Logic and Foundations II

Part 7. Real Analysis and Reverse Mathematics

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Real analysis and reverse mathematics (9 lectures?)
- Part 8. Second order arithmetic and non-standard methods (6 lectures?)

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining real numbers in RCA_0
- Apr. 23, (3) Completeness of the reals and ACA_0
- Apr. 25, (4) Continuous functions and WKL_0
- Apr. 30, (5) Continuous functions and WKL_0, II
- May 9, (6) König's lemma and Ramsey's theorem
- May 14, (7) Determinacy of infinite games I
- May 16, (8) Determinacy of infinite games II
- to be continued

Recap

The system of **recursive comprehension axioms** (RCA_0) consists of:

(1) first-order logic with axioms of equality for numbers plus basic arithmetic such as $\mathsf{Q}_<$.

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and neither includes X as a free variable.

(3) Σ^0_1 induction: $\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

The system of **arithmetical comprehension axioms** (ACA_0) is RCA_0 plus

$$(\Pi^1_0\text{-}\mathsf{CA}) : \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is an arithmetical formula, which does not have X as a free variable.

ACA_0 is a conservative extension of Peano Arithmetic PA.(Lemma 3.2)

In RCA_0, the following are equivalent (Lemma 3.3)
(1) ACA_0, (2) (Σ^0_1-CA), (3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

WKL_0

The system WKL_0 is RCA_0 plus **weak König's lemma**: every infinite tree $T \subset \mathrm{Seq}_2$ has an infinite path.

In RCA_0, WKL_0 is equivalent to $(\Sigma^0_1\text{-SP})$(Separation Principle). (Lemma 3.6)

WKL_0 is strictly between RCA_0 and ACA_0. (Lemma 3.7)

Theorem 3.12. The following assertions are pairwise equivalent in RCA_0:

(1) WKL_0,

(2) A continuous function $f : [0,1] \to \mathbb{R}$ is uniformly continuous,

(3) A continuous function $f : [0,1] \to \mathbb{R}$ is bounded.

Theorem 3.1 (Shioji-T.)

Brouwer's Fixed-Point Theorem is equivalent to WKL_0 over RCA_0.

§4. König's Lemma and Ramsey's theorem

Let Seq denote the set of finite sequences from $\mathbb{N}$, that is, the set of functions with domain $\{i \in \mathbb{N} : i < n\}$ for some $n \in \mathbb{N}$.

König's Lemma asserts that "every infinite, finitely branching tree in Seq has an infinite path."

Theorem 4.1

Over RCA_0, the following are pairwise equivalent:

(1) ACA_0

(2) König's Lemma

(3) An infinite tree T, such that each node $s \in T$ has at most two children $s^\frown m \in T$ $(m \in \mathbb{N})$, has an infinite path.

Note: In the above (3), it is crucial that m such that $s^\frown m \in T$ is not bounded over T. If m were bounded, the assertion would be equivalent to weak König's Lemma.

Ramsey's Theorem

For a set $X \subseteq \mathbb{N}$, we denote by $[X]^k$ the set of all sequences $(m_1, \ldots, m_k)$ of k elements from X such that $m_1 < \ldots < m_k$. **Ramsey's theorem** RT^k_l states that for a coloring of $[\mathbb{N}]^k$ into l colors, there exists an infinite subset $X \subseteq \mathbb{N}$ such that $[X]^k$ is monochromatic[1]. Such an X is called a **homogeneous** set.

Definition 4.2 (Ramsey's Theorem)

Let $k, l > 0$ be natural numbers. RT^k_l is the following assertion:

$$\forall f : [\mathbb{N}]^k \to \{0, 1, \ldots, l-1\} \exists X \subseteq \mathbb{N}(X \text{ is infinite} \wedge f \text{ is constant on } [X]^k).$$

If we consider the statement of painting any finite number of colors, we denote it as RT^k, i.e., $\mathsf{RT}^k \equiv \forall l \in \mathbb{N}(\mathsf{RT}^k_l)$. Although RT^k_l for any standard natural number $l \geq 2$ can be deduced from RT^k_2 by meta-induction in RCA_0, the equivalence of RT^k to RT^k_2 may require Π^1_2-induction, since RT^k_l is a Π^1_2 formula.

[1] Finite Ramsey's Theorem, denoted $m \to (n)^k_l$, is the statement that if $[\{0, \ldots, n-1\}]^k$ is painted in l colors, there exists a subset $X \subseteq \{0, \ldots, n-1\}$ of m elements such that $[X]^k$ is monochromatic.

We first consider the strength of RT^1, which is also known as the **pigeonhole principle** (PHP). For a standard natural number $l \geq 1$, RT^1_l obviously holds even in RCA_0.

Recall: the collection principle $(\mathsf{B}\varphi)$ for $\varphi(x, y_1, \cdots, y_k)$ in $\mathcal{L}_{\mathrm{OR}}$ is as follows

$$\forall x < u \exists y_1 \cdots \exists y_k \varphi(x, y_1, \cdots, y_k) \to \exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \cdots, y_k).$$

$\mathrm{B}\Pi^0_1$ denotes $\{(\mathsf{B}\varphi) \mid \varphi \in \Pi^0_1\}$. $\mathrm{B}\Pi^0_1$ is equivalent to $\mathrm{B}\Sigma^0_2$, and $\mathsf{I}\Sigma_1 \subsetneq \mathrm{B}\Sigma_2 \subsetneq \mathsf{I}\Sigma_2$.

Theorem 4.3 (J. Hirst)

In RCA_0, RT^1 is equivalent to $\mathrm{B}\Pi^0_1$.

The above theorem indicates that the strength of RT^1 is intermediate between ACA_0 and RCA_0, and it is incomparable with WKL_0. The strength of RT^2 becomes even more difficult to specify.

Theorem 4.4

In ACA_0, both RT^1 and $\forall k(\mathsf{RT}^k \to \mathsf{RT}^{k+1})$ are provable.

Proof RT^1 is clear from the above theorem. We now assume RT^k, and prove RT^{k+1}. Let $f : [\mathbb{N}]^{k+1} \to \{0, 1, \ldots, l-1\}$ be a coloring function. We will construct a homogeneous set X for this f by König's lemma. We first define a tree T as follows: $t \in T \Leftrightarrow$ for any $n < \mathrm{leng}(t)$, $t(n)$ is the least j such that

(1) $\max\{t(m) : m < n\} < j$,

(2) For any $m_1 < \ldots < m_k < m \leq n$,

$$f(t(m_1), \ldots, t(m_k), j) = f(t(m_1), \ldots, t(m_k), t(m)).$$

This tree T is called the **Erdős–Rado tree**.

Hence, (3) if $\max\{t(m) : m < n\} < j < t(n)$ then there exists $m_1 < \ldots < m_k < n$ s.t.

$$f(t(m_1), \ldots, t(m_k), j) \neq f(t(m_1), \ldots, t(m_k), t(n)).$$

First, we show that T is a finitely branching tree. Choose a node $t \in T$ with $\mathrm{leng}(t) = n$. For $j > t(n-1)$, define a function $\hat{f}_j : [0, \dots, n-1]^k \to \{0, \dots, l-1\}$ by

$$\hat{f}_j(m_1, \dots, m_k) = f(t(m_1), \dots, t(m_k), j).$$

Then, for $j \neq j'$ such that $t^\frown j \in T$ and $t^\frown j' \in T$, we can show that $\hat{f}_j \neq \hat{f}_{j'}$ as follows. If $t(n-1) < j < j'$, then by condition (3) (with $t(n) = j'$), we have $f_j \neq f_{j'}$. The same applies to the case $t(n-1) < j' < j$.

The number of functions from $[0, \cdots, n-1]^k$ to $\{0, \cdots, l-1\}$ is finite. This implies $t^\frown j \in T$ for only a finite number ($\leq l^{n^k}$) of j.

Next, to assert that T is infinite, we show that any $j \in \mathbb{N}$ appears in some sequence s in T.

So, fix a j and take a longest element t of T satisfying the following conditions:

(1°) $\max\{t(m) : m < \mathrm{leng}(t)\} < j$,

(2°) For any $m_1 < \ldots < m_k < m < \mathrm{leng}(t)$,

$$f(t(m_1), \dots, t(m_k), m) = f(t(m_1), \dots, t(m_k), j),$$

The empty sequence satisfies conditions (1°), (2°), and T has at most $j!$ elements that satisfies condition (1°), hence there exists a longest sequence t satisfying these conditions.

Let $t' = t^\frown j$. We will show $t' \in T$. First, we can easily see that t' satisfies conditions (1) and (2), since t satisfies conditions (1°) and (2°), respectively.

By way of contradiction, assume there exists the smallest such number $j' < j$. Then $t^\frown j'$ also satisfies conditions (1°), (2°), which contradicts the maximal length of t.

Thus, the Erdős–Rado tree T is an infinite finitely branching tree, and by König's lemma, it has an infinite path g. First note that g is a monotone increasing function $(m < n \to g(m) < g(n))$ from (1).

Now, define a function $\hat{f} : \mathbb{N}^k \to \{0, \dots, l-1\}$ as follows:

$$\hat{f}(m_1, \dots, m_k) = f(g(m_1), \dots, g(m_k), g(m)),$$

where $m_1 < \dots < m_k < m$. This definition does not depend on the choice of m, which is ensured by condition (2).

Using the assumption RT^k, we can find an infinite homogeneous set X' for $\hat{f}$. Finally, setting $X = \{g(m) : m \in X'\}$, it is clear that X becomes an infinite homogeneous set for f. □

Lemma 4.5

Within RCA_0, ACA_0 can be derived from RT^3_2.

Theorem 4.6

For any standard natural numbers $k \geq 3,\ l \geq 2$, RT^k_l, RT^k, and ACA_0 are equivalent within RCA_0.

Finally, concerning RT^2 and RT^2_2, it is known that both are between ACA_0 and RCA_0, and are incomparable with WKL_0. Within RCA_0, RT^2 implies $\mathsf{B}\Pi^0_2$, but RT^2_2 does not.

Introducing ATR_0 and Π^1_1-CA_0

Definition 5.1

The system Π^k_j-CA_0($k = 0, 1,\ j \in \omega$) is obtained from RCA_0 by adding the following Π^k_j **Comprehension Axiom** (Π^k_j-CA): for any Π^k_j formula $\varphi(n)$,

$$\exists X \forall n (n \in X \Leftrightarrow \varphi(n)),$$

where $\varphi(n)$ may contain set variables other than X as parameters.

In particular, Π^1_1-CA_0 is one of the BIG FIVE. Π^1_1 Comprehension Axiom (Π^1_1-CA) asserts the existence of sets defined by Π^1_1 formulas.

Note Since Π^0_2 sets can be defined by using (Π^0_1-CA) twice, (Π^0_1-CA) and (Π^0_2-CA) are equivalent when set parameters are allowed. However, even if (Π^1_1-CA) is used many times, only Δ^1_2 sets are defined. So, (Π^1_1-CA) and (Π^1_2-CA) are not equivalent.

For a formula $\varphi(n, X)$ and a set A, define the sequence of sets $A_0 = A$, $A_{i+1} = \{n : \varphi(n, A_i)\}$ for $i \in \omega$. Then, set $A_\omega = \{(i, n) : n \in A_i\}$ and continue with $A_{\omega+i+1} = \{n : \varphi(n, A_{\omega+i})\}$ to create the sequence $A_{\omega+1}, A_{\omega+2}, \ldots$.
Transfinite recursion requests such operations to be repeated up to any countable ordinal.

Definition 5.2

The system Π^k_j-$\mathsf{TR}_0$$(k = 0, 1,\ j \in \omega)$ is obtained from RCA_0 by adding the following Π^k_j **Transfinite Recursion Axiom** (Π^k_j-TR): for any Π^k_j formula $\varphi(n, X)$, for any set A and any well-order $\prec$, there exists a set H satisfying the following conditions:

(1) If b is the minimal element in $\prec$, then
$(H)_b = A$,

(2) If b is the successor of a with respect to $\prec$, then
$\forall n(n \in (H)_b \Leftrightarrow \varphi(n, (H)_a))$,

(3) If b is a $\prec$-limit, then for all $a \prec b$
$\forall n(n \in ((H)_b)_a \Leftrightarrow n \in (H)_a)$,

where $(X)_a = \{n : (a, n) \in X\}$.

The well-order $\prec$ is defined as a binary relation on $\mathbb{N}$ ($\prec \subseteq \mathbb{N} \times \mathbb{N}$) that is a linear order and contains no infinite descending sequences. In other words, it represents the order type of the countable ordinal numbers expressible within the system.

In RCA_0, it is clear that (Π^i_j)-TR $\to$ (Π^i_j)-CA.

Π^0_1-TR_0 is called the system of **arithmetical transfinite recursion** ATR_0, one of the BIG FIVE. For any non-zero natural number j, the strength of (Π^0_j-TR) remains the same as in arithmetical comprehension axioms (see Lemma 3.3). However, this is not the case for (Π^1_j-TR).

ATR_0 was introduced by H. Friedman in 1974. To my knowledge, its generalization (Π^i_j-TR) was first presented in my doctoral dissertation in 1986.

Π^1_1-CA implies ATR_0. This fact will be proved indirectly by their equivalent statements.

Strictness of ATR_0 and Π^1_1-CA

Π^1_1-CA$_0$ is strictly stronger than ATR$_0$. To see this, let's reconsider the axiom of the arithmetical transfinite recursion. Here, we omit the description of the parameter A and summarize conditions (1), (2), (3) into an arithmetic statement $\theta_\prec(H)$, so that the axiom asserts the existence of a set H satisfying:

$$\prec \text{ is a well-ordering } \Rightarrow \theta_\prec(H).$$

It can be rewritten as a Σ^1_1 formula: $\prec$ is not well-ordered $\vee \exists H \theta_\prec(H)$. Here $\prec$ is a kind of set variable, and the axiom may contain other parameters, so its universal closure turns into a Π^1_2 sentence.

Let (M, S) be a model of ATR$_0$. $A \in S$ can express $\langle A_n \mid n \in M \rangle \subset S$. Then, A is called (a countably coded) β-**model** if

$$(M, \{A_n\}) \models \varphi \Leftrightarrow (M, S) \models \varphi$$

for any Σ^1_1 formula φ with parameters from $\{A_n\}$.

The existence of (a coded) β-models is ensured by Π^1_1-CA$_0$ (via the strong Σ^1_1 dependent choice axiom [Simpson, Theorem VII. 6. 9]). Since ATR$_0$ is a Σ^1_1 formula, any β-models are models of ATR$_0$. Hence, the consistency of ATR$_0$ can be derived from Π^1_1-CA$_0$.

Gale-Stewart games

The games considered here are perfect information two-player games, similar to chess or Go. Although it's not realistic for players to continue indefinitely in real games, Zermelo argued in 1913 that it's natural to treat games like chess as infinite games in theory. Various infinite games have been conceived since then but in the 1950's, Gale and Stewart formulated a general infinite game where two players alternately choose natural numbers, and the outcome is decided by the infinite sequence produced.

Definition 5.3

In the **Gale-Stewart game** G, two players I and II alternately choose natural numbers, constructing an infinite sequence (called a **play**)

$$\begin{array}{lllllllll} \mathrm{I} & n_0 & & n_2 & & n_4 & & \ldots \\ \mathrm{II} & & n_1 & & n_3 & & n_5 & \ldots \end{array}$$

If the resulting sequence $(n_0, n_1, n_2, \ldots)$ is in a predetermined **winning set** $G \subseteq \mathbb{N}^{\mathbb{N}}$, then player I **wins**; otherwise, player II wins. The winning set is also referred to as the pay-off set.

Gale-Stewart games

A game is often identified with its winning set G and is simply treated as set. A game or set G is said to be **determined** if, given the winning set, one of the players can always win by playing smartly. Let's give a more precise definition of this concept.

Definition 5.4

A **strategy** for player I is a function $\sigma : \cup_{i \in \mathbb{N}} \mathbb{N}^{2i} \to \mathbb{N}$, and a **strategy** for player II is a function $\tau : \cup_{i \in \mathbb{N}} \mathbb{N}^{2i+1} \to \mathbb{N}$. If the players obey their strategies σ and τ, a play $(n_0, n_1, n_2, \ldots)$ is uniquely determined as follows:

$$\begin{array}{cllllllll} \mathrm{I} & n_0 = \sigma(\varnothing) & & n_2 = \sigma(n_0, n_1) & & n_4 = \sigma(n_0, n_1, n_2, n_3) & & \ldots & \\ \mathrm{II} & & \mathsf{n}_1 = \tau(n_0) & & n_3 = \tau(n_0, n_1, n_2) & & n_5 = \tau(n_0, n_1, n_2, n_3, n_4) & & \ldots \end{array}$$

Here, the resulting play is denoted by $\sigma \otimes \tau$. Then, σ is called a **winning strategy** for player I if for any τ, $\sigma \otimes \tau$ belongs to G, that is, player I can win the game with σ whatever II plays. A **winning strategy** for player II is defined similarly. When one of the players has a winning strategy, the game G is said to be **determined, determinate**.

Until now, formulas in second-order arithmetic are mainly used to define subsets of $\mathbb{N}$, but they can also define classes of subsets of $\mathbb{N}$. Indeed, $\mathbb{R}$ has been defined by a formula, and subsets of $\mathbb{R}$, the **Cantor space** $\{0,1\}^n$, and the **Baire space** $\mathbb{N}^{\mathbb{N}}$ can also be handled in second-order arithmetic.

We treat the topology of the Baire space in second-order arithmetic. An open subset G of the Baire space can be expressed as a union of some basic open sets $[s] = \{f \in \mathbb{N}^{\mathbb{N}} | s \subset f\}$ ($s \in$ Seq), that is, there exists some $W \subseteq$ Seq such that

$$G = \bigcup_{s \in W} [s]$$

or

$$f \in G \Leftrightarrow \exists n (f \upharpoonright n \in W)$$

where $f \upharpoonright n$ is the sequence $(f(0), \cdots, f(n-1)) \in$ Seq[2]. Thus, an open set G can be described by a Σ^0_1 formula including a parameter W. Conversely, a Σ^0_1 formula containing a parameter A, written as $\exists n \theta(f \upharpoonright n, A \upharpoonright n)$, defines an open set.

[2]Strictly speaking, $\mathcal{L}^2_{\mathrm{OR}}$ does not include function variable f, so it is necessary to translate this into a set variable.

Subsets of the Baire space defined by a Σ^i_j formula containing any parameter are called $\underset{\sim}{\Sigma}^i_j$ sets[3]. Consequently, $\underset{\sim}{\Sigma}^0_1$ sets coincide with **open sets**. Similarly, $\underset{\sim}{\Pi}^0_1$ sets coincide with closed sets, and $\underset{\sim}{\Sigma}^0_2$ sets correspond to $\mathcal{F}_\sigma$ sets, which are countable unions of closed sets. Thus, $\underset{\sim}{\Sigma}^0_j$ corresponds to the finite ranks of the **Borel hierarchy**.

Furthermore, $\underset{\sim}{\Sigma}^1_1$ coincides with **analytic sets**, which are projections of Borel sets, and $\underset{\sim}{\Sigma}^1_j$ corresponds to the **projective hierarchy**. Additionally, $\underset{\sim}{\Delta}^i_j = \underset{\sim}{\Sigma}^i_j \cap \underset{\sim}{\Pi}^i_j$.

Gale and Stewart (1953) showed that all $\underset{\sim}{\Sigma}^0_1$ games are determined, followed by determinacy of $\underset{\sim}{\Sigma}^0_2$, $\underset{\sim}{\Sigma}^0_3$, $\underset{\sim}{\Sigma}^0_4$ games proved by Wolff (1955), Davis (1964), and Paris (1972), respectively. Finally, Martin (1975) proved the determinacy of Borel($\underset{\sim}{\Delta}^1_1$) games within ZFC set theory. On the other hand, Harrington showed (1978) that the determinacy of analytic sets ($\underset{\sim}{\Sigma}^1_1$) requires the assumption of large cardinals, which is independent of ZFC set theory. Furthermore, the existence of undecidable games can be demonstrated from the axiom of choice.

[3]It is common to represent $\underset{\sim}{\Sigma}$ in boldface $\mathbf{\Sigma}$. However, to clearly distinguish between lightface Σ and boldface $\mathbf{\Sigma}$, we adopt $\underset{\sim}{\Sigma}$ instead of $\mathbf{\Sigma}$, which is commonly used in blackboard notation.

Propositions asserting such determinacy can be expressed in second-order arithmetic. Therefore, it is natural to consider what axioms of second-order arithmetic are necessary for their proof. The first step in this direction was taken by J. Steel in his thesis (1976), namely, the equivalence between determinacy of $\underset{\sim}{\Sigma}^0_1$ games and ATR_0. However, he discussed this equivalence over the system RCA with full induction.

Subsequently, I developed a proof method based on the game semantics (a debate between Pro and Con) to derive ATR_0 from $\underset{\sim}{\Sigma}^0_1$ games within RCA_0 (discussed later).

To demonstrate the determinacy of $\underset{\sim}{\Sigma}^0_1$ games from ATR_0, the method of the pseudo-hierarchy used in Steel's proof can be directly applied, so I will first give a brief explanation of this proof.

Theorem 5.5

ATR_0 proves $\underset{\sim}{\Sigma}^0_1$ determincay.

Note: the determinacy of $\underset{\sim}{\Sigma}^0_1$ games is equivalent to the determinacy of $\underset{\sim}{\Pi}^0_1$ games.

Proof ...

Thank you for your attention!