Logic and Foundations II

Part 7. Real Analysis and Reverse Mathematics

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Real analysis and reverse mathematics (9 lectures)
- Part 8. Second order arithmetic and non-standard methods (6 lectures)

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining real numbers in RCA_0
- Apr. 23, (3) Completeness of the reals and ACA_0
- Apr. 25, (4) Continuous functions and WKL_0
- Apr. 30, (5) Continuous functions and WKL_0, II
- May 9, (6) König's lemma and Ramsey's theorem
- May 14, (7) Determinacy of infinite games I
- May 16, (8) Determinacy of infinite games II
- May 21, (9) Determinacy of infinite games III

Recap

The system of **recursive comprehension axioms** (RCA_0) consists of:

(1) first-order logic with axioms of equality for numbers plus basic arithmetic $\mathsf{Q}_<$.

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0).

(3) Σ^0_1 induction.

The system of **arithmetical comprehension axioms** (ACA_0) is RCA_0 plus (Π^1_0-CA).

In RCA_0, the following are equivalent (Lemma 3.3)
(1) ACA_0, (2) (Σ^0_1-CA), (3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

The system WKL_0 is RCA_0 plus **weak König's lemma**: every infinite tree $T \subset \mathrm{Seq}_2$ has an infinite path.

In RCA_0, WKL_0 is equivalent to (Σ^0_1-SP)(Separation Principle). (Lemma 3.6)

WKL_0 is strictly between RCA_0 and ACA_0. (Lemma 3.7)

§4. König's Lemma and Ramsey's theorem

Theorem 4.1

Over RCA_0, the following are pairwise equivalent: (1) ACA_0

(2) **König's Lemma**: every infinite, finitely branching tree in Seq has an infinite path.

(3) An infinite tree T, such that each node $s \in T$ has at most two children $s^\frown m \in T$ $(m \in \mathbb{N})$, has an infinite path.

For a set $X \subseteq \mathbb{N}$, we denote by $[X]^k$ the set of all sequences $(m_1, \ldots, m_k)$ of k elements from X such that $m_1 < \ldots < m_k$.

Definition 4.2 (Ramsey's Theorem)

Let $k, l > 0$ be natural numbers. **Ramsey's theorem** RT^k_l states

$$\forall f : [\mathbb{N}]^k \to \{0, 1, \ldots, l-1\}\ \exists X \subseteq \mathbb{N}\ (X \text{ is infinite} \wedge f \text{ is constant on } [X]^k).$$

Such an X is called a **homogeneous** set for f. RT^k_l for any standard $l \geq 2$ is equivalent to RT^k_2 in RCA_0, but equivalence between $\mathsf{RT}^k \equiv \forall l \in \mathbb{N}(\mathsf{RT}^k_l)$ and RT^k_2 does not hold in RCA_0.

RT^1 is also known as the **pigeonhole principle**. For a standard $l \geq 1$, RT^1_l holds in RCA_0.

Theorem 4.3 (J. Hirst)

In RCA_0, RT^1 is equivalent to $\mathsf{B}\Pi^0_1$.

Theorem 4.4

In ACA_0, both RT^1 and $\forall k(\mathsf{RT}^k \to \mathsf{RT}^{k+1})$ are provable.

Lemma 4.5

Within RCA_0, ACA_0 can be derived from RT^3_2.

Theorem 4.6

For any standard natural numbers $k \geq 3,\ l \geq 2$, RT^k_l, RT^k, and ACA_0 are equivalent within RCA_0.

Finally, concerning RT^2 and RT^2_2, it is known that both are between ACA_0 and RCA_0, and are incomparable with WKL_0. Within RCA_0, RT^2 implies $\mathsf{B}\Pi^0_2$, but RT^2_2 does not.

§5. Determinacy of Infinite Games

We introduce the remaining two of the BIG FIVE: ATR_0 and Π^1_1-CA_0.

Definition 5.1

The system Π^k_j-$\mathsf{CA}_0$$(k = 0, 1,\ j \in \omega)$ is obtained from RCA_0 by adding the following Π^k_j **Comprehension Axiom** (Π^k_j-CA): for any Π^k_j formula $\varphi(n)$,

$$\exists X \forall n (n \in X \Leftrightarrow \varphi(n)),$$

where $\varphi(n)$ may contain set variables other than X as parameters.

Definition 5.2

The system Π^k_j-$\mathsf{TR}_0$$(k = 0, 1,\ j \in \omega)$ is obtained from RCA_0 by adding the following Π^k_j **Transfinite Recursion Axiom** (Π^k_j-TR): for any Π^k_j formula $\varphi(n, X)$, for any set A and any well-order $\prec$, there exists a set H satisfying the following conditions:

(1) If b is the minimal element in $\prec$, then $(H)_b = A$, where $(X)_a = \{n : (a, n) \in X\}$.

(2) If b is the successor of a with respect to $\prec$, then $\forall n (n \in (H)_b \Leftrightarrow \varphi(n, (H)_a))$.

(3) If b is a $\prec$-limit, then for all $a \prec b$, $\forall n (n \in ((H)_b)_a \Leftrightarrow n \in (H)_a)$.

Π^0_1-TR$_0$ is called the system of **arithmetical transfinite recursion** ATR$_0$. For any $j > 0$, the strength of (Π^0_j-TR) is the same as (Π^0_1-TR), but it is not the case for (Π^1_j-TR).

Π^1_1-CA implies ATR$_0$. This fact will be proved indirectly by their equivalent statements.

Π^1_1-CA$_0$ is strictly stronger than ATR$_0$. To see this, let's observe that the axiom of arith. transfinite recursion can be written as a Σ^1_1 formula: $\prec$ is well-ordered $\to \exists H \theta_\prec(H)$.

Let (M, S) be a model of ATR$_0$. $A \in S$ can express $\langle A_n \mid n \in M \rangle \subset S$. Then, A is called (a countably coded) β-**model** if

$$(M, \{A_n\}) \models \varphi \Leftrightarrow (M, S) \models \varphi$$

for any Σ^1_1 formula φ with parameters from $\{A_n\}$.

The existence of (a coded) β-models is ensured by Π^1_1-CA$_0$ (via the strong Σ^1_1 dependent choice axiom [Simpson, Theorem VII. 6. 9]). Since ATR$_0$ is a Σ^1_1 formula, any β-models are models of ATR$_0$. Hence, the consistency of ATR$_0$ can be derived from Π^1_1-CA$_0$.

Gale-Stewart games

The games considered here are perfect information two-player games, similar to chess or Go. Although it's not realistic for players to continue indefinitely in real games, Zermelo argued in 1913 that it's natural to treat games like chess as infinite games in theory. Various infinite games have been conceived since then but in the 1950's, Gale and Stewart formulated a general infinite game where two players alternately choose natural numbers, and the outcome is decided by the infinite sequence produced.

Definition 5.3

In the **Gale-Stewart game** G, two players I and II alternately choose natural numbers, constructing an infinite sequence (called a **play**)

$$\begin{array}{cccccccc} \mathrm{I} & n_0 & & n_2 & & n_4 & & \ldots \\ \mathrm{II} & & n_1 & & n_3 & & n_5 & \ldots \end{array}$$

If the resulting sequence $(n_0, n_1, n_2, \ldots)$ is in a predetermined **winning set** $G \subseteq \mathbb{N}^{\mathbb{N}}$, then player I **wins**; otherwise, player II wins. The winning set is also referred to as the pay-off set.

Gale-Stewart games

A game is often identified with its winning set G and is simply treated as set. A game or set G is said to be **determined** if, given the winning set, one of the players can always win by playing smartly. Let's give a more precise definition of this concept.

Definition 5.4

A **strategy** for player I is a function $\sigma : \cup_{i \in \mathbb{N}} \mathbb{N}^{2i} \to \mathbb{N}$, and a **strategy** for player II is a function $\tau : \cup_{i \in \mathbb{N}} \mathbb{N}^{2i+1} \to \mathbb{N}$. If the players obey their strategies σ and τ, a play $(n_0, n_1, n_2, \ldots)$ is uniquely determined as follows:

$$\begin{array}{lllll} \mathrm{I} & n_0 = \sigma(\varnothing) \quad n_2 = \sigma(n_0, n_1) \quad n_4 = \sigma(n_0, n_1, n_2, n_3) \quad \ldots \\ \mathrm{II} & \quad \mathsf{n}_1 = \tau(n_0) \quad n_3 = \tau(n_0, n_1, n_2) \quad n_5 = \tau(n_0, n_1, n_2, n_3, n_4) \quad \ldots \end{array}$$

Here, the resulting play is denoted by $\sigma \otimes \tau$. Then, σ is called a **winning strategy** for player I if for any τ, $\sigma \otimes \tau$ belongs to G, that is, player I can win the game with σ whatever II plays. A **winning strategy** for player II is defined similarly. When one of the players has a winning strategy, the game G is said to be **determined, determinate**.

We treat the topology of the Baire space in second-order arithmetic.
An open subset G is a union of some basic open sets $[s] = \{f \in \mathbb{N}^{\mathbb{N}} | s \subset f\}$ ($s \in$ Seq), that is, there exists some $W \subseteq$ Seq such that

$$G = \bigcup_{s \in W} [s].$$

Equivalently,

$$f \in G \Leftrightarrow \exists n (f \upharpoonright n \in W), \text{ more generally } \exists n\, \theta(f \upharpoonright n, W \upharpoonright n)(\theta \in \Sigma^0_0),$$

where $f \upharpoonright n$ is the sequence $(f(0), \cdots, f(n-1)) \in$ Seq.
Subsets of the Baire space defined by a Σ^i_j formula with parameters are called $\utilde{\Sigma}^i_j$ sets. Consequently, $\utilde{\Sigma}^0_1$ sets coincide with **open sets**. Similarly, $\utilde{\Pi}^0_1$ sets coincide with closed sets, and $\utilde{\Sigma}^0_2$ sets correspond to $\mathcal{F}_\sigma$ sets, which are countable unions of closed sets. Thus, $\utilde{\Sigma}^0_j$ corresponds to the finite ranks of the **Borel hierarchy**.

Furthermore, $\utilde{\Sigma}^1_1$ coincides with **analytic sets**, which are projections of Borel sets, and $\utilde{\Sigma}^1_j$ corresponds to the **projective hierarchy**. Additionally, $\utilde{\Delta}^i_j = \utilde{\Sigma}^i_j \cap \utilde{\Pi}^i_j$.

Theorem 5.5

ATR_0 proves $\underset{\sim}{\Sigma}^0_1$ determincay.

Note: the determinacy of $\underset{\sim}{\Sigma}^0_1$ games is equivalent to the determinacy of $\underset{\sim}{\Pi}^0_1$ games.

Proof For a $\underset{\sim}{\Sigma}^0_1$ game G, there exists a set of finite sequences W such that,

$$f \in G \Leftrightarrow \exists x \; f \restriction x \in W.$$

Namely, W is the set of positions at which player I has already won, or she will never lose however they play afterwards.

Now, we set $W_0 = \{s \in \cup_{i \in \mathbb{N}} \mathbb{N}^{2i} : \exists x \; s \restriction x \in W\}$, and define W_1 as follows:

$$t \in W_1 \Leftrightarrow \exists m \forall n (t^\frown m^\frown n \in W_0).$$

Then, W_1 is the set of positions where player I chooses a next move and can get into W_0 in two steps. Next, W_2 is the set of player I's positions where she can get into W_1 in two steps. And so on.

More generally, in ATR_0, given a well-order $\prec$, we define $\{W_a\}$ as follows: Initially, for the least element 0 of $\prec$, set W_0 as above. Then, for any a in $\prec$, we define W_a as

$$t \in W_a \Leftrightarrow \exists m \forall n \exists b \prec a (t^\frown m^\frown n \in W_b)).$$

Now, suppose there exists a well-order $\prec$ and a corresponding $\{W_a\}$ such that the empty sequence $\varnothing \in W_{a_0}$ for some a_0. Then, we can show that player I has a winning strategy.

If $a_0 = 0$, then player I has already won. If $a_0 \neq 0$, there exists a move m by player I such that for any move n by player II, there exists $a_1 \prec a_0$ with $m^\frown n \in W_{a_1}$. If $a_1 = 0$, player I wins at $m^\frown n$. If $a_1 \neq 0$, there exists a move m such that for any move n, there exists $a_2 \prec a_1$ with the sequence falling into W_{a_2}.

Repeating this process, since $\prec$ is well-founded, it eventually reaches W_0, and so player I wins. This becomes player I's winning strategy.

Next, suppose such a well-order $\prec$ does not exist. Recall that the axiom of arithmetical transfinite recursion can be written as a Σ^1_1 formula: $\prec$ is well-ordered $\to \exists H\ \theta_\prec(H)$. Then our assumption is expressed as: $\prec$ is well-ordered $\to \exists W\ (\theta_\prec(W) \wedge \forall a\ \varnothing \notin W_a)$.

Then, there must exist a non-well-founded linear order $\prec$ s.t. $\exists W\ (\theta_\prec(W) \wedge \forall a\ \varnothing \notin W_a)$ holds. Otherwise, for a linear order $\prec$, we had

$$\prec \text{ is well-founded } \Leftrightarrow \exists W\ (\theta_\prec(W) \wedge \forall a\ \varnothing \notin W_a),$$

which implies that well-foundedness is expressed by a Σ^1_1 formula. This is a contradiction. See the lecture slides "logic and computation II" 06-06, p.15.

So, there must exist a non-well-founded linear order $\prec$ s.t. $\exists W\ (\theta_{\prec}(W) \wedge \forall a\ \varnothing \notin W_a)$ holds. Such a set W is called a **pseudo-hierarchy**.

Now, let $a_0 \succ a_1 \succ a_2 \succ \cdots$ be an infinite descending sequence in the non-well-founded part of W. And recall

$$t \in W_a \Leftrightarrow \exists m \forall n \exists b \prec a(t^\frown m^\frown n \in W_b)).$$

First, note $\varnothing \notin W_{a_0}$. Then, by the definition of W, for any move m by player I, there exists a move n by player II such that for all $a \prec a_0$, $m^\frown n \notin W_a$. So take a as $a_1 \prec a_0$. Then, for any move m' by player I, there exists a move n' by player II such that $m^\frown n^\frown m'^\frown n' \notin W_{a_1}$. Repeating this along the infinite descending sequence, it never enters W. This becomes player II's winning strategy, since any resulting infinite play $f \notin G$. Thus, the determinacy of $\underset{\sim}{\Sigma}^0_1$ games is demonstrated in ATR_0. □

The idea of the converse proof is as follows. Two players engage in a debate on the hierarchy $\{H_a\}$ claimed to exist by the transfinite recursion axiom. Player II wins the game by making correct assertions thoroughly. Since player II's winning strategy accurately describes the hierarchy $\{H_a\}$, the strategy allows $\{H_a\}$ to be constructed within RCA_0.

Theorem 5.6

$\underset{\sim}{\Sigma}^1_1$ determincay is equivalent to ATR_0 over RCA_0.

Proof We have proved ATR_0 implies $\underset{\sim}{\Sigma}^0_1$ determincy by Lemma 5.5. Before demonstrating the converse, we note the following. As we will see later that $\underset{\sim}{\Sigma}^1_1$ determinacy implies ACA_0, we may work within ACA_0. A well-order $\prec$ and an initial set $(H)_0 = A$ are given. The Π^0_1 formula $\varphi(n, X)$ is expressed as $\forall x\ \theta(n, X \restriction x)$ with a Σ^0_0 formula $\theta(n, h)$.

Our game proceeds as follows: First, player I chooses (b, y) intending to pose a question of whether $y \in (H)_b$ or not. Player II answers with Yes ("1") or No ("0").

First, consider the case that b is the minimum element.

> If the answer is Yes, then player II wins iff $y \in A$.
> If the answer is No, then player II wins iff $y \notin A$.

Then, the winner is decided, although the players continue to play meaninglessly.

Next, consider the case that b is the successor of a with respect to $\prec$.

Player II answers "Yes" to mean $y \in (H)_b$, that is, $\varphi(y, (H)_a)$, i.e., $\forall x\ \theta(y, (H)_a \upharpoonright x))$. So, if for any finite $h \subseteq (H)_a$ player I selects, $\theta(y, h)$ holds, then player II wins. Alternatively, player I may cheat by selecting $h \not\subseteq (H)_a$. In such a case, player II chooses $y' \in \mathrm{dom}\ h$ and asserts $y' \in (H)_a$ if $h(y') = 0$, or $y' \notin (H)_a$ if $h(y') = 1$. Then, the players begin the next debate round on $y' \in (H)_a$.

Player II answers "No" to mean $y \notin (H)_b$, that is, $\neg\varphi(y, (H)_a))$, i.e., $\exists x \neg\theta(y, (H)_a \upharpoonright x)$. Then, Player II must choose a finite sequence $h \subseteq (H)_a$ and they begin the next debate round on $y' \in (H)_a$ for I's move y'.

Finally, consider the case where b is a limit. In this case, if there does not exist an $a \prec b$ such that $y = (a, z)$, then Player I immediately loses. If $y = (a, z)$, then they begin the next debate round on $z \in (H)_a$, that is, $y = (a, z) \in (H)_b$.

This game always terminates in a finite number of steps, since the debate progresses by selecting lower elements a for $(H)_a$ according to the well-ordering $\prec$. From this, it follows that the winning set is Δ^0_1.

Moreover, we can show that Player I does not have a winning strategy as follows. By way of contradiction, Player I had a winning strategy σ. At the first move, Player I chooses (b, y) following σ. Then, Player I must win whether Player II answers Yes or No. However, we show this is impossible.

First, if b is a minimal element, II can win by answering Yes if $y \in A$, and No if $y \notin A$. If b is a limit, we may assume that there exists an $a \prec b$ such that $y = (a, z)$, and then the problem reduces to $z \in (H)_a$. Hence, we can assume that b is the successor of a.

In this case, if Player II answers Yes, then Player I selects a finite h^* according to σ. On the other hand, if Player II answers No, Player II can choose the finite h^*. Then, Player I chooses $y^* \in \mathrm{dom}\ h^*$ following σ for the next round of the question $y^* \in (H)_a$. Now, consider the case that Player II chooses the same y^* after Player II answers Yes and Player I selects a finite h^* in the first round. Then, Player I must still win whether Player II answers Yes or No for $y^* \in (H)_a$.

Continuing this way, we can construct two plays in which Player I must defend opposite claims. Since the game will terminate in a finite steps, it leads to a contradiction. Therefore, Player I cannot have a winning strategy.
Thus, by $\underset{\sim}{\Delta}^0_1$ determinacy, Player II has a winning strategy τ, and
$H = \{(b, y) : \tau(b, y) = 1\ (\text{"yes"})\}$ becomes the desired set.

□

Next, we prove that determinacy of $\utilde{\Sigma}^0_1 \wedge \utilde{\Pi}^0_1$ games is equivalent to Π^1_1-CA$_0$. Here, a $\utilde{\Sigma}^0_1 \wedge \utilde{\Pi}^0_1$ game is defined by a formula $\varphi \wedge \psi$ where φ is a Σ^0_1 formula and ψ a Π^0_1 formula both with parameters.

Theorem 5.7

The determinacy of $\utilde{\Sigma}^0_1 \wedge \utilde{\Pi}^0_1$ games and Π^1_1-CA$_0$ are equivalent over RCA$_0$.

Proof First, we demonstrate the determinacy of $\utilde{\Sigma}^0_1 \wedge \utilde{\Pi}^0_1$ games within Π^1_1-CA$_0$. Consider a game $A(f)$ in the form $\psi_1(f) \wedge \psi_2(f)$, where $\neg\psi_1$ is a Π^0_1 and ψ_2 is a Σ^0_1 formula (both including parameters). For ψ_2, there exists a Π^0_0 formula θ_2 such that, $\psi_2(f) \equiv \exists x \theta_2(f \upharpoonright x)$. Hence, according to Π^1_1-CA$_0$, there exists a Σ^1_1 set W.[1]

$$W = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \theta_2(s) \text{ and I has a winning strategy for } \psi_1 \text{ at } s.\}$$

Here, "I has a winning strategy for ψ_1 at s" means starting the game from the position s and ensuring a win in ψ_1.

[1]The statement "I has a winning strategy in the Π^1_1 game ψ_1 at s" can be restated as "there exists a strategy τ such that following τ all plays f satisfy $\psi_1(f)$". Since $\psi_1(f) \equiv \forall x \theta_1(f \upharpoonright x)$, "all plays f" can be translated as "all finite plays $f \upharpoonright x$", which makes the statement Σ^1_1.

Then consider the following Σ^0_1 game

$$W^* = \{f \in \mathbb{N}^{\mathbb{N}} \mid \exists x(f \restriction x \in W)\}.$$

If Player I has a winning strategy for W^*, then by following it, Player I will eventually enter W, and from the position s, using a winning strategy for ψ_1, $\psi_1(f) \wedge \psi_2(f)$ will hold.

On the other hand, suppose Player I has no winning strategy for W^*. Then Player II can make a play out of W throughout the game. So $\theta_2(s)$ will not hold forever, or Player I does not have a winning strategy for ψ_1. Therefore, by ATR_0, Player II has a winning strategy, and by following that strategy from then on, $\neg\psi_1$ will hold. That is, it is possible to ensure that $\neg\psi_1 \vee \neg\psi_2$ holds. Thus, the game $A(f)$ is determined.

Conversely, from the determinacy of $\underset{\sim}{\Sigma}^0_1 \wedge \underset{\sim}{\Pi}^0_1$ games, we prove Π^1_1-CA_0. First, let $\varphi(n)$ be a Π^1_1 logical formula of the form $\forall f \exists x \theta(n, f \upharpoonright x)$, where θ is a Σ^0_0 logical formula. Consider the following game G:

First, Player I chooses n. Then,

if player II believes that $\varphi(n)$ holds, II answers 1 (Yes),

and

if player II believes that $\neg\varphi(n)$ holds, II answer 0 (No).

If Player II answers Yes, Player I generates an (infinite) sequence f until Player II stops it. If at the stopping point (step x) $\theta(n, f \upharpoonright x)$ holds, then Player II wins; if it does not hold, or if Player II never stops the sequence, then Player I wins.

If Player II answers No, the roles of the players are reversed, and Player II generates an infinite sequence.

This game is in $\underset{\sim}{\Sigma}^0_1 \wedge \underset{\sim}{\Pi}^0_1$, and it is not possible for Player I to have a winning strategy. Therefore, Player II must have a winning strategy τ. Consequently, the set defined by $\varphi(n)$ will be $\{n : \tau(n) = 1\}$, and this exists in RCA_0.

□

From Theorem 5.7, we can see that the determinacy of games defined by the Boolean combination of $\underset{\sim}{\Sigma}^0_1$ sets can also be obtained through iterations of Π^1_1-CA$_0$.

Therefore, when analyzing $\underset{\sim}{\Delta}^0_2$ games, if we consider $\underset{\sim}{\Delta}^0_2$ sets as transfinite combinations of $\underset{\sim}{\Sigma}^0_1$ sets, it works well. For this purpose, we make the Hausdorff-Kuratowski Theorem on the difference hierarchy of $\underset{\sim}{\Delta}^0_n$ sets applicable within second-order arithmetic.[2]

Remark. We show that $\underset{\sim}{\Sigma}^0_1$ determinacy implies ACA$_0$. For any Σ^0_1 formula $\exists x\theta(n,x)$, consider the following game. Player I chooses n and player II answers Yes with a witness x, or No. If II answers No, then I must select a witness x. Then, Player II wins if he answers Yes and $\theta(n,x)$ holds, or No and $\neg\theta(n,x)$. Obviously, Player I can not have a winning strategy. By Σ^0_1 determinacy, Player II has a winning strategy τ, and in RCA$_0$, $\{n : \exists x\theta(n,x)\} = \{n : \tau(n) = (\text{Yes}, x)\}$ exists.

[2]K. Kuratowski, Topology, vol.1, Academic Press, 1966

Thank you for your attention!