from which we have $(1)_{T_{\sigma_b}}$.

If $\sigma_b(a) = 1$, then $T_{(\sigma_b \lceil a)^\frown 0}$ is finite, and thus $(2)_{T_{\sigma_b \lceil a}}$ and also $(2)_{T_{\sigma_b}}$ holds.

From all the above, $T_{\sigma_b} \in D_b$, which proves that D_b is dense. □

Fix a generic path G for $T \in \mathbb{P}$, and let

$$S^T = \{X \subseteq M \mid X \text{ is definable in } (M, S \cup \{G\}) \text{ by a } \Delta^0_1 \text{ formula}\}.$$

Lemma 1.8

$(M, S^T) \models \mathsf{RCA}_0 + T$ has an infinite path.

Proof For a Σ^0_1 formula φ with parameters from S^T, there exists an equivalent Σ^0_1 formula ψ with parameters only from $S \cup \{G\}$, which is obtained from the former by replacing a parameter X of S^T with a Δ^0_1 formula defining it. Recall that the same argument was used to show that RCA_0 is a conservative extension of $\mathsf{I}\Sigma_1$ (in part 7, Lemma 1.3). Then, by Lemma 1.7, $(M, S^T) \models \mathsf{RCA}_0$. Also, in (M, S^T), T has an infinite path G. □

Notice that if (M, S) is countable, then S^T is also countable. In the following lemma, this process is repeated to construct a model (M, S_∞) of WKL_0, which is also countable.

Lemma 1.9

For any countable model (M, S) of RCA_0, there exists a countable set S_∞ such that $S \subseteq S_\infty \subseteq \mathcal{P}(M)$ and $(M, S_\infty) \models \mathsf{WKL}_0$.

Proof Construct $S_0 \subseteq S_1 \subseteq \cdots$ as follows: $S_0 = S$, and

$$S_{(n,m)+1} = S^T_{(n,m)}, \text{ where } T \text{ is the } m\text{-th infinite tree in } S_n (\subseteq S_{(n,m)}).$$

Here, $(n, m) = \frac{(n+m)(n+m+1)}{2} + n$, and so $(n, m) \geq n$. Finally, let $S_\infty = \bigcup_{i \in \omega} S_i$. It is clear from the definition that this is the desired set. □

Theorem 1.10 (Harrington)

For any Π^1_1 sentence σ, $\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{RCA}_0 \vdash \sigma$.

Proof Suppose σ is a Π^1_1 sentence that is not provable in RCA_0. By Gödel's completeness theorem, there exists a countable model $(M, S) \models \mathsf{RCA}_0 + \neg\sigma$. Now, $\neg\sigma$ can be expressed as $\exists X \varphi(X)$ with $\varphi \in \Pi^1_0$. Then there exists $A \in S$ such that $(M, S) \models \mathsf{RCA}_0 + \varphi(A)$. By constructing S_∞ by Lemma 1.9, we have $(M, S_\infty) \models \mathsf{WKL}_0 + \varphi(A)$. Note that since $\varphi(X)$ is arithmetical, the truth value of $\varphi(A)$ depends only on M and A. Therefore, $(M, S_\infty) \models \mathsf{WKL}_0 + \neg\sigma$, which implies $\mathsf{WKL}_0 \nvdash \sigma$. □

§8.2. Semi-Regular Cuts and Friedman's Theorem

The goal of this section is to prove a theorem of H. Friedman that "WKL_0 is Π^0_2 conservative over PRA." First of all, we introduce a formal system of finitistic arithmetic PRA, which stands for Primitive Recursive Arithmetic, to handle all primitive recursive functions on the natural numbers. Its language consists of symbols for the primitive recursive functions and its axioms are their defining equations, along with Σ_0 induction. A model of PRA is of the form $(M, \mathtt{f}_0^M, \mathtt{f}_1^M, \cdots)$, also denoted as (M, F) or just M.

Now, we fix a nonstandard model (M, F) of PRA (i.e., $M \neq \omega$). Also, let $p \in F$ be a primitive recursive function that lists the prime numbers in the ascending order, i.e., $p(0) = 2, p(1) = 3, p(2) = 5, \cdots$.

Definition 2.1

A set $X (\subseteq M)$ has a **code** $c \in M$ or is coded by c, if

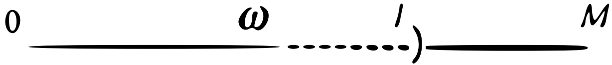
$$X = \{n \in M : M \models \exists d < c \; (c = p(n) \cdot d)\}.$$

Such a set X is called **M-finite**, and the number of elements in X is denoted by $|X|$ or $|c|$.

Note $|x|$ is a primitive recursive function on M, i.e., $|x| \in F$. Also, if $X (\neq \varnothing)$ has a code c, the largest element of X can be denoted by $\max(c)$, and $\max(x) \in F$.

Definition 2.2

A proper initial segment I of M is called a **cut** of M, denoted $I \subseteq_e M$[4], if it is closed under the successor function (i.e., $a \in I \Rightarrow a+1 \in I$).
Furthermore, a cut $I \subseteq_e M$ is called a **semi-regular cut**, if $X \cap I$ is bounded within I for any M-finite set X with $|X| \in I$.



Note. If X is an M-finite set and $X \cap I$ is bounded in I, then $X \cap I$ is also M-finite, and so the largest element of $X \cap I$ exists.

The analogy between the semi-regular cuts of nonstandard models of arithmetic and the regular cardinals in models of set theory was discovered by Paris and his colleagues in the UK in the mid-1970s. Recall that a regular cardinal is a cardinal such that the range of any function from a smaller cardinal to it is always bounded.

[4]In Chapter 5, all initial segments were denoted by $\subseteq_e$.

Let (M, F) be a nonstandard model of PRA.

Theorem 2.3 (Kirby-Paris)

If $I \subseteq_e M$ is a semi-regular cut, then $(I, F\lceil I) \models$ PRA, where $F\lceil I$ is the set of functions obtained by restricting the domain of each function f in F to I.

Proof First, we show that I is closed under primitive recursive functions. For each $n \in \omega$, define the unary primitive recursive function g_n as follows:

$$\mathrm{g}_0(x) = x + 1,$$

$$\mathrm{g}_{n+1}(x) = \overbrace{\mathrm{g}_n \mathrm{g}_n \cdots \mathrm{g}_n}^{x+2}(x)^{5}$$

For any primitive recursive function symbol $\mathtt{f}$, there exists some $n \in \omega$ such that

$$\text{PRA} \vdash \mathtt{f}(x_1, x_2, \cdots, x_k) < \mathrm{g}_n(\max\{x_1, x_2, \cdots, x_k\})$$

(where for $k = 0$, the value of $\max$ is taken as 0). Let's briefly demonstrate this fact.

[5] To see $\mathrm{g}_{n+1}(x)$ is primitive recursive, we first introduce a two-variable function $\mathrm{g}'_{n+1}(x, y)$ as follows: $\mathrm{g}'_{n+1}(x, 0) = 0, \mathrm{g}'_{n+1}(x, y+1) = \mathrm{g}_n(\mathrm{g}'_{n+1}(x, y))$. Then $\mathrm{g}_{n+1}(x) = \mathrm{g}'_{n+1}(x, x+2)$ is primitive recursive. Compare with the Ackermann function in part 1.

For the three initial functions of primitive recursion, we have $\mathrm{Z}() = 0 < \mathsf{g}_0(0)$, $\mathrm{S}(x) = x + 1 < 2x + 2 = \mathsf{g}_1(x)$, and $\mathrm{P}^n_i(x_1, \ldots, x_n) = x_i < \mathsf{g}_0(\max\{x_1, x_2, \ldots, x_k\})$. For function composition, we consider one-variable functions for simplicity. If $h_1(x) < \mathsf{g}_n(x)$ and $h_2(x) < \mathsf{g}_n(x)$, then their composite function $h_1(h_2(x)) < \mathsf{g}_n(\mathsf{g}_n(x)) \leq \mathsf{g}_{n+1}(x)$. For primitive recursion $f(x, y + 1) = h(x, y, f(x, y))$ defined by $f(x, 0) < \mathsf{g}_n(x)$ and $h(x, y, z) < \mathsf{g}_n(\max\{x, y, z\})$, we have $f(x, y) < \mathsf{g}_n^{y+2}(\max\{x, y\}) \leq \mathsf{g}_{n+1}(\max\{x, y\})$. Hence, every primitive recursive function is bounded by some g_n.

To confirm that I is closed under all primitive recursive functions, it suffices to show closure for each g_n. By definition 2.2, I is closed under the successor function, so the case $n = 0$ holds. Now, by way of contradiction, assume it is closed under g_n, but not g_{n+1}. Then choose $a \in I$ such that $\mathsf{g}^M_{n+1}(a) \notin I$, and define

$$X = \{\mathsf{g}^M_n(a), \mathsf{g}^M_n\mathsf{g}^M_n(a), \cdots, \overbrace{\mathsf{g}^M_n\mathsf{g}^M_n\cdots\mathsf{g}^M_n}^{a+2}(a)\}.$$

Since X is an M-finite set with $|X| = a + 2 \in I$, so $X \cap I$ is bounded and has a maximum element b. However, since I is closed under g_n, we have $\mathsf{g}^M_n(b) \in X \cap I$, contradicting the maximality of b.

From the above, $(I, F\lceil I)$ can be considered a substructure of (M, F), and thus the truth values of Σ_0 formulas are the same in both structures. Finally, to show $(I, F\lceil I) \models \Sigma_0$-ind., let $\varphi(x)$ be a Σ_0 formula and assume $(I, F\lceil I) \models \varphi(0) \land \forall x(\varphi(x) \to \varphi(x+1))$. Choose any $c \in I$ and let $\psi(x) = \varphi(x) \lor c < x$. Then, $\psi(x)$ is also Σ_0 and it is easy to see $(M, F) \models \psi(0) \land \forall x(\psi(x) \to \psi(x+1))$. Since $(M, F) \models \Sigma_0$-induction, $(M, F) \models \forall x \psi(x)$ and so $(M, F) \models \psi(c)$, which means $(M, F) \models \varphi(c)$, and thus $(I, F\lceil I) \models \varphi(c)$. Since $c \in I$ is arbitrary, we obtain $(I, F\lceil I) \models \forall x \varphi(x)$. Therefore, $(I, F\lceil I) \models \Sigma_0$-induction, and so $(I, F\lceil I) \models$ PRA. □

Definition 2.4

Let $I \subseteq_e M$ and let S be the set of all M-finite sets. A set $B \subseteq I$ is called an **M-coded set** if there exists a $X \in S$ such that $B = X \cap I$. Then, B is also coded by a code c of X. We denote the set of all M-coded subsets of I by $S\lceil I$.

Note. We can consider $(I, S\lceil I)$ as a structure of second-order arithmetic, with basic operations $+^I, \cdot^I$, etc., which are obtained by restricting the corresponding operations (primitive recursive functions) on M to I.

Lemma 2.5

If $I \subseteq_e M$ is a semi-regular cut, then $(I, S\lceil I) \models \mathsf{WKL}_0$.

Proof It is clear that $(I, S\lceil I)$ satisfies the basic axioms of arithmetic. Therefore, what we need to show is that it satisfies (Δ^0_1-CA), weak König's lemma (WKL), and Σ^0_1 induction. Let's start with Σ^0_1 induction. Similarly to Lemma 8.1.7 1.7, it suffices to show (bounded Σ^0_1-CA).

First, we consider how to transform a formula θ in $(I, S\lceil I)$ into a formula θ^* in (M, F). For each set parameter $B \in S\lceil I$ in θ, let c_B be a code of B or equivalently a code of X such that $B = X \cap I$. θ^* is obtained from θ by replacing every subformula "$t \in B$" with "$\exists d < c_B\ (c_B = p(t) \cdot d)$". Then, if θ is Σ^0_0, also is θ^*. It is easy to see that for any $a \in I$,

$$(I, S\lceil I) \models \theta(a) \ \Leftrightarrow\ (M, F) \models \theta^*(a)$$

Next, consider a Σ^0_1 formula $\varphi(x) = \exists y \theta(x, y)$ (where θ is Σ^0_0 in $(I, S\lceil I)$). The goal is to show that for arbitrary $c \in I$, $\{x < c \mid \varphi(x)\} \in S\lceil I$. Take any $d \in M - I$ and define

$$Z = \{(a, b) : a <_M c,\ b <_M d \text{ and } (M, F) \models \theta^*(a, b) \land \forall x < b \neg\theta^*(a, x)\}.$$

That is, if $(a, b) \in Z$, b is the smallest element in M such that $\theta^*(a, b)$ holds. It is evident that Z is M-finite with $|Z| \leq_M c$. From the semi-regularity of I, $Z \cap (I \times I)$ is bounded, and so there exists $d' \in I$, such that for all $a <_M c$,

$$\exists b \in I\ (a, b) \in Z \ \Leftrightarrow\ \exists b <_M d'\ (a, b) \in Z.$$

Since (M, F) satisfies Σ_0 induction (the least number principle), for all $a <_M c$,

$$\exists b \in I\ (M, F) \models \theta^*(a, b) \ \Leftrightarrow\ \exists b \in I\ (a, b) \in Z$$
$$\Leftrightarrow\ \exists b <_M d'\ (a, b) \in Z \ \Leftrightarrow\ \exists b <_M d'\ (M, F) \models \theta^*(a, b)$$

Therefore, for all $a <_M c$,

$$(I, S\lceil I) \models \varphi(a) \Leftrightarrow \exists b \in I \ (I, S\lceil I) \models \theta(a, b)$$

$$\Leftrightarrow \exists b \in I \ (M, F) \models \theta^*(a, b)$$

$$\Leftrightarrow \exists b <_M d' \ (M, F) \models \theta^*(a, b)$$

$$\Leftrightarrow (M, F) \models \exists y < d' \ \theta^*(a, y)$$

Since the last expression $\exists y < d' \ \theta^*(a, y)$ is a Σ_0 formula,

$$\{a < c : (I, S\lceil I) \models \varphi(a)\} = \{a < c : (M, F) \models \exists y < d' \ \theta^*(a, y)\}$$

is an M-coded set, belonging to $S\lceil I$.

Since $(\Delta^0_1\text{-CA})$ + weak König's lemma is equivalent to $(\Sigma^0_1\text{-SP})$ combined[6], it suffices to show that $(I, S\lceil I) \models (\Sigma^0_1\text{-SP})$. Let $\varphi_i(x) = \exists y \theta_i(x, y)$, $\theta_i(x, y) \in \Sigma^0_0$ $(i = 0, 1)$, and assume $(I, S\lceil I) \models \neg\exists x(\varphi_0(x) \wedge \varphi_1(x))$. Similar to the above, let θ^*_i be the Σ_0 formula obtained by replacing the set parameters of θ_i with their definitions. Now, fix any $d \in M - I$, and define

$$Y = \{a <_M d \mid \exists b <_M d \ (M, F) \models \theta^*_0(a, b) \wedge \forall x < b \neg\theta^*_1(a, x)\}$$

That is, Y is the set of element a such that, when b increases from below, $\theta^*_0(a, b)$ holds before $\theta^*_1(a, b)$. Obviously, Y is M-finite, so $Y \cap I \in S\lceil I$. Then, it is easy to see

$$(I, S\lceil I) \models \forall a[(\varphi_0(a) \to a \in Y \cap I) \wedge (\varphi_1(a) \to a \notin Y \cap I)].$$

Hence, $(I, S\lceil I) \models (\Sigma^0_1\,\text{SP})$. From all the above, $(I, S\lceil I) \models \text{WKL}_0$ is proved. □

[6]See Lemma 3.6 in part 7

Thank you for your attention!