Logic and Foundation II

Part 7. Real Anasis and Reverse Mathematics

Kazuyuki Tanaka

BIMSA

April 30, 2024

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Theory of reals and reverse mathematics (9 lectures?)
- Part 8. Second order arithmetic and non-standard methods (6 lectures?)

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining real numbers in RCA_0
- Apr. 23, (3) Completeness of the reals and ACA_0
- Apr. 25, (4) Continuous functions and WKL_0
- Apr. 30, (5) Continuous functions and WKL_0, II
- May 9, (6) König's lemma and Ramsey's theorem
- May 14, (7) Determinacy of infinite games I
- May 16, (8) Determinacy of infinite games II
- to be continued

Recap

Reverse Mathematics: Which axioms are needed to prove a theorem?

The Reverse Mathematics Phenomenon

Many theorems of mathematics are either provable in RCA_0, *or logically equivalent (over* RCA_0*) to one of* WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Definition 1.2 The system of **recursive comprehension axioms** (RCA_0) consists of:

(0) Axioms and inference rules of first-order logic with axioms of equality for numbers.

(1) Basic arithmetic axioms: Same as $\mathsf{Q}_<$ (Chapter 4).

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and neither includes X as a free variable.

(3) Σ^0_1 induction: $\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

ACA_0

The **system of arithmetical comprehension axioms** (ACA_0) is RCA_0 plus

$$(\Pi^1_0\text{-CA}) : \exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is an arithmetical formula, which does not have X as a free variable.

ACA_0 is a conservative extension of Peano Arithmetic PA.(Lemma 3.2)

In RCA_0, the following are equivalent (Lemma 3.3)

(1) ACA_0, (2) $(\Sigma^0_1\text{-CA})$, (3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

Theorem 3.4

The followings are pairwise equivalent over RCA_0.

(1) ACA_0,

(2) The Bolzano-Weierstrass theorem: Every bounded sequence of real numbers has a convergent subsequence,

(3) Every Cauchy sequence converges,

(4) Every bounded sequence of real numbers has a supremum,

(5) The monotone convergence theorem: Every bounded increasing sequence converges.

WKL_0

Definition 3.5

Weak König's lemma is the statement that every infinite tree $T \subset \mathrm{Seq}_2$ has an infinite path. The system WKL_0 is RCA_0 plus weak König's lemma.

- In RCA_0, WKL_0 is equivalent to $(\Sigma^0_1\text{-SP})$(Separation Principle). (Lemma 3.6)
- WKL_0 is strictly stronger than RCA_0. (Lemma 3.7)
- ACA_0 is strictly stronger than WKL_0.
- **Heine-Borel (Covering) Theorem** states that if an open set U covers the closed interval $[0, 1]$, then there exists a finite subset U' of U that also covers $[0, 1]$.

Theorem 3.9

In RCA_0, the Heine-Borel Theorem is equivalent to WKL_0.

The Heine-Borel property of $[0,1]$ allows us to derive various properties of continuous functions $f:[0,1]\to\mathbb{R}$.

Lemma 3.10

In WKL_0, a continuous function $f:[0,1]\to\mathbb{R}$ is uniformly continuous.

Proof Fix any $n\in\mathbb{N}$. We want to show the existence of $d>0$ such that

$$\forall x,y\in[0,1](\mid x-y\mid<d\to\mid f(x)-f(y)\mid<2^{-n}).$$

Let F be the code for the continuous function f, and denote the open interval with code i as (p_i,q_i). Then, define the open set U as follows:

$$i\in U\Leftrightarrow\exists j<i((p_i,q_i,p_j,q_j)\in F\wedge q_j-p_j<2^{-n-1}).$$

First, we show that U is a covering of $[0,1]$. For any real number $x\in[0,1]$, since $x\in\mathrm{dom}f$, there exists $(p_k,q_k,p_j,q_j)\in F$, such that

$$p_k<x<q_k\wedge q_j-p_j<2^{-n-1}.$$

Furthermore, there are infinitely many i such that $p_k\le p_i<x<q_i\le q_k$, so taking such an $i>j$, we have $i\in U$ with $p_i<x<q_i$. Therefore, U forms an open covering of $[0,1]$.

By the Heine-Borel Theorem, U has a finite subcover U'.

Let d be the minimum width $q_i - p_i$ among the intervals (p_i, q_i) in U'. We shall show that this d satisfies the uniform convergence condition.

Now, choose any real numbers $x, y \in [0, 1]$ such that $\mid x - y \mid < d$. Then, there must exist intervals $(p_i, q_i), (p_{i'}, q_{i'})$ in U' such that $x \in (p_i, q_i)$, $y \in (p_{i'}, q_{i'})$ and they have a common point z.

Otherwise, take an interval $(p_i, q_i) \ni x$ in U' with maximum q_i, and an interval $(p_{i'}, q_{i'}) \ni y$ in U' with minimum $p_{i'}$. If there is no common point, $q_i < p_{i'}$. Since U' is a covering, there exists $q_i \in (p_k, q_k)$ in U'. By the maximality of q_i, $x \notin (p_k, q_k)$. From $|q_k - p_k| \geq d > |x - y|$, we have $y \in (p_k, q_k)$, which contradicts with the minimality of $p_{i'}$.

By the definition of U, we have $\mid f(x) - f(z) \mid < 2^{-n-1}$ and $\mid f(y) - f(z) \mid < 2^{-n-1}$, thus $\mid f(x) - f(y) \mid < 2^{-n}$, which fulfills the lemma. □

Lemma 3.11

In WKL_0, a continuous function $f : [0,1] \to \mathbb{R}$ attains a maximum value.

Proof First, we show that the supremum M of the range of f exists.

As in the proof of the previous lemma, we define U by a Σ^0_0 formula:

$$i \in U \Leftrightarrow \exists j < i((p_i, q_i, p_j, q_j) \in F \wedge q_j - p_j < 2^{-n-1}).$$

We can finitely calculate whether or not a given finite set of open rational intervals covers $[0,1]$. Therefore, by arranging all finite subsets of U and checking sequentially whether they cover $[0,1]$, we eventually obtain a finite subcover U'. That is, in WKL_0, we can construct a function extracting U' according to n.

For each $i \in U'$, select $j_i < i$ such that $(p_i, q_i, p_{j_i}, q_{j_i}) \in F \wedge q_{j_i} - p_{j_i} < 2^{-n-1}$, and let $M_n = \max\{q_{j_i} : i \in U'\}$. Then, $\{M_n\}$ itself is a real number, and it is clear that it is the supremum M of the range of f.

What remains is to show that the existence of a point $x = a$ such that $f(a) = M$. For the sake of the following argument, we redefine $M_n = \max\{p_{j_i} : i \in U'\}$. This ensures that for any n, $M_n \leq M = \{M_n\}$.

By way of contradiction, assume that $f(x) < M$ for all $x \in [0, 1]$. Then, we define an open set V as follows:

$$i \in V \Leftrightarrow \exists j < i \ \exists n < i((p_i, q_i, p_j, q_j) \in F \wedge q_j < M_n).$$

To show that this set forms a covering of $[0, 1]$, take any real number $x \in [0, 1]$. Since $f(x) < M$, there exists n such that $f(x) < M_n \leq M$, and hence there exists $(p_k, q_k, p_j, q_j) \in F$ and n such that

$$p_k < x < q_k \ \wedge \ p_j \leq f(x) \leq q_j < M_n \leq M.$$

As there are infinitely many i such that $p_k \leq p_i < x < q_i \leq q_k$, taking $i > j, n$ ensures $i \in V$ with $p_i < x < q_i$. Therefore, V forms an open covering of $[0, 1]$.

Again, by the Heine-Borel Theorem, V has a finite subcover V'. Let M' be the maximum of q_i for (p_i, q_i) in V'. Then, by the definition of values of a continuous function, obviously M' is an upper bound of the range. However, due to the finiteness of V', for some n, $M' < M_n \leq M$, which contradicts the fact that M is the supremum. □

Conversely, the properties described in the two lemmas above allow us to derive WKL_0. In sum, the following theorem holds:

Theorem 3.12

The following assertions are pairwise equivalent in RCA_0:

(1) WKL_0,

(2) A continuous function $f : [0, 1] \to \mathbb{R}$ is uniformly continuous,

(3) A continuous function $f : [0, 1] \to \mathbb{R}$ is bounded,

(4) A bounded continuous function $f : [0, 1] \to \mathbb{R}$ has a supremum,

(5) A continuous function $f : [0, 1] \to \mathbb{R}$ that has a supremum attains its maximum value.

Proof By Lemmas 3.10 and 3.11, we can deriving (2), (3), (4), and (5) from (1). Hence, it suffices to obtain counterexamples for (2), (3), (4) and (5) from the negation of (1). Now, assume the negation of (1). Then, there exists an infinite tree $T \subseteq \mathrm{Seq}_2$ without infinite paths.

As shown in the proof of Heine-Borel's theorem, for each $s \in \mathrm{Seq}_2$, define the two rational numbers a_s and b_s as follows:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{s(i)}{2^{i+1}},$$

$$b_s = a_s + \frac{1}{2^{\mathrm{leng}(s)}}.$$

Let B be the infinite set of all minimal binary sequences not in T,

$$s \in B \Leftrightarrow s \notin T \wedge \forall t \subset s(t \neq s \to t \in T)$$

and J be the set of closed intervals $[a_s, b_s]$ for all $s \in B$.

Each real number $x \in [0, 1]$ is either an interior point of exactly one interval in J or an endpoint of one or two intervals. Such an infinite set J is called a **singular closed cover**.

$\neg\,\mathsf{WKL}_0 \to \neg\,$(3) bounded.

We will construct a counterexample for (3) using this singular closed cover J. This also serves as a counterexample for (2) since (2) implies (3). We define a continuous function f_s for each interval $[a_s, b_s]$ in J as follows:

$$f_s(x) = \begin{cases} \mathrm{leng}(s)\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ \mathrm{leng}(s)\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s. \end{cases}$$

That is, f_s takes 0 at the endpoints $x = a_s, b_s$, takes $\mathrm{leng}(s)$ at the midpoint $x = \frac{a_s+b_s}{2}$, and is linearly interpolated otherwise.

Let f be a function obtained by composing all such functions f_s. Then, it is clearly continuous but unbounded. (It is left as an exercise for the reader to construct a continuous function code for f.)

$\neg\,\mathsf{WKL}_0 \rightarrow \neg$ (5) a maximum value.

A counterexample for (5) can be constructed in the way similar to that for (3) in the previous slide. We just replace the maximum value of f_s from $\mathrm{leng}(s)$ to $1 - 2^{-\mathrm{leng}(s)}$ as follows:

$$f_s(x) = \begin{cases} (1 - 2^{-\mathrm{leng}(s)})\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ (1 - 2^{-\mathrm{leng}(s)})\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s. \end{cases}$$

Then, a composed function f clearly has 1 as its supremum, but it can not attain the maximum value 1 in $[0, 1]$.

$\neg\,\mathsf{WKL}_0 \to \neg$ (4) a supremum.

Recall:

Theorem 3.4.(5)

($\mathsf{RCA}_0 \vdash$) $\mathsf{ACA}_0 \Leftrightarrow$ (4) Every bounded increasing sequence of reals has a supremum.

Negating WKL_0, we have the negation of ACA_0, which implies the existence of a bounded increasing sequence of rational numbers $\{c_n\}$ that lacks a supremum.
Then, replace the maximum value of f_s with $c_{\mathrm{leng}(s)}$ and proceed similarly.

$$f_s(x) = \begin{cases} c_{\mathrm{leng}(s)} \frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ c_{\mathrm{leng}(s)} \frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s, \end{cases}$$

Problem

Show that in the theorem 3.12 (4) and (5), "continuous function" can be replaced with "uniformly continuous function". Hint: It is beneficial to use a singular closed cover for the ternary set.

Complete separable metric spaces

We will deal with continuous functions in more general spaces than $\mathbb{R}$.
Let's briefly look into complete separable metric spaces.

Let A be a non-empty subset of $\mathbb{N}$. Suppose $d : A \times A \to \mathbb{R}$ is a (pseudo) metric on A, i.e.,

(1) $d(a, a) = 0$, (2) $d(a, b) = d(b, a)$, (3) $d(a, b) + d(b, c) \geq d(a, c)$.

A sequence $\{a_n\}$ from A satisfying $\forall n \forall i d(a_n, a_{n+i}) \leq 2^{-n}$ is called a point of $\hat{A}$, and we write $\{a_n\} \in \hat{A}$. For $x = \{a_n\}$, $y = \{b_n\}$ define $d(x, y) = \lim_n (a_n, b_n)$.

Then $\hat{A}$ can be viewed as a **complete separable metric space**. Here, $\hat{A}$ is complete in the following sense. If $\{x_n\}$ is a sequence from $\hat{A}$ such that for some sequence $\{r_n\}$ of reals, $d(x_n, x_{n+i}) \leq r_n$ and $\lim r_n = 0$, then $\lim x_n$ exists in RCA_0. Also, $\hat{A}$ is separable, since A is dense in $\hat{A}$ if $a \in A$ is identified with $\{a_n\} \in \hat{A}$ where $a_n = a$.

Example 1. If $A = \mathbb{Q}$ and $d(p,q) = |p-q|$, then $\hat{A}$ is nothing but $\mathbb{R}$.
Also, if $A = \mathbb{Q}^2$ and $d((p,q),(p',q')) = \sqrt{(p-p')^2 + (q-q')^2}$, then $\hat{A}$ is $\mathbb{R}^2$.

Example 2. Given an infinite sequence of spaces $\hat{A}_i$, $i \in \mathbb{N}$. For simplicity, we assume that $0 \in A_i$ for all i. We then define the product space $\prod_i \hat{A}_i$ as the completion of (A, d),

$$A = \bigcup_{m=0}^{\infty} (A_0 \times \cdots \times A_m), \quad d(\langle a_i : i \leq m \rangle, \langle b_i : i \leq n \rangle = \sum_{i=0}^{\infty} \frac{d_i(a'_i, b'_i)}{1 + d_i(a'_i, b'_i)} \cdot \frac{1}{2^i},$$

where $\langle a'_i : i \in \mathbb{N} \rangle$ is $\langle a_i : i \leq m \rangle$ followed by infinitely many 0's, and similarly for $\langle b'_i \rangle$.
Then, in RCA_0, we can define the Cantor space $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$, the Baire space $\mathbb{N}^{\mathbb{N}}$, the Hilbert cube $[0,1]^{\mathbb{N}}$, a Fréchet space $\mathbb{R}^{\mathbb{N}}$, etc.

In a metric space $\hat{A}$, an **open ball** $B_r(a)$ centered at $a \in A$ with a rational radius $r > 0$ is coded by the pair $(a, r)(\in A \times \mathbb{Q}^+)$. An **open set** is a set of codes of open balls.

The code F of a **continuous function** f from a metric space $\hat{A}$ to a metric space $\hat{B}$ is a subset of $A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$, fulfilling conditions similar to those for a continuous function from $\mathbb{R}$ to $\mathbb{R}$, by which

$$(a, r, b, s) \in F \text{ means } x \in B_r(a) \to f(x) \in \overline{B_s(b)} \text{ (closed ball)}.$$

Brouwer's Fixed-Point Theorem

Brouwer's Fixed-Point Theorem states that any continuous function $f : [0,1]^n \to [0,1]^n$ has a fixed point, i.e., a point x such that $f(x) = x$.

While the case $n = 1$ can be directly derived from the Intermediate Value Theorem and thus holds in RCA_0, the case $n > 1$ is not provable within RCA_0.

Theorem 3.13

Brouwer's Fixed-Point Theorem is equivalent to WKL_0 over RCA_0.

Proof idea There are various proofs known for Brouwer's Fixed-Point Theorem, most of which utilize the uniform continuity of a given function f to reduce the problem to a finite combinatorial issue (e.g., Sperner's Lemma).

In WKL_0, it can be proved that any continuous function $f : [0,1]^n \to [0,1]^n$ is uniform continuous, in a way similar to the proof for Theorem 3.12. So, the rest of a proof can proceed as a standard argument for Brouwer's theorem.

For the converse direction, it is enough to show that the case $n = 2$ implies WKL_0, since the other cases obviously implies that of $n = 2$. So, we negate WKL_0, and construct a continuous function $h : [0,1]^2 \to [0,1]^2$ that does not have a fixed point.

By the negation of WKL_0, we have a singular closed cover J for $[0,1]$, which is given in the proof of Theorem 3.12. Using this, we construct a **retraction** f from $[0,1]^2$ to its boundary B (a continuous function invariant on B). If such an f exists, combining it with the operation g that rotates B by 90° results in a continuous function $h = g \circ f$ without fixed points.

Let $J = \{I_i : i \in \mathbb{N}\}$ be a singular closed cover J for $[0,1]$. For convenience, we assume the left end of I_0 is 0 and the right end of I_1 is 1.

Set $A_k = \bigcup_{i \leq k}(I_i \times I_k \cup I_k \times I_i)$. Then, $[0,1]^2 = \bigcup_k A_k$.

We construct a retraction f by induction on a subset A_k of its domain. Assuming f is defined on $\bigcup_{i<k} A_i$, we show how to define it on A_k.

Divide A_k into connected rectangular parts $P_0, P_1, \ldots, P_m$. If P_l $(l \leq m)$ adjoins $\bigcup_{i<k} A_i$ or the boundary B of $[0,1]^2$, f should map the adjoining edge of P_l to B as already determined. However, we can easily observe that at least one edge of P_l does not adjoin $\bigcup_{i<k} A_i$ or B. So, we can construct a retraction of P_l onto the sides on which the values of f are determined.

Thus, we can define a continuous mapping from each P_l to B by composition of such a retraction of P_l and f on $\bigcup_{i<k} A_i \cup B$. If P_l has no constrained edge, f can map P_l to B anyway continuously.

Combining all such functions on P_l's, we have a continuous mapping from A_k to the boundary B.

Finally, f thus defined is a retraction from $[0,1]^2$ to its boundary B. Therefore, by negating WKL_0, a counterexample to the fixed-point theorem is obtained. □

In WKL_0, Brouwer's Fixed-Point Theorem can be extended to the infinite-dimensional space $[0,1]^{\mathbb{N}}(\subseteq \mathbb{R}^{\mathbb{N}})$, which is known as the **Tychonoff-Schauder fixed-point theorem**.

By this fixed-point theorem, the **Cauchy-Peano theorem** for the existence of local solutions to ordinary differential equations can be proved within WKL_0, and the converse is also provable.

It is worth mentioning that the standard proof of the Cauchy-Peano Theorem involves constructing a sequence of piecewise linear approximations to the solution and using the Ascoli-Arzelà Lemma to argue for the existence of the solution; however, since the Ascoli-Arzelà Lemma cannot be proved in WKL_0[1], this approach does not fit within WKL_0.

Various fixed-point theorems in WKL_0 and their applications have been developed by N. Shioji and K. Tanaka [Fixed point theory in weak second-order arithmetic *Ann. Pure Appl. Logic, 47, 167-188, 1990].*

[1]In Theorem 3.4, the Bolzano-Weierstrass theorem, which has been shown to be equivalent to ACA_0, can be derived as a special case from the Ascoli-Arzelà lemma. Indeed, it is known that the Ascoli-Arzelà lemma is equivalent to ACA_0.

König's Lemma

Let's begin with the general König's Lemma, not the "weak" version.

The set of natural number sequences of length n, that is, the set of functions (or their codes) with domain $\{i \in \mathbb{N} : i < n\}$, is denoted by Seq.

A subset T of Seq, which is closed under initial segment, is called a **tree**.

A tree T where each element $s \in T$ has at most finitely many immediate successors $s^\frown m \in T (m \in \mathbb{N})$, or

$$\forall s(s \in T \to \exists n \forall m(s^\frown m \to m < n))$$

is called a **finitely branching tree**. In Seq_2, the "tree" is also a special tree in Seq.

Moreover, a subtree of T that does not branch is called a **path** of T.

König's Lemma asserts that "every infinite, finitely branching tree has at least on path." Of course, a tree consisting only of binary sequences is a finitely branching tree, so the weak König's Lemma is a special case of König's Lemma. However, as will be shown, König's Lemma is equivalent to ACA_0, and thus, it is not possible to derive König's Lemma from the weak König's Lemma.

Theorem 3.14

Over RCA_0, the following are pairwise equivalent:

(1) ACA_0

(2) König's Lemma

(3) An infinite tree T, where each element $s \in T$ has at most two immediate successors $s^\frown m \in T (m \in \mathbb{N})$, has an infinite path.

Note: In the above (3), it is crucial that the size of m for immediate successors $s^\frown m \in T$ is not bounded. If the size of m were bounded across the tree, it would result in an assertion equivalent to the weak König's Lemma.

Proof $(1) \Rightarrow (2)$. Given an infinite finitely branching tree T, collect the points $s \in T$ that have an infinite number of descendants $t \supseteq s$ to form T' (by (Π^1_0-CA)).

Then, using primitive recursion, define a path g in T' as follows:

$$g(0) = \text{empty sequence}, \ g(n+1) = g(n)^\frown m,$$

where m is the smallest number such that $g(n)^\frown m \in T'$.

$(2) \Rightarrow (3)$ is trivial. To show $(3) \Rightarrow (1)$, assume (3) and show the existence of range of a given 1-1 function $f : \mathbb{N} \to \mathbb{N}$, which is equivalent to ACA_0, Lemma 3.3.(3).

Define a tree T as follows: $s \in T \Leftrightarrow$

(a) $\forall m, n < \mathrm{leng}(s)(f(m) = n \leftrightarrow s(n) = m + 1)$,

(b) $\forall n < \mathrm{leng}(s)(s(n) > 0 \to f(s(n) - 1) = n)$.

Then, each element $s \in T$ has at most two immediate successors $s^\frown k \in T$. That is, from (b), $k = 0$ or, for $f(m) = \mathrm{leng}(s)$, $k = m + 1$.

Next, show that the tree T is an infinite set. For this, it suffices to show that for any $k \in \mathbb{N}$, there exists a sequence $s \in T$ with $\mathrm{leng}(s) = k$. First, by bounded (Σ^0_1-CA), the subset $Y = \{n \in \mathrm{ran} f : n < k\}$ exists. Then, define a sequence s of length k as follows. For $n < k$,

$$s(n) = \begin{cases} 0 & \text{if } n \notin Y \\ m + 1 & \text{if } n \in Y \wedge f(m) = n \end{cases}$$

In this case, it is clear that $s \in T$.

Now, by assumption (3), the tree T has an infinite path g. From the definition of T (a),

$$\forall m, n(f(m) = n \leftrightarrow g(n) = m + 1).$$

Thus, setting $X = \{n : g(n) > 0\}$, we have $X = \mathrm{ran} f$. □

Thank you for your attention!