Logic and Foundation II

Part 7. Real Anasis and Reverse Mathematics

Kazuyuki Tanaka

BIMSA

April 25, 2024

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Theory of reals and reverse mathematics (9 lectures?)
- Part 8. Second order arithmetic and non-standard methods (6 lectures?)

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining real numbers in RCA_0
- Apr. 23, (3) Completeness of the reals and ACA_0
- Apr. 25, (4) Continuous functions and WKL_0
- Apr. 30, (5) König's lemma and Ramsey's theorem
- May 9, (6) Determinacy of infinite games I
- May 14, (7) Determinacy of infinite games II
- to be continued

Recap

Reverse Mathematics: Which axioms are needed to prove a theorem?

The Reverse Mathematics Phenomenon

Many theorems of mathematics are either provable in RCA_0*, or logically equivalent (over* RCA_0*) to one of* WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Definition 1.2 The system of **recursive comprehension axioms** (RCA_0) consists of:

(0) Axioms and inference rules of first-order logic with axioms of equality for numbers.

(1) Basic arithmetic axioms: Same as $\mathsf{Q}_<$ (Chapter 4).

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and neither includes X as a free variable.

(3) Σ^0_1 induction: $\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

Real numbers and continuous functions

Primitive recursive functions (e.g., sequence numbers and Gödel numbers) are available in RCA_0. Note that RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$.

So, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and their arithmetical operations are naturally defined in RCA_0.

A sequence of rational numbers $\{q_n\}$ is a **real number**, $\{q_n\} \in \mathbb{R}$, if it satisfies

$$\forall n \forall i (|q_n - q_{n+i}| \leq 2^{-n}).$$

A set $\Phi \subseteq \mathbb{Q}^4$ that satisfies the following conditions is called the **code** for a **continuous function** $f : \mathrm{dom}\ f (\subseteq \mathbb{R}) \to \mathbb{R}$.

(1) $(p, q, r, s) \in \Phi \to p < q \wedge r \leq s$,

(2) $(p, q, r, s), (p', q', r', s') \in \Phi, p' < q \wedge p < q' \ \to \ r' \leq s \wedge r \leq s'$.

Intuitively, $(p, q, r, s) \in \Phi$ means $\forall x (p < x < q \to r \leq f(x) \leq s)$.

A real number x belongs to the **domain** of a continuous function f coded by Φ, if

$$\forall n \exists (p, q, r, s) \in \Phi (p < x < q \wedge s - r < 2^{-n}), \text{ denoted } x \in \mathrm{dom} f.$$

It is provable in RCA_0 that if $x \in \mathrm{dom} f$, there exists a unique real y such that $\forall (p, q, r, s) \in \Phi (p < x < q \to r \leq y \leq s)$. We denote this y as $f(x)$.

ACA_0

The **system of arithmetical comprehension axioms** (ACA_0) is RCA_0 plus

$$(\Pi^1_0\text{-CA}) : \exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is an arithmetical formula, which does not have X as a free variable.

ACA_0 is a conservative extension of Peano Arithmetic PA.(Lemma 3.2)

In RCA_0, the following are equivalent (Lemma 3.3)
(1) ACA_0, (2) $(\Sigma^0_1\text{-CA})$, (3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

Theorem 3.4

The followings are pairwise equivalent over RCA_0.

(1) ACA_0,

(2) The Bolzano-Weierstrass theorem: Every bounded sequence of real numbers has a convergent subsequence,

(3) Every Cauchy sequence converges,

(4) Every bounded sequence of real numbers has a supremum,

(5) The monotone convergence theorem: Every bounded increasing sequence converges.

WKL_0

Definition 3.5

Weak König's lemma is the statement that every infinite tree $T \subset \mathrm{Seq}_2$ has an infinite path. The system WKL_0 is RCA_0 plus weak König's lemma.

Lemma 3.6

In RCA_0, WKL_0 is equivalent to the following statement:

$$(\Sigma^0_1\text{-SP}) : \forall n(\varphi(n) \to \psi(n)) \to \exists X \forall n\{(\varphi(n) \to n \in X) \land (n \in X \to \psi(n))\},$$

where $\varphi(n)$ is Σ^0_1 and $\psi(n)$ is Π^0_1. SP stands for the **Separation Principle**.

Corollary 3.7

WKL_0 is strictly stronger than RCA_0.

There are various ways to show that ACA_0 is strictly stronger than WKL_0. WKL_0 and RCA_0 are conservative over $\mathsf{I}\Sigma_1$, whereas ACA_0 is over PA.

Hine-Borel theorem in WKL_0

An **open interval** with rational endpoints p, q $(p < q)$ is represented by the natural number code for (p, q).

An **open set** of $\mathbb{R}$ is defined (encoded) as a set of codes of open intervals.

Now, we say an open set U of $\mathbb{R}$ **covers** the closed interval $[0, 1]$ if, for any real number $x \in [0, 1]$, there exists a code $(p, q) \in U$ such that $p < x < q$.

Heine-Borel (Covering) Theorem states that if an open set U covers the closed interval $[0, 1]$, then there exists a finite subset U' of U that also covers $[0, 1]$.

Lemma 3.8

The Heine-Borel Theorem can be proved in WKL_0.

Proof. For each $s \in \mathrm{Seq}_2$, we associate the rational open interval (a_s, b_s) defined as:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{s(i)}{2^{i+1}}, \quad b_s = a_s + \frac{1}{2^{\mathrm{leng}(s)}}.$$

In this case, if $s \subseteq t$, then $(a_t, b_t) \subseteq (a_s, b_s)$.

Now, consider an open covering U of $[0,1]$. For intuition, let's denote the open interval with code i as (p_i, q_i). Then, define a tree $T \subseteq \mathrm{Seq}_2$ as follows:

$$s \in T \leftrightarrow \neg\exists i \leq \mathrm{leng}(s)(i \in U \wedge p_i < a_s < b_s < q_i).$$

We first show that T has no infinite path. By way of contradiction, we suppose there exists a path $f \subseteq T$. By the nested interval property, there exists a (unique) real number x such that $a_s \leq x \leq b_s$ for all $s \in f$. Since the open set U covers $[0,1]$, there exists some $i \in U$ such that the real number x is contained in the open interval (p_i, q_i). Then, there exists an $s \in f$ with $\mathrm{leng}(s) \geq i$ such that $p_i < a_s \leq x \leq b_s < q_i$, which implies $s \notin T$, a contradiction.

If T has no infinite path, then by weak König's lemma, T is a finite set. This means that there exists a sufficiently large n such that all sequences in T have a length shorter than n. Thus,

$$\forall s(\mathrm{leng}(s) = n \to \exists i \leq n(i \in U \wedge p_i < a_s < b_s < q_i)).$$

Therefore, $\{i \in U : i \leq n\}$ forms a finite covering of $[0,1]$. □

Theorem 3.9

In RCA_0, the Heine-Borel Theorem is equivalent to WKL_0.

Proof We have already shown that the Heine-Borel Theorem holds in WKL_0. Now, we assume the Heine-Borel Theorem and derive the weak König's lemma.

First, let's discuss the idea behind the proof. The Heine-Borel Theorem implies the compactness of $[0, 1]$, which leads to the compactness of a closed subset

$$\left\{ \sum_{i=0}^{\infty} f(i) \cdot 3^{-i-1} \mid f \in \{0, 2\}^{\mathbb{N}} \right\}$$

(the ternary set), and hence also the compactness of the Cantor space $\{0, 1\}^{\mathbb{N}}$ since it is homeomorphic to the ternary set. Finally, the compactness of $\{0, 1\}^{\mathbb{N}}$ implies WKL_0.

For preparation, for each $s \in \mathrm{Seq}_2$, we associate the rational open interval (a_s, b_s) defined as follows:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{2s(i)}{3^{i+1}},$$

$$b_s = a_s + \frac{1}{3^{\mathrm{leng}(s)}}.$$

Let $s^\frown i$ simply denote the binary sequence s followed by $i = 0, 1$, i.e., $s \cup \{(\mathrm{leng}(s), i)\}$.

Then, the closed intervals $[a_{s^\frown 0}, b_{s^\frown 0}]$ and $[a_{s^\frown 1}, b_{s^\frown 1}]$ respectively become the left and right thirds of the closed interval $[a_s, b_s]$.

Thus, for any real number x not belonging to the ternary set $\{\sum_{i=0}^{\infty} f(i) \cdot 3^{-i-1} : f \in \{0, 2\}^{\mathbb{N}}\}$, there exists exactly one open interval $(b_{s^\frown 0}, a_{s^\frown 1})$ containing it. Especially,

$$\bigcup \{(b_{s^\frown 0}, a_{s^\frown 1}) \mid s \in \mathrm{Seq}_2\}$$

is the complement of the ternary set.

Furthermore, for each $s \in \mathrm{Seq}_2$, define

$$a'_s = a_s - \frac{1}{3^{\mathrm{leng}(s)+1}},$$

$$b'_s = b_s + \frac{1}{3^{\mathrm{leng}(s)+1}}.$$

Then, for any real number x in the ternary set, there exists a unique $f \in \{0,1\}^{\mathbb{N}}$ such that: for any finite initial sequence $s \subset f, x \in (a'_s, b'_s)$. Note that two open intervals (a'_s, b'_s) and (a'_t, b'_t) intersect only if either s or t is an initial segment of the other.

Now, let's consider any (nonempty) tree $T \subseteq \mathrm{Seq}_2$ without infinite paths and show that T is finite.

Let B be the set of minimal binary sequences not in T, that is,

$$s \in B \Leftrightarrow s \notin T \wedge \forall t \subset s(t \neq s \to t \in T).$$

It's clear that any infinite path $f \subseteq T$ shares exactly one element $s \in B$ and $s \subset f$.

Thus, if we set

$$U = \bigcup\{(a'_s, b'_s) : s \in B\} \cup \bigcup\{(b_{s^\frown 0}, a_{s^\frown 1}) : s \in \mathrm{Seq}_2\},$$

then, it forms an open cover of $[0, 1]$.

By the Heine-Borel Theorem, there exists a finite subcover U'.

Since for any $s \in B$, (a'_s, b'_s) does not intersect with any other $(a'_t, b'_t) \in U$ and is not a subset of $\bigcup\{(b_{s^\frown 0}, a_{s^\frown 1}) : s \in \mathrm{Seq}_2\}$, U' must contain $\{(a'_s, b'_s) : s \in B\}$. Therefore, B is finite.

Since T is obtained from the set of all initial segments of elements in B by removing the elements of B, it is also finite. $\square$

The Heine-Borel property of $[0,1]$ allows us to derive various properties of continuous functions $f:[0,1]\to\mathbb{R}$.

Lemma 3.10

In WKL_0, a continuous function $f:[0,1]\to\mathbb{R}$ is uniformly continuous.

Proof Fix any $n\in\mathbb{N}$. We want to show the existence of $d>0$ such that

$$\forall x,y\in[0,1](|\, x-y\,|< d \to|\, f(x)-f(y)\,|< 2^{-n}).$$

Let F be the code for the continuous function f, and denote the open interval with code i as (p_i,q_i). Then, define the open set U as follows:

$$i\in U\Leftrightarrow \exists j<i((p_i,q_i,p_j,q_j)\in F\wedge q_j-p_j<2^{-n-1}).$$

First, we show that U is a covering of $[0,1]$. For any real number $x\in[0,1]$, since $x\in\mathrm{dom} f$, there exists $(p_k,q_k,p_j,q_j)\in F$, such that

$$p_k<x<q_k\wedge q_j-p_j<2^{-n-1}.$$

Furthermore, there are infinitely many i such that $p_k\le p_i<x<q_i\le q_k$, so taking such an $i>j$, we have $i\in U$ with $p_i<x<q_i$. Therefore, U forms an open covering of $[0,1]$.

By the Heine-Borel Theorem, U has a finite subcover U'.

Let d be the minimum width $q_i - p_i$ among the intervals (p_i, q_i) in U'. We shall show that this d satisfies the uniform convergence condition.

Now, choose any real numbers $x, y \in [0, 1]$ such that $\mid x - y \mid < d$. Then, there must exist intervals $(p_i, q_i), (p_{i'}, q_{i'})$ in U' such that $x \in (p_i, q_i)$, $y \in (p_{i'}, q_{i'})$ and they have a common point z.

Otherwise, take an interval $(p_i, q_i) \ni x$ in U' with maximum q_i, and an interval $(p_{i'}, q_{i'}) \ni y$ in U' with minimum $p_{i'}$. If there is no common point, $q_i < p_{i'}$. Since U' is a covering, there exists $q_i \in (p_k, q_k)$ in U'. By the maximality of q_i, $x \notin (p_k, q_k)$. From $|q_k - p_k| \geq d > |x - y|$, we have $y \in (p_k, q_k)$, which contradicts with the minimality of $p_{i'}$.

By the definition of U, we have $\mid f(x) - f(z) \mid < 2^{-n-1}$ and $\mid f(y) - f(z) \mid < 2^{-n-1}$, thus $\mid f(x) - f(y) \mid < 2^{-n}$, which fulfills the lemma. □

Lemma 3.11

In WKL_0, a continuous function $f : [0,1] \to \mathbb{R}$ attains a maximum value.

Proof First, we show that the supremum M of the range of f exists.

As in the proof of the previous lemma, we define U by a Σ^0_0 formula:

$$i \in U \Leftrightarrow \exists j < i((p_i, q_i, p_j, q_j) \in F \wedge q_j - p_j < 2^{-n-1}).$$

We can finitely calculate whether or not a given finite set of open rational intervals covers $[0,1]$. Therefore, by arranging all finite subsets of U and checking sequentially whether they cover $[0,1]$, we eventually obtain a finite subcover U'. That is, in WKL_0, we can construct a function extracting U' according to n.

For each $i \in U'$, select $j_i < i$ such that $(p_i, q_i, p_{j_i}, q_{j_i}) \in F \wedge q_{j_i} - p_{j_i} < 2^{-n-1}$, and let $M_n = \max\{q_{j_i} : i \in U'\}$. Then, $\{M_n\}$ itself is a real number, and it is clear that it is the supremum M of the range of f.

What remains is to show that the existence of a point $x = a$ such that $f(a) = M$. For the sake of the following argument, we redefine $M_n = \max\{p_{j_i} : i \in U'\}$. This ensures that for any n, $M_n \leq M = \{M_n\}$.

By way of contradiction, assume that $f(x) < M$ for all $x \in [0,1]$. Then, we define an open set V as follows:

$$i \in V \Leftrightarrow \exists j < i\ \exists n < i((p_i, q_i, p_j, q_j) \in F \wedge q_j < M_n).$$

To show that this set forms a covering of $[0,1]$, take any real number $x \in [0,1]$. Since $f(x) < M$, there exists n such that $f(x) < M_n \leq M$, and hence there exists $(p_k, q_k, p_j, q_j) \in F$ and n such that

$$p_k < x < q_k\ \wedge\ p_j \leq f(x) \leq q_j < M_n \leq M.$$

As there are infinitely many i such that $p_k \leq p_i < x < q_i \leq q_k$, taking $i > j, n$ ensures $i \in V$ with $p_i < x < q_i$. Therefore, V forms an open covering of $[0,1]$.

Again, by the Heine-Borel Theorem, V has a finite subcover V'. Let M' be the maximum of q_i for (p_i, q_i) in V'. Then, by the definition of values of a continuous function, obviously M' is an upper bound of the range. However, due to the finiteness of V', for some n, $M' < M_n \leq M$, which contradicts the fact that M is the supremum. □

Conversely, the properties described in the two lemmas above allow us to derive WKL_0. In sum, the following theorem holds:

Theorem 3.12

The following assertions are pairwise equivalent in RCA_0:

(1) WKL_0,

(2) A continuous function $f : [0, 1] \to \mathbb{R}$ is uniformly continuous,

(3) A continuous function $f : [0, 1] \to \mathbb{R}$ is bounded,

(4) A bounded continuous function $f : [0, 1] \to \mathbb{R}$ has a supremum,

(5) A continuous function $f : [0, 1] \to \mathbb{R}$ that has a supremum attains its maximum value.

Proof By Lemmas 3.10 and 3.11, we can deriving (2), (3), (4), and (5) from (1). Hence, it suffices to obtain counterexamples for (2), (3), (4) and (5) from the negation of (1). Now, assume the negation of (1). Then, there exists an infinite tree $T \subseteq \mathrm{Seq}_2$ without infinite paths.

As shown in the proof of Heine-Borel's theorem, for each $s \in \mathrm{Seq}_2$, define the two rational numbers a_s and b_s as follows:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{s(i)}{2^{i+1}},$$

$$b_s = a_s + \frac{1}{2^{\mathrm{leng}(s)}}.$$

Let B be the infinite set of all minimal binary sequences not in T,

$$s \in B \Leftrightarrow s \notin T \wedge \forall t \subset s(t \neq s \rightarrow t \in T)$$

and J be the set of closed intervals $[a_s, b_s]$ for all $s \in B$.

Each real number $x \in [0, 1]$ is either an interior point of exactly one interval in J or an endpoint of one or two intervals. Such an infinite set J is called a **singular closed cover**.

$\neg\,\mathsf{WKL}_0 \to \neg$ (3) bounded.

We will construct a counterexample for (3) using this singular closed cover J. This also serves as a counterexample for (2) since (2) implies (3). We define a continuous function f_s for each interval $[a_s, b_s]$ in J as follows:

$$f_s(x) = \begin{cases} \mathrm{leng}(s)\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ \mathrm{leng}(s)\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s. \end{cases}$$

That is, f_s takes 0 at the endpoints $x = a_s, b_s$, takes $\mathrm{leng}(s)$ at the midpoint $x = \frac{a_s+b_s}{2}$, and is linearly interpolated otherwise.

Let f be a function obtained by composing all such functions f_s. Then, it is clearly continuous but unbounded. (It is left as an exercise for the reader to construct a continuous function code for f.)

$\neg\,\mathsf{WKL}_0 \to \neg$ (5) a maximum value.

A counterexample for (5) can be constructed in the way similar to that for (3) in the previous slide. We just replace the maximum value of f_s from $\mathrm{leng}(s)$ to $1 - 2^{-\mathrm{leng}(s)}$ as follows:

$$f_s(x) = \begin{cases} (1 - 2^{-\mathrm{leng}(s)})\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ (1 - 2^{-\mathrm{leng}(s)})\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s. \end{cases}$$

Then, a composed function f clearly has 1 as its supremum, but it can not attain the maximum value 1 in $[0,1]$.

$\neg\,\mathsf{WKL}_0 \to \neg$ (4) a supremum.

Recall:

Theorem 3.4.(5)

($\mathsf{RCA}_0 \vdash$) $\mathsf{ACA}_0 \Leftrightarrow$ (4) Every bounded increasing sequence of reals has a supremum.

Negating WKL_0, we have the negation of ACA_0, which implies the existence of a bounded increasing sequence of rational numbers $\{c_n\}$ that lacks a supremum.
Then, replace the maximum value of f_s with $c_{\mathrm{leng}(s)}$ and proceed similarly.

$$f_s(x) = \begin{cases} c_{\mathrm{leng}(s)} \frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ c_{\mathrm{leng}(s)} \frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s, \end{cases}$$

Problem

Show that in the theorem 3.12 (4) and (5), "continuous function" can be replaced with "uniformly continuous function". Hint: It is beneficial to use a singular closed cover for the ternary set.

Thank you for your attention!