Logic and Foundations II

Part 7. Real Anasis and Reverse Mathematics

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Theory of reals and reverse mathematics (9 lectures?)
- Part 8. Second order arithmetic and non-standard methods (6 lectures?)

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining real numbers in RCA_0
- Apr. 23, (3) Completeness of the reals and ACA_0
- Apr. 25, (4) Continuous functions and WKL_0
- Apr. 30, (5) Continuous functions and WKL_0, II
- May 9, (6) König's lemma and Ramsey's theorem
- May 14, (7) Determinacy of infinite games I
- May 16, (8) Determinacy of infinite games II
- to be continued

Recap

The system of **recursive comprehension axioms** (RCA_0) consists of:

(1) first-order logic with axioms of equality for numbers plus basic arithmetic such as $\mathsf{Q}_<$.

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and neither includes X as a free variable.

(3) Σ^0_1 induction: $\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

The system of **arithmetical comprehension axioms** (ACA_0) is RCA_0 plus

$$(\Pi^1_0\text{-CA}) : \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is an arithmetical formula, which does not have X as a free variable.

ACA_0 is a conservative extension of Peano Arithmetic PA.(Lemma 3.2)

In RCA_0, the following are equivalent (Lemma 3.3)
(1) ACA_0, (2) (Σ^0_1-CA), (3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

WKL_0

The system WKL_0 is RCA_0 plus **weak König's lemma**: every infinite tree $T \subset \mathrm{Seq}_2$ has an infinite path.

In RCA_0, WKL_0 is equivalent to (Σ^0_1-SP)(Separation Principle). (Lemma 3.6)

WKL_0 is strictly between RCA_0 and ACA_0. (Lemma 3.7)

Theorem 3.12. The following assertions are pairwise equivalent in RCA_0:
(1) WKL_0,
(2) A continuous function $f : [0, 1] \to \mathbb{R}$ is uniformly continuous,
(3) A continuous function $f : [0, 1] \to \mathbb{R}$ is bounded.

Let A be a non-empty subset of $\mathbb{N}$. Suppose $d : A \times A \to \mathbb{R}$ is a (pseudo) metric on A. A sequence $\{a_n\}$ from A satisfying $\forall n \forall i\ d(a_n, a_{n+i}) \le 2^{-n}$ is called a point of $\hat{A}$, and we write $\{a_n\} \in \hat{A}$. $\hat{A}$ can be viewed as a **complete separable metric space**.

Example 1. If $A = \mathbb{Q}$ and $d(p, q) = |p - q|$, then $\hat{A}$ is nothing but $\mathbb{R}$. Also, if $A = \mathbb{Q}^2$ and $d((p, q), (p', q')) = \sqrt{(p - p')^2 + (q - q')^2}$, then $\hat{A}$ is $\mathbb{R}^2$.

Example 2. Given an infinite sequence of spaces $\hat{A}_i$, $i \in \mathbb{N}$. For simplicity, we assume that $0 \in A_i$ for all i. We then define the product space $\prod_i \hat{A}_i$ as the completion of (A, d),

$$A = \bigcup_{m=0}^{\infty} (A_0 \times \cdots \times A_m), \quad d(\langle a_i : i \leq m \rangle, \langle b_i : i \leq n \rangle = \sum_{i=0}^{\infty} \frac{d_i(a'_i, b'_i)}{1 + d_i(a'_i, b'_i)} \cdot \frac{1}{2^i},$$

where $\langle a'_i : i \in \mathbb{N} \rangle$ is $\langle a_i : i \leq m \rangle$ followed by infinitely many 0's, and similarly for $\langle b'_i \rangle$. Then, in RCA_0, we can define the Cantor space $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$, the Baire space $\mathbb{N}^{\mathbb{N}}$, the Hilbert cube $[0, 1]^{\mathbb{N}}$, a Fréchet space $\mathbb{R}^{\mathbb{N}}$, etc.

In a metric space $\hat{A}$, an **open ball** $B_r(a)$ centered at $a \in A$ with a rational radius $r > 0$ is coded by the pair $(a, r)(\in A \times \mathbb{Q}^+)$. An **open set** is a set of codes of open balls.

The code F of a **continuous function** f from a metric space $\hat{A}$ to a metric space $\hat{B}$ is a subset of $A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$, fulfilling conditions similar to those for a continuous function from $\mathbb{R}$ to $\mathbb{R}$, by which

$$(a, r, b, s) \in F \text{ means } x \in B_r(a) \to f(x) \in \overline{B_s(b)} \text{ (closed ball).}$$

Brouwer's Fixed-Point Theorem

Brouwer's Fixed-Point Theorem states that any continuous function $f : [0,1]^n \to [0,1]^n$ has a fixed point, i.e., a point x such that $f(x) = x$.

Theorem 3.13 (Shioji-T.)

Brouwer's Fixed-Point Theorem is equivalent to WKL_0 over RCA_0.

In WKL_0, Brouwer's Fixed-Point Theorem can be extended to the infinite-dimensional space $[0,1]^{\mathbb{N}} (\subseteq \mathbb{R}^{\mathbb{N}})$, which is known as the **Tychonoff-Schauder fixed-point theorem**.

By utilizing this fixed-point theorem, the **Cauchy-Peano theorem** for the existence of local solutions to ordinary differential equations can be derived within WKL_0, and the converse is also provable.

Various fixed-point theorems and their applications (e.g., the Hahn-Banach theorem) have been studied by N. Shioji and K. Tanaka [Fixed point theory in weak second-order arithmetic *Ann. Pure Appl. Logic*, 47, 167-188, 1990].

§4. König's Lemma and Ramsey's theorem

We begin with König's Lemma, not the "weak" version.

Let Seq denote the set of finite sequences from $\mathbb{N}$, that is, the set of functions with domain $\{i \in \mathbb{N} : i < n\}$ for some $n \in \mathbb{N}$.

A subset T of Seq, which is closed under initial segment, is called a **tree**.

A tree T is said to be **finitely branching**, if each node $s \in T$ has at most finitely many children, i.e.,

$$\forall s(s \in T \to \exists n \forall m(s^\frown m \in T \to m < n))$$

A subtree of T that never branches is called a **path** of T.

König's Lemma asserts that "every infinite, finitely branching tree has an infinite path." Weak König's Lemma is König's Lemma about special trees consisting of binary sequences. As we will see, König's Lemma is equivalent to ACA_0, and thus it is properly stronger than weak König's Lemma.

Theorem 4.1

Over RCA_0, the following are pairwise equivalent:

(1) ACA_0

(2) König's Lemma

(3) An infinite tree T, such that each node $s \in T$ has at most two children $s^\frown m \in T$ $(m \in \mathbb{N})$, has an infinite path.

Note: In the above (3), it is crucial that m such that $s^\frown m \in T$ is not bounded over T. If m were bounded, the assertion would be equivalent to weak König's Lemma.

Proof $(1) \Rightarrow (2)$. Given an infinite finitely branching tree T, let T' be the set of $s \in T$ that have an infinitely many descendants $t \supseteq s$ (by (Π^1_0-CA)).

Then, using primitive recursion, define a path g in T' as follows:

$g(0) =$ empty sequence,

$g(n+1) = g(n)^\frown m,$ where m is the smallest number such that $g(n)^\frown m \in T'$.

(2) $\Rightarrow$ (3) is trivial. To show (3) $\Rightarrow$ (1), assume (3) and show the existence of range of a given 1-1 function $f : \mathbb{N} \to \mathbb{N}$, which is equivalent to ACA_0, by Lemma 3.3.(3).

Define a tree T as follows: $s \in T \Leftrightarrow$

(a) $\forall m, n < \mathrm{leng}(s)(f(m) = n \leftrightarrow s(n) = m + 1)$,

(b) $\forall n < \mathrm{leng}(s)(s(n) > 0 \to f(s(n) - 1) = n)$.

Then, each node $t \in T$ has at most two children $t^\frown k \in T$. This is because letting $s = t^\frown k$, $n = \mathrm{leng}(s) - 1$ in (b), we have $k = 0$ or $k = f^{-1}(n) + 1$ if $f^{-1}(n)$ exists.

Next, show that T is infinite. For this, it suffices to show that for any $k \in \mathbb{N}$, there exists a sequence $s \in T$ with $\mathrm{leng}(s) = k$. First, by bounded $(\Sigma^0_1\text{-CA})$, $Y = \mathrm{ran} f \cap k$, that is, $\{n \in \mathrm{ran} f : n < k\}$ exists. Then, define a sequence s of length k as follows: for $n < k$,

$$s(n) = \begin{cases} 0 & \text{if } n \notin Y \\ m+1 & \text{if } n \in Y \wedge f(m) = n \end{cases}$$

Obviously, $s \in T$. So, T satisfies the conditions of (3).

Now, by (3), the tree T has an infinite path g. From the condition (a) of T,

$$\forall m, n(f(m) = n \leftrightarrow g(n) = m + 1).$$

Thus, setting $X = \{n : g(n) > 0\}$, we have $X = \mathrm{ran} f$. □

Ramsey's Theorem

Ramsey's Theorem was first invented by F. Ramsey in order to settle Hilbert's decision problem for first-order logic, though he only succeeded partially and we have no space to explain his original motivation and results.

For a set $X \subseteq \mathbb{N}$, we denote by $[X]^k$ the set of all sequences $(m_1, \ldots, m_k)$ of k elements from X such that $m_1 < \ldots < m_k$. Somewhat naïvely, (infinite) Ramsey's theorem RT^k_l states that for a coloring of $[\mathbb{N}]^k$ into l colors, there exists an infinite subset $X \subseteq \mathbb{N}$ such that $[X]^k$ is monochromatic[1]. More precisely, we state it as follows.

Definition 4.2 (Ramsey's Theorem)

Let $k, l > 0$ be natural numbers. Ramsey's Theorem RT^k_l is the following assertion:

$$\forall f : [\mathbb{N}]^k \to \{0, 1, \ldots, l-1\} \exists X \subseteq \mathbb{N}(X \text{ is infinite} \wedge f \text{ is constant on } [X]^k).$$

[1] Finite Ramsey's Theorem, denoted $m \to (n)^k_l$, is the statement that if $[\{0, \ldots, n-1\}]^k$ is painted in l colors, there exists a subset $X \subseteq \{0, \ldots, n-1\}$ of m elements such that $[X]^k$ is monochromatic. The finite version can be derived from the infinite version by the compactness argument.

For example, RT^2_l can be interpreted as follows: If all pairs $\{m, n\}$ of natural numbers are painted in l colors, then there always exists an infinite set X such that all pairs of elements from X are painted the same color. Such an X is called a **homogeneous** set.

If we consider the statement of painting any finite number of colors, we denote it as RT^k, i.e., $\mathsf{RT}^k \equiv \forall l \in \mathbb{N}(\mathsf{RT}^k_l)$.

Although RT^k_l for any standard natural number $l \geq 2$ can be deduced from RT^k_2 by meta-induction in RCA_0, the equivalence of RT^k to RT^k_2 may require Π^1_2-induction, since RT^k_l is a Π^1_2 formula.

So, we first consider the strength of RT^1, which is also known as the (infinite version of) **pigeonhole principle** (PHP). For a standard natural number $l \geq 1$, RT^1_l obviously holds even in RCA_0. The question is how much restricted induction is needed to derive $\forall l\, \mathsf{RT}^1_l$.

Recall: the collection principle ($\mathsf{B}\varphi$) for $\varphi(x, y_1, \cdots, y_k)$ in $\mathcal{L}_{\mathsf{O}R}$ is as follows

$$\forall x < u \exists y_1 \cdots \exists y_k \varphi(x, y_1, \cdots, y_k) \to \exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \cdots, y_k).$$

$\mathsf{B}\Pi^0_1$ denotes $\{(\mathsf{B}\varphi) \mid \varphi \in \Pi^0_1\}$. $\mathsf{B}\Pi^0_1$ is equivalent to $\mathsf{B}\Sigma^0_2$, and $\mathsf{I}\Sigma_1 \subsetneq \mathsf{B}\Sigma_2 \subsetneq \mathsf{I}\Sigma_2$. $\mathsf{B}\Pi^0_1$ is not provable in WKL_0, but obviously provable in ACA_0.

Theorem 4.3 (J. Hirst)

In RCA_0, RT^1 is equivalent to $\mathsf{B}\Pi^0_1$.

Proof. First, we derive $\mathsf{B}\Pi^0_1$ from RT^1. Take a Π^0_1 formula $\forall z \varphi(x,y,z)$ (where $\varphi \in \Sigma^0_0$), and assume $\forall x < u \exists y \forall z \varphi(x,y,z)$. We want to show $\exists v \forall x < u \exists y < v \forall z \varphi(x,y,z)$.

Now, consider the Σ^0_0 function $f : \mathbb{N} \to \mathbb{N}$ defined as

$$f(w) = \mu v < w(\forall x < u \exists y < v \forall z < w \varphi(x,y,z)).$$

Here, if no such v exists that the condition holds, we set $f(w) = w$.

If the range of f is finite, RT^1 ensures the existence of an infinite set H where $f(w)$ takes a constant value v_0. Thus, $\forall w \in H(\forall x < u \exists y < v_0 \forall z < w \varphi(x,y,z))$, which yields

$$\forall w(\forall x < u \exists y < v_0 \forall z < w \varphi(x,y,z)).$$

From the contrapositive of $\mathsf{B}\Sigma^0_0$, which holds in RCA_0,

$$\forall w \exists y < v_0 \forall z < w \varphi(x,y,z) \text{ implies } \exists y < v_0 \forall z \varphi(x,y,z).$$

Therefore, we have $\forall x < u \exists y < v_0 \forall z \varphi(x,y,z)$, which proves $\mathsf{B}\Pi^0_1$.

If the range of f is infinite, we choose a monotone increasing sequence $\{t_n\}$ such that $f(t_n) < f(t_{n+1})$. We first observe that $\forall x < u\ \exists y < f(t_n) - 1\ \forall z < t_n\ \varphi(x, y, z)$. Then, we define a function $g : \mathbb{N} \to \{0, 1, \ldots, u-1\}$ as

$$g(n) = \mu x < u \forall y < f(t_n) - 1\ \exists z < t_n \neg\varphi(x, y, z).$$

RT^1 ensures the existence of an infinite set H such that $g(n)$ takes a constant value x_0. Since H is infinite, for any y, there exists $n \in H$ such that $y < f(t_n) - 1$. Then, we have $\exists z < t_n \neg\varphi(x_0, y, z)$, which means $\forall y \exists z \neg\varphi(x_0, y, z)$, contradicting the initial assumption.

Next, we derive RT^1 from $\mathsf{B}\Pi^0_1$. We want to show that for any function $f : \mathbb{N} \to \{0, 1, \ldots, u-1\}$, there exists an x such that $f^{-1}(x)$ is infinite.

By way of contradiction, assume for all x, $f^{-1}(x)$ is finite, that is,

$$\forall x < u \exists y \forall z (z > y \to f(z) \neq x).$$

By $\mathsf{B}\Pi^0_1$, $\exists v \forall x < u \exists y < v \forall z (z > y \to f(z) \neq x)$, hence $\exists v \forall x < u \forall z > v - 1 (f(z) \neq x)$, so $\exists v \forall x < u (f(v) \neq x)$, which is clearly absurd. Thus, the proof is completed. □

The above theorem indicates that the strength of RT^1 is intermediate between ACA_0 and RCA_0, and it is incomparable with WKL_0. The strength of RT^2 becomes even more difficult to specify. First, we see the next theorem.

Theorem 4.4

In ACA_0, both RT^1 and $\forall k(\mathsf{RT}^k \to \mathsf{RT}^{k+1})$ are provable.

Proof RT^1 is clear from the above theorem. We now assume RT^k, and prove RT^{k+1}. Let $f : [\mathbb{N}]^{k+1} \to \{0, 1, \ldots, l-1\}$ be a coloring function. We will construct a homogeneous set X for this f by König's lemma. We first define a tree T as follows: $t \in T \Leftrightarrow$ for any $n < \mathrm{leng}(t)$, the following holds,

(1) $\max\{t(m) : m < n\} < t(n)$,

(2) For any $m_1 < \ldots < m_k < m < n$,

$$f(t(m_1), \ldots, t(m_k), m) = f(t(m_1), \ldots, t(m_k), t(n)),$$

(3) If $\max\{t(m) : m < n\} < j < t(n)$ then there exists $m_1 < \ldots < m_k < n$ such that,

$$f(t(m_1), \ldots, t(m_k), j) \neq f(t(m_1), \ldots, t(m_k), t(n)).$$

This tree T is called the **Erdős–Rado tree**.

First, we show that T is a finitely branching tree. Choose a node $t \in T$ with $\mathrm{leng}(t) = n$. For $j > t(n-1)$, define a function $\hat{f}_j : [0, \ldots, n-1]^k \to \{0, \ldots, l-1\}$ by

$$\hat{f}_j(m_1, \ldots, m_k) = f(t(m_1), \ldots, t(m_k), j).$$

Then, for $j \neq j'$ such that $t^\frown j \in T$ and $t^\frown j' \in T$, we can show that $\hat{f}_j \neq \hat{f}_{j'}$ as follows. If $t(n-1) < j < j'$, then by condition (3) (with $t(n) = j'$), we have $f_j \neq f_{j'}$. The same applies to the case $t(n-1) < j' < j$.

The number of functions from $[0, \cdots, n-1]^k$ to $\{0, \cdots, l-1\}$ is finite. This implies $t^\frown j \in T$ for only a finite number of j.

Next, to assert that T is infinite, we show that any $j \in \mathbb{N}$ appears in some sequence s in T.

So, fix j and take a longest element t of T satisfying the following conditions:

(1°) $\max\{t(m) : m < \mathrm{leng}(t)\} < j$,

(2°) For any $m_1 < \ldots < m_k < m < \mathrm{leng}(t)$,

$$f(t(m_1), \ldots, t(m_k), m) = f(t(m_1), \ldots, t(m_k), j),$$

The empty sequence satisfies conditions (1°), (2°), and T has at most $j!$ elements that satisfies condition (1°), hence there exists a longest sequence t satisfying these conditions.

Let $t' = t^\frown j$. We will show $t' \in T$. First, we can easily see that t' satisfies conditions (1) and (2), since t satisfies conditions (1°) and (2°), respectively.

By way of contradiction, we assume that t' does not satisfy condition (3).
Let $n = \mathrm{leng}(t)$. Then, there exists $j' < j$ such that $\max\{t(m) : m < n\} < j'$ and for any $m_1 < \ldots < m_k < n$,

$$f(t(m_1), \ldots, t(m_k), j') = f(t(m_1), \ldots, t(m_k), j).$$

Choosing j' as the smallest such number, then $t^\frown j'$ belongs to the tree T. Furthermore, $t^\frown j'$ also satisfies conditions (1°), (2°), which contradicts the maximal length of t.
Therefore, T is an infinite set.

Thus, the Erdős–Rado tree T is an infinite finitely branching tree, and by König's lemma, it has an infinite path g.

First note that g is a monotone increasing function ($m < n \to g(m) < g(n)$) from (1).

Now, define a function $\hat{f} : \mathbb{N}^k \to \{0, \ldots, l-1\}$ as follows:

$$\hat{f}(m_1, \ldots, m_k) = f(g(m_1), \ldots, g(m_k), g(m)),$$

where $m_1 < \ldots < m_k < m$. This definition does not depend on the choice of m, which is ensured by condition (2).

Using the assumption RT^k, we can find an infinite homogeneous set X' for $\hat{f}$. Finally, setting $X = \{g(m) : m \in X'\}$, it is clear that X becomes an infinite homogeneous set for f. □

Since RT^k is a Π^1_2 statement, the theorem above does not allow us to derive $\forall k\,\mathsf{RT}^k$ within induction of ACA_0. Paris and Harrington formulated a proposition PH in the language of first-order arithmetic to express something like $\forall k\,\mathsf{RT}^k$, and proved that PH is independent from PA.

Lemma 4.5

Within RCA_0, ACA_0 can be derived from RT^3_2.

Proof Assuming RT^3_2, we prove (Σ^0_1-CA). Let $\varphi(m)$ be any Σ^0_1 formula $\exists n \theta(m,n)$ with $\theta(m,n) \in \Sigma^0_0$.

Now, define a 2-color function $f : \mathbb{N}^3 \to \{0,1\}$ as follows:

$$f(a,b,c) = \begin{cases} 1 & \text{if } \forall m < a(\exists n < c\ \theta(m,n) \to \exists n < b\ \theta(m,n)) \\ 0 & \text{otherwise} \end{cases}$$

This definition is Σ^0_0 and the existence of function f is assured within RCA_0.

By RT^3_2, there exists an infinite homogeneous set X for f. Then, the value of f on $[X]^3$ is either always 1 or always 0.

By contradiction, we show that it cannot be always 0. Select any element a from X, and choose $a+2$ elements from X larger than a, denoted as $a < b_0 < b_1 < \ldots < b_{a+1}$. For each $i < a+1$, since $f(a, b_i, b_{i+1}) = 0$, there exists some $m < a$ such that $\exists n < b_{i+1} \theta(m,n)$ and $\neg\exists n < b_i \theta(m,n)$. Let m_{i+1} be the smallest such m.

If $i < j$, then $\exists n < b_j \theta(m_{i+1}, n)$ since $\exists n < b_{i+1} \theta(m_{i+1}, n)$. Then, $m_{i+1} \neq m_{j+1}$ since $\neg\exists n < b_j \theta(m_{j+1}, n)$. Namely, if $i \neq j$, then $m_{i+1} \neq m_{j+1}$.

However, there are only a elements less than a, so $m_1, \ldots, m_{a+1}$ cannot be all distinct. Thus, f cannot always take the value 0.

Therefore, for any $(a, b, c) \in [X]^3$, it holds that $\forall m < a(\exists n < c\theta(m, n) \to \exists n < b\theta(m, n))$.

Since c can be arbitrarily large, we finally have $\forall m < a(\exists n\theta(m, n) \to \exists n < b\theta(m, n))$. Consequently, we have

$$\exists n\theta(m, n) \leftrightarrow \forall a \forall b((a \in X \land b \in X \land m < a < b) \to \exists n < b\theta(m, n)).$$

The above shows that $\exists n\theta(m, n)$ is Δ^0_1. Hence, from (Δ^0_1-CA), there exists a set Y such that $\forall m(m \in Y \leftrightarrow \varphi(m))$.

□

Theorem 4.6

For any standard natural numbers $k \geq 3,\ l \geq 2$, RT^k_l, RT^k, and ACA_0 are equivalent within RCA_0.

Proof This follows immediately from Theorem4.4 and Lemma4.5. □

Finally, concerning RT^2 and RT^2_2, it is known that both are between ACA_0 and RCA_0, and are incomparable with WKL_0. Within RCA_0, RT^2 implies $\mathsf{B}\Pi^0_2$, but RT^2_2 does not. The intricate web of propositions between ACA_0 and RCA_0 is displayed as the "Reverse Mathematics Zoo" on websites such as that of Damir Dzhafarov.

Thank you for your attention!