Logic and Foundations II

Part 7. Real Analysis and Reverse Mathematics

Kazuyuki Tanaka

BIMSA

May 21, 2024

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Real analysis and reverse mathematics (9 lectures)
- Part 8. Second order arithmetic and non-standard methods (6 lectures)

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining real numbers in RCA_0
- Apr. 23, (3) Completeness of the reals and ACA_0
- Apr. 25, (4) Continuous functions and WKL_0
- Apr. 30, (5) Continuous functions and WKL_0, II
- May 9, (6) König's lemma and Ramsey's theorem
- May 14, (7) Determinacy of infinite games I
- May 16, (8) Determinacy of infinite games II
- May 21, (9) Determinacy of infinite games III + Introduction to Part 8

Gale-Stewart games

In a **Gale-Stewart game** G, two players I and II alternately choose natural numbers, constructing an infinite sequence (called a **play**)

$$\begin{array}{cccccccc} \mathrm{I} & n_0 & & n_2 & & n_4 & & \ldots \\ \mathrm{II} & & n_1 & & n_3 & & n_5 & \ldots \end{array}$$

If the resulting sequence $(n_0, n_1, n_2, \ldots)$ is in a predetermined **winning set** $G \subseteq \mathbb{N}^{\mathbb{N}}$, then player I **wins**; otherwise, player II wins.

A **strategy** for player I is a function $\sigma : \cup_{i \in \mathbb{N}} \mathbb{N}^{2i} \to \mathbb{N}$, and a **strategy** for player II is a function $\tau : \cup_{i \in \mathbb{N}} \mathbb{N}^{2i+1} \to \mathbb{N}$. If the players obey their strategies σ and τ, a play $(n_0, n_1, n_2, \ldots)$, denoted $\sigma \otimes \tau$, is uniquely determined as follows:

$$\begin{array}{llllllll} \mathrm{I} & n_0 = \sigma(\varnothing) & & n_2 = \sigma(n_0, n_1) & & n_4 = \sigma(n_0, n_1, n_2, n_3) & \ldots & \\ \mathrm{II} & & n_1 = \tau(n_0) & & n_3 = \tau(n_0, n_1, n_2) & & n_5 = \tau(n_0, n_1, n_2, n_3, n_4) & \ldots \end{array}$$

Then, σ is called a **winning strategy** for player I if for any τ, $\sigma \otimes \tau$ belongs to G, that is, player I can win the game with σ whatever II plays. A **winning strategy** for player II is defined similarly. When one of the players has a winning strategy, the game G is said to be **determined**, or **determinate**.

Fact 1: The determinacy of $\underset{\sim}{\Sigma}^0_1$ games is equivalent to the determinacy of $\underset{\sim}{\Pi}^0_1$ games. $\because$ By thinking that the first move of player I has no effect to the rest of game, the game may be considered to start with player II, which makes a $\underset{\sim}{\Sigma}^0_1$ game a $\underset{\sim}{\Pi}^0_1$ game, and vice versa.

Lemma 5.5

ATR_0 proves $\underset{\sim}{\Sigma}^0_1$-Det.

Proof If there exists a well-order $\prec$ along which we can define a set $\mathcal{W}$ of sure winning positions such that $\varnothing \in \mathcal{W}$. Then, player I can win by keeping in $\mathcal{W}$ and eventually reaching W_0.

If such a well-order $\prec$ never exists, there must exist a non-well-founded linear order $\prec$ and a $\prec$-ordered set $\mathcal{W}$ of sure winning positions such that $\varnothing \notin \mathcal{W}$. Such a set W is called a **pseudo-hierarchy**. Then, player II can win by keeping out of $\mathcal{W}$, which becomes player II's winning strategy. Thus, $\underset{\sim}{\Sigma}^0_1$ games are determined in ATR_0. $\square$

Fact 2: We show that $\underset{\sim}{\Sigma}^0_1$-Det implies ACA_0. For any Σ^0_1 formula $\exists x \theta(n, x)$, consider the following game. Player I chooses n and player II answers Yes with a witness x, or No. If II answers No, then I must select a witness x. Then, player II wins if he answers Yes and $\theta(n, x)$ holds, or No and $\neg\theta(n, x)$. Since player I can not win both the cases, player II has a winning strategy τ. Hence in RCA_0, $\{n : \exists x \theta(n, x)\} = \{n : \tau(n) = (\text{Yes}, x)\}$ exists.

Theorem 5.6

In RCA_0, $\underset{\sim}{\Delta}^0_1$-Det implies ATR_0, and thus $\underset{\sim}{\Delta}^0_1$-Det, $\underset{\sim}{\Sigma}^0_1$-Det and ATR_0 are equivalent.

Proof We may work within ACA_0. A well-order $\prec$, an initial set $(H)_0 = A$ and Π^0_1 formula $\varphi(n, X) \equiv \forall x\ \theta(n, X \restriction x)$ are given. Two players engage in a debate on the hierarchy $\{H_a\}$ claimed to exist by ATR_0. Player II wins the game by making correct assertions thoroughly. Since player II's winning strategy accurately describes the hierarchy $\{H_a\}$, the strategy allows $\{H_a\}$ to be constructed within RCA_0.

Our game proceeds as follows: First, player I chooses (b, y) intending to pose a question of whether $y \in (H)_b$ or not. Player II answers with Yes ("1") or No ("0").

The debate progresses by selecting lower elements a for $(H)_a$ according to the well-ordering $\prec$, and so it always terminates in a finite number of steps. Hence, the winning set can be written as a Δ^0_1 formula.

Moreover, we can see that player I does not have a winning strategy, since it is impossible for player I to win the debate whether player II answers Yes or No. Thus, by $\underset{\sim}{\Delta}^0_1$-Det, player II has a winning strategy τ, and $H = \{(b, y) : \tau(b, y) = 1\ (\text{"yes"})\}$ becomes the desired set. □

Fact 3: We show Π^1_1-CA$_0$ implies ATR$_0$. The axiom of arithmetical transfinite recursion can be written as a Σ^1_1 formula: $\prec$ is well-ordered $\to \exists H\ \theta_\prec(H)$. So, this can be shown by Σ^1_1 transfinite induction over ACA$_0$, hence also by Π^1_1-CA$_0$.

Theorem 5.7

The determinacy of $\underset{\sim}{\Sigma}^0_1 \wedge \underset{\sim}{\Pi}^0_1$ games and Π^1_1-CA$_0$ are equivalent over RCA$_0$.

Proof First, we demonstrate the determinacy of $\underset{\sim}{\Sigma}^0_1 \wedge \underset{\sim}{\Pi}^0_1$ games within Π^1_1-CA$_0$. Consider a game $A(f) \equiv \psi_1(f) \wedge \psi_2(f)$, where ψ_1 is $\psi_1(f) \equiv \exists x\theta_1(f \restriction x)$ and $\psi_2 \equiv \forall x\theta_2(f \restriction x)$. Then the following set W is Σ^1_1.

$$W = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \theta_1(s) \text{ and I has a winning strategy for } \psi_2 \text{ at } s.\}$$

Here, "I has a winning strategy for ψ_2 at s" can be restated as "there exists a strategy τ at s such that all plays f following τ satisfy $\psi_2(f)$". Moreover, "all plays f" can be translated as "all finite plays $f \restriction x$". Hence, according to Π^1_1-CA$_0$, W exists.

Then consider the Σ^0_1 (in W) game $W^* = \{f \in \mathbb{N}^\mathbb{N} \mid \exists x(f \restriction x \in W)\}$. If player I has a winning strategy for W^*, then by following it, player I will eventually enter W, and from the position s, using a winning strategy for ψ_2, finally $\psi_1(f) \wedge \psi_2(f)$ will hold.

On the other hand, suppose player I has no winning strategy for W^*. Then player II can make a play out of W throughout the game. So $\theta_1(s)$ never holds, i.e., $\neg\psi_1$, or player I does not have a winning strategy for ψ_2, i.e., by $\underset{\sim}{\Sigma}^0_1$-Det, player II has a winning strategy, and so $\neg\psi_1$ will hold. That is, we have $\neg\psi_1 \vee \neg\psi_2$. Thus, $A(f)$ is determined.

Conversely, from the determinacy of $\underset{\sim}{\Sigma}^0_1 \wedge \underset{\sim}{\Pi}^0_1$ games, we prove Π^1_1-CA$_0$. First, let $\varphi(n)$ be $\forall f \exists x \theta(n, f \restriction x)$, where θ is Σ^0_0. Consider the following game G:

First, player I chooses n. Then, player II answers Yes or No. If player II answers Yes, she generates an (infinite) sequence f until player I stops it. If player II answers No, player I generates f and player II stops. If at the stopping point (step x) $\theta(n, f \restriction x)$ holds, then the player generating f wins.

This game is in $\underset{\sim}{\Sigma}^0_1 \wedge \underset{\sim}{\Pi}^0_1$, and it is not possible for player I to have a winning strategy. Therefore, player II must have a winning strategy τ. Consequently, the set defined by $\varphi(n)$ will be $\{n : \tau(n) = \text{Yes}\}$, and this exists in RCA$_0$.

□

By generalizing Theorem 5.7, we can also show that the determinacy of games defined by Boolean combinations of $\undertilde{\Sigma}^0_1$ sets can be obtained through iterations of Π^1_1-CA_0.

For example, consider the determinacy of a game $A(f) \equiv B(f) \vee \psi(f)$, where B is $\Sigma^0_1 \wedge \Pi^0_1$ and $\psi \equiv \forall x \theta(f \upharpoonright x)$. As in the proof of the above theorem, the set W of player I's sure winning positions for $B(f)$ can be written as a Σ^1_1 formula, and hence it exists in Π^1_1-CA_0. Next, define the following Σ^1_1 (in W) set

$$V = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \neg\theta(s) \text{ and II has a winning strategy for } \neg W^* \text{ at } s.\}$$

If player II has a winning strategy for $V^* = \{f \in \mathbb{N}^{\mathbb{N}} \mid \exists x(f \upharpoonright x \in V)\}$, then by following it, player II will eventually enter V, and from that position s, using a winning strategy for $\neg W^*$, finally $\neg\psi(f) \wedge \neg B(f) \equiv \neg A(f)$ will hold.

If player I has a winning strategy for $\neg V^*$, then $\neg\theta(s)$ never holds, or I has a winning strategy for W. So $\psi(f) \vee B(f) \equiv A(f)$ holds.
As we will see in the following slides, $\undertilde{\Delta}^0_2$ sets are expressed as transfinite combinations of $\undertilde{\Sigma}^0_1$ sets, and so $\undertilde{\Delta}^0_2$-Det can be deduced from transfinite iterations of Π^1_1-CA_0, i.e., Π^1_1-TR_0

For this purpose, I introduced the effective version of the Hausdorff-Kuratowski theorem on ambiguous Borel sets in my dissertation in 1986.

Now, we define the effective difference hierarchy over Π^0_n formulas. For a well-order $\prec$ on $\mathbb{N}$, define the well-order $\prec^*$ on $\mathbb{N} \times \{0,1\}$ as follows:

$$(x,i) \prec^* (y,j) \quad \text{iff} \quad x \prec y \vee (x = y \wedge i < j).$$

A formula $\varphi(n,i,f)$ is said to **decreasing** along $\prec^*$ if it satisfies:

$$\forall f \in \mathbb{N}^{\mathbb{N}}\ \forall n \forall i \forall m \forall j\ (((m,j) \prec^* (n,i) \wedge \varphi(n,i,f)) \to \varphi(m,j,f)).$$

Then, the difference hierarchy $\mathcal{D}^0_{n+1}$ is defined as follows.

Definition 5.8 (Effective Difference Hierarchy)

For $n \geq 1$, $A \subseteq \mathbb{N}^{\mathbb{N}}$ belongs to $\mathcal{D}^0_{n+1}$ iff there exists a Π^0_n formula $\varphi(x,i,f)$ decreasing along a well-order $\prec^*$ such that

$$A(f) \Leftrightarrow \exists x(\neg\varphi(x,1,f) \wedge \varphi(x,0,f)).$$

For effective hierarchies, a well-order $\prec$ may be assumed to be recursive. But for instance, when you consider $\underset{\sim}{\Delta}^0_2$-Det, $\prec$ must be recursive in existing parameters.

Theorem 5.9 (Effective Difference Hierarchy Theorem)

In ACA_0, $\mathcal{D}^0_n = \Delta^0_n$ $(n \geq 2)$.

Theorem 5.10

$\underset{\sim}{\Delta}{}^0_2$-Det is equivalent to Π^1_1-TR_0 in RCA_0.

For the details and proof of the above definition and theorems, see: K.Tanaka, Weak axioms of determinacy and subsystems of analysis I: Δ^0_2 games, *Zeitschr. f. math. Logik und Grundlaten d. Math.*, **36**, 481-491,1990.

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Part 8. Schedule

- May 21, (1) Introduction to forcing
- May 23, (2) Harrington's conservation result on WKL_0
- May 28, (3) H.Fridman's conservation result on WKL_0
- May 30, (4)
- June 04, (5)
- June 06, (6)
- June 11, (7)

§8.1. Forcing and Harrington's Theorem

In this section, we introduce Harrington's theorem that "WKL_0 is a Π^1_1 conservative extension of RCA_0." The forcing argument of adding infinite paths of an infinite tree as generic paths to a ground model was invented by Jockusch and Soare (Π^0_1 classes and degrees of theories, *Trans. of the A. M. S.* 173 (1972), pp.35–56). Subsequently, Harrington cleverly applied it to non-ω models in second-order arithmetic.

The basic idea of forcing is to generate something that does not exist in the world without causing confusion. First, a set of conditions $\mathbb{P}$ for what to generate is given, and a partial order is defined on $\mathbb{P}$. Ways to interpret these conditions varies depending on applications, and we first proceed without giving particular meanings.

Fix an arbitrary partially ordered set $(\mathbb{P}, <)$, and let $p, q, r, \ldots$ denote elements of $\mathbb{P}$. A set $G \subseteq \mathbb{P}$ is called an **open set**, if it satisfies the following condition

$$\forall p, q\ (q < p \land p \in G \to q \in G).$$

Thus, $(\mathbb{P}, <)$ becomes a topological space. Now, let

$$[p] = \{q \in \mathbb{P} \mid q \leq p\}.$$

Any open set G coincides with $\bigcup_{p \in G}[p]$, and so $\{[p] \mid p \in \mathbb{P}\}$ forms a basis for the topology.

Any set $D \subseteq \mathbb{P}$ is called a **dense** set, if it has a non-empty intersection with every non-empty open set. The condition for D to be dense is equivalent to

$$\forall p \in \mathbb{P}\ \ [p] \cap D \neq \emptyset, \text{ in other words, } \forall p \in \mathbb{P}\ \ \exists d \in D\ \ d \leq p.$$

Definition 1.1

A set $F \subseteq \mathbb{P}$ is called a **filter**, if it satisfies the following conditions:

1) $p \in F \land p < q \to q \in F$,
2) $\forall p, q \in F\ \ [p] \cap [q] \cap F \neq \emptyset$.

Definition 1.2

Given a family of sets $\mathcal{D}$, a filter G is called a **$\mathcal{D}$-generic filter** if it intersects every dense set $D \subseteq \mathbb{P}$ belonging to $\mathcal{D}$.

Lemma 1.3

If $\mathcal{D}$ contains at most countably many dense subsets of $\mathbb{P}$, then for any $p \in \mathbb{P}$, there exists a $\mathcal{D}$-generic filter G that contains p.

Proof Enumerate the dense subsets of $\mathbb{P}$ contained in $\mathcal{D}$ as $D_0, D_1, \cdots, D_i, \cdots (i \in \omega)$. For a given $p \in \mathbb{P}$, construct a decreasing sequence $p_0 \geq p_1 \geq \cdots$ from $\mathbb{P}$ as follows: $p_0 = p$, and $p_n \in [p_{n-1}] \cap D_{n-1}$ for each $n > 0$. Then, we set $G = \{q \mid \exists i \ p_i \leq q\}$. Thus, it is obvious that $p \in G$ and G is a $\mathcal{D}$-generic filter. □

Now, we will introduce the forcing conditions used in Harrington's proof.
Let $\mathfrak{M} = (M, S)$ be a countable model of RCA_0. Here, M is the first-order part (the domain corresponding to the natural numbers), and S is the second-order part consisting of subsets of M, that is, $S \subseteq \mathcal{P}(M)$. Then, set

$$\mathbb{P} = \{T \in S \mid \mathfrak{M} \models \text{“}T (\subseteq \mathrm{Seq}_2) \text{ is an infinite binary tree”}\},$$

and define a partial order on $\mathbb{P}$ by

$$T_1 \le T_2 \Leftrightarrow T_1 \subseteq T_2.$$

For each $T \in \mathbb{P}$, we want to generate an infinite path and put it into S. But if we bring in an arbitrary path of T from outside, it might break the condition of $\mathcal{P}(M)$ such as induction axiom. Instead, we approximate an infinite path by $T' \le T$, and for this purpose, the concept of density is important, namely

$$D \subseteq \mathbb{P} \text{ is dense } \Leftrightarrow \forall T \in \mathbb{P}\ \exists T' \in D\ T' \le T.$$

$E \subseteq \mathbb{P}$ is said to be **definable in $\mathfrak{M}$** if there exists a formula $\varphi(X)$ (with parameters from $M \cup S$) such that $E = \{T \in \mathbb{P} \mid \mathfrak{M} \models \varphi(T)\}$. The totality of such sets is denoted by $\mathrm{Def}(\mathfrak{M})$. Since we only consider a countable model $\mathfrak{M} = (M, S)$ in a countable language, $\mathrm{Def}(\mathfrak{M})$ is a countable set. By Lemma 1.3, any $T \in \mathbb{P}$ is contained in some $\mathrm{Def}(\mathfrak{M})$-generic filter. Such a filter is simply referred to as an $\mathfrak{M}$-generic filter.

Lemma 1.4

If $F \subseteq \mathbb{P}$ is an $\mathfrak{M}$-generic filter, then there exists a unique infinite path $G = \cap F = \cap_{T \in F} T$ common to all $T \in F$. That is, F is contained in the principal filter generated by G.

Proof For each $k \in M$, let $E_k = \{T \in \mathbb{P} \mid \exists! s \in \{0,1\}^k \ s \in T\}$ be dense and definable in $\mathfrak{M}$. If F is an $\mathfrak{M}$-generic filter, then for each k, there exists some $s_k \in \{0,1\}^k$ such that there is $T_k \in F$ with $T_k \cap \{0,1\}^k = \{s_k\}$. Moreover, if $k < k'$, then s_k is an initial segment of $s_{k'}$, and $s_{k'} \in T_k$. If not, $[T_k] \cap [T_{k'}] = \emptyset$[1], which would contradict the filter condition of F. Thus, let $G = \bigcup_{k \in M} s_k$; then $G = \bigcap_k T_k$ as well. Finally, to show $G = \cap F$, if $G \not\subseteq T \in F$, then there exists some k such that $s_k \notin T$, and $[T] \cap [T_k] = \varnothing$, which contradicts the filter condition of F. $\square$

[1] Here, $[T]$ denotes $\{T' \in \mathbb{P} \mid T' \subset T\}$. In the latter half of part 8, the same notation $[T]$ represents the set of infinite paths of T. Since both are conventional, we would use both as they are.

Definition 1.5

$G(\subseteq M)$ is called an **$\mathfrak{M}$-generic path**, if for every dense set $D \in \mathrm{Def}(\mathfrak{M})$, there exists a tree $T \in D$ such that G is an infinite path through T.

Lemma 1.6

Every $T \in \mathbb{P}$ has an $\mathfrak{M}$-generic path G.

Proof By Lemma 1.3, every T is contained in some $\mathfrak{M}$-generic filter F. Furthermore, by Lemma 1.4, there is a common infinite path G in the trees of F. It is clear from the definition that this G is an $\mathfrak{M}$-generic path. □

From now on, an $\mathfrak{M}$-generic path will simply be referred to as a generic path.

Lemma 1.7

If G is a generic path, then $(M, S \cup \{G\}) \models \Sigma^0_1$-induction.

Proof Let $\varphi(i, X)$ be any Σ^0_1 formula, and choose any $b \in M$, and we will show that $A = \{a \leq_M b \mid \varphi(a, G)\} \in S$ [2]. If $A \in S$, induction on $\varphi(n, G)$ can be shown as follows.

[2] See Lemma 1.8 of part 7 for (bounded Σ^0_1-CA).

Suppose $A \in S$. Then, $B = \{a \mid a \in A \vee a >_M b\} \in S$ since $\mathfrak{M} \models (\Delta^0_1\text{-CA})$. Now, assume $\varphi(0, G)$ and $\forall n(\varphi(n, G) \to \varphi(n+1, G))$. Then, we have $0 \in B$ and $\forall m(m \in B \to m+1 \in B)$. Since $\mathfrak{M} \models \Sigma^0_1$-induction, by induction on B, we have $B = M$. Therefore, $b \in A$, that is, $\varphi(b, G)$. Since $b \in M$ is arbitrary, we get $\forall n \varphi(n, G)$.

Now we show $A \in S$. Let $\varphi(i, X) \equiv \exists j \theta(i, X\lceil j)$ (where $\theta \in \Sigma^0_0$)[3], and set

$$D_b = \{T \in \mathbb{P} \mid \mathfrak{M} \models \forall a \leq b \ (1)\ \forall t \in T \neg\theta(a, t) \vee$$
$$(2)\ \exists k \forall t \in T \cap \{0,1\}^k \exists s \subseteq t \theta(a, s)\}.$$

Of course, D_b is definable in $\mathfrak{M}$. Here, note that if $T \in D_b$ and $T' \subseteq T$, then $T' \in D_b$. And as shown below, D_b is dense, so there exists a tree T_0 in D_b that has G as an infinite path. Fix such a T_0. For simplicity, we write $(1)_{T_0}$ for above condition (1) with $T = T_0$, and $(2)_{T_0}$ for condition (2) with $T = T_0$.

[3] $X \restriction j$ represents the code of the initial segment $(f(0), \cdots, f(j-1))$ of the characteristic function f of X. The truth value of the Σ^0_0 formula $\theta(X)$ depends only on a finite part of X, so for sufficiently large j, X can be replaced by $X \restriction j$. See [Simpson, Theorem II.2.7] for details.

Then, for each $a \leq_M b$,

$$\mathfrak{M} \models (1)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \neg\varphi(a, G),$$
$$\mathfrak{M} \models (2)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \varphi(a, G).$$

Since $\mathfrak{M} \models (1)_{T_0} \vee (2)_{T_0}$, we have

$$\mathfrak{M} \models (2)_{T_0} \Leftrightarrow (M, S \cup \{G\}) \models \varphi(a, G)$$

Since (2) is a Σ^0_1 formula, and $\mathfrak{M} \models (\text{bounded}\Sigma^0_1\text{-CA})$ (Lemma 1.8, Chapter 7),
$A = \{a \leq_M b \mid \mathfrak{M} \models (2)_{T_0}\} \in S$.

Finally, we show that D_b is dense. Choose any $\tilde{T} \in \mathbb{P}$. For each $\sigma \in \{0,1\}^{\leq b}$, define a tree T_σ inductively as follows:

$T_\varnothing = \tilde{T}$,
$T_{\sigma^\frown 0} = \{t \in T_\sigma \mid \forall s \subseteq t\ \neg\theta(a, s)\}$, where $a = \text{leng}(\sigma)$,
$T_{\sigma^\frown 1} = T_\sigma$.

Here, $\varnothing$ is the empty sequence, and $\sigma^\frown i$ denotes the sequence σ followed by $i (= 0, 1)$.

Next, let $S_b = \{\sigma \in \{0,1\}^{b+1} \mid T_\sigma$ is an infinite tree$\}$. Then, since "T_σ is an infinite tree" is expressed by a Π^0_1 formula $\forall n \exists \tau \in \{0,1\}^n \ \tau \in T_\sigma$, by (bounded Σ^0_1-CA), we have $S_b \in S$. Also, since $\overbrace{<1,1,\cdots,1>}^{b+1} \in S_b$, we get $S_b \neq \varnothing$.
Thus, let σ_b be the lexicographically first element in S_b. Take any $a \leq_M b$.

If $\sigma_b(a) = 0$, then $(\sigma_b \lceil a)^\frown 0 \subset \sigma_b$, so

$$T_{\sigma_b} \subseteq T_{(\sigma_b \lceil a)^\frown 0} \subseteq \{t \mid \neg\theta(a,t)\},$$

from which we have $(1)_{T_{\sigma_b}}$.

If $\sigma_b(a) = 1$, then $T_{(\sigma_b \lceil a)^\frown 0}$ is finite, so $(2)_{T_{\sigma_b \lceil a}}$ and hence $(2)_{T_{\sigma_b}}$ also holds.

From all the above, $T_{\sigma_b} \in D_b$, and it has been shown that D_b is dense. □

Thank you for your attention!