Logic and Foundation II

Part 7. Theory of reals and Reverse Mathematics

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April 23, 2024

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued) (5 lectures)
- Part 6. Real-closed ordered fields: completeness and decidability (4 lectures)
- Part 7. Theory of reals and reverse mathematics (9 lectures?)
- Part 8. Second order arithmetic and non-standard methods (6 lectures?)

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining real numbers in RCA_0
- Apr. 23, (3) Completeness of the reals and ACA_0
- Apr. 25, (4) Continuous functions and WKL_0
- Apr. 30, (5) Knig's lemma and Ramsey's theorem
- May 9, (6) Determinacy of infinite games I
- May 14, (7) Determinacy of infinite games II
- to be continued

Recap

Reverse Mathematics: Which axioms are needed to prove a theorem?

The Reverse Mathematics Phenomenon

Many theorems of mathematics are either provable in RCA_0, *or logically equivalent (over* RCA_0*) to one of* WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Definition 1.2 The system of **recursive comprehension axioms** (RCA_0) consists of:

(0) Axioms and inference rules of first-order logic with axioms of equality for numbers.

(1) Basic arithmetic axioms: Same as $\mathsf{Q}_<$ (Chapter 4).

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and neither includes X as a free variable.

(3) Σ^0_1 induction: $\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

Real numbers in RCA_0

Primitive recursive functions (e.g., sequence numbers and Gödel numbers) are available in RCA_0. Note that RCA_0 is a conservative extension of first-order arithmetic $\mathrm{I}\Sigma_1$.

So, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and their arithmetical operations are naturally defined in RCA_0.

A sequence of rational numbers $\{q_n\}$ is a **real number**, $\{q_n\} \in \mathbb{R}$, if it satisfies

$$\forall n \forall i (|q_n - q_{n+i}| \leq 2^{-n}).$$

Moreover, we define

$$\begin{aligned}
\{p_n\} = \{q_n\} &\leftrightarrow \forall n (|p_n - q_n| \leq 2^{-n+1}), \\
\{p_n\} < \{q_n\} &\leftrightarrow \exists n (q_n - p_n > 2^{-n+1}), \\
\{p_n\} + \{q_n\} &= \{p_{n+1} + q_{n+1}\} \\
\{p_n\} \cdot \{q_n\} &= \{p_{n+m} \cdot q_{n+m}\} \text{ for a large enough } m
\end{aligned}$$

Then, it is provable in RCA_0 that $(\mathbb{R}, +, \cdot, 0, 1, <, =)$ is an **Archimedean ordered field**.

Sequences and continuous functions

We define a **sequence of real numbers** as a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that:
for each n, letting $f_n(m) = f(n, m)$, $f_n : \mathbb{N} \to \mathbb{Q}$ is a real number.

- The nested interval property of $\mathbb{R}$ and uncountability of $\mathbb{R}$ is provable in RCA_0.

We will introduce **continuous functions** on $\mathbb{R}$ in RCA_0. A set $\Phi \subseteq \mathbb{Q}^4$ that satisfies the following conditions is called the **code** for a continuous function $f : \mathrm{dom}\, f(\subseteq \mathbb{R}) \to \mathbb{R}$.

(1) $(p, q, r, s) \in \Phi \to p < q \wedge r \leq s$,

(2) $(p, q, r, s), (p', q', r', s') \in \Phi, p' < q \wedge p < q' \ \to \ r' \leq s \wedge r \leq s'$.

Intuitively, $(p, q, r, s) \in \Phi$ means $\forall x(p < x < q \to r \leq f(x) \leq s)$.
A real number x belongs to the **domain** of a continuous function f coded by Φ, if

$$\forall n \exists (p, q, r, s) \in \Phi(p < x < q \wedge s - r < 2^{-n}), \text{ denoted } x \in \mathrm{dom} f.$$

It is provable in RCA_0 that if $x \in \mathrm{dom} f$, there exists a unique real y such that $\forall(p, q, r, s) \in \Phi(p < x < q \to r \leq y \leq s)$. We denote this y as $f(x)$.

- The intermediate value theorem is provable in RCA_0.

Introducing ACA_0

In the previous section, we proved the nested interval property of $\mathbb{R}$ in RCA_0, but we will see that other important properties such as sequential compactness require a strictly stronger systems ACA_0 (or often another system WKL_0, which lies between RCA_0 and ACA_0).

Definition 3.1

The **system of arithmetical comprehension axioms** (ACA_0) is RCA_0 extended with the following axiom:

$$(\Pi^1_0\text{-CA}) : \exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is an arithmetical formula (Π^1_0 formula), which does not have X as a free variable.

In the definition of ACA_0, (Π^1_0-CA) can be replaced with (Σ^0_1-CA). See Lemma 3.3 below.

Lemma 3.2

ACA_0 is a conservative extension of Peano Arithmetic PA. That is, all theorems of PA are provable in ACA_0, and a sentence in $\mathcal{L}_{\mathrm{OR}}$ provable in ACA_0 is a theorem of PA.

Proof To prove that all theorems of PA are provable in ACA_0, it suffices to show that induction axiom for any arithmetic formula can be proved in ACA_0.
Let $\varphi(n)$ be any arithmetic formula, and assume

$$\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1)).$$

By the arithmetical comprehension axiom, there exists X such that

$$\forall n(n \in X \leftrightarrow \varphi(n)).$$

For this X, it follows that

$$0 \in X \land \forall n(n \in X \to n+1 \in X),$$

and applying Σ^0_1 induction yields $\forall n(n \in X)$, hence $\forall n \varphi(n)$.

For the converse direction, to prove that a proposition of first-order arithmetic provable in ACA_0 is a theorem of PA, we can use the same method that was used to show RCA_0 is a conservative extension of $\mathrm{I}\Sigma_1$.

□

Let σ be a sentence in $\mathcal{L}_{\mathrm{OR}}$ such that PA $\nvdash \sigma$. By the completeness theorem, there exists a model $\mathfrak{M}$ of PA where $\mathfrak{M} \models \neg\sigma$.

For each arithmetical formula $\varphi(x, y_1, \ldots, y_k)$ and $b_1, \ldots, b_k \in M$, we put

$$A_{\varphi,b_1,\ldots,b_k} = \{a \in M : \mathfrak{M} \models \varphi(a, b_1, \ldots, b_k)\}.$$

Then let S be the set of arithmetically definable subsets $A_{\varphi,b_1,\ldots,b_k}$ of M.
We show $(\mathfrak{M}, S)$ forms a model of ACA$_0$.

We first note that in any arithmetical formula with set parameters from S, a parameter $A_{\varphi,\bar{c}}$ can be eliminated by replacing $t \in A_{\varphi,\bar{c}}$ with $\varphi(t, \bar{c})$.

Then, Σ^0_1 induction of $(\mathfrak{M}, S)$ can be derived from arithmetical induction of $\mathfrak{M}$. Also, Σ^0_1 comprehension holds in $(\mathfrak{M}, S)$, since any set definable by Σ^0_1 comprehension is also definable without set parameters, and so already belongs to S. Thus, $(\mathfrak{M}, S)$ is a model of RCA$_0$.

Finally, since σ does not contain set variables, its truth value is independent of S, and hence $(\mathfrak{M}, S) \models \neg\sigma$. Therefore, ACA$_0 + \neg\sigma$ is consistent, which implies ACA$_0 \nvdash \sigma$. This completes the proof. □

Since PA is a proper extension of IΣ_1, it follows that ACA$_0$ is a proper extension of RCA$_0$.

Lemma 3.3

In RCA_0, the following are equivalent:

(1) ACA_0,

(2) (Σ^0_1-CA),

(3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

Proof $(1) \Rightarrow (2)$ is obvious. $(2) \Rightarrow (3)$ is also clear since the range of f is Σ^0_1. $(3) \Rightarrow (2)$ immediately follows from Lemma 1.7 and Lemma 1.8 of Lec07-01.

To show $(2) \Rightarrow (1)$, we prove $(\Sigma^0_1\text{-CA}) \to (\Sigma^0_k\text{-CA})$ by meta-induction on k. For $k \leq 1$, it is clear. Let $\varphi(n)$ be any Σ^0_{k+1} formula. Then, we can write $\varphi(n)$ as

$$\varphi(n) \leftrightarrow \exists m \neg \theta(m,n) \text{ ,where } \theta(m,n) \text{ is } \Sigma^0_k$$

By (Σ^0_k-CA), the set $Y = \{(m,n) : \theta(m,n)\}$ exists. Then by (Σ^0_1-CA), the set

$$X = \{n : \exists m \neg((m,n) \in Y)\}$$

also exists. Thus,

$$n \in X \leftrightarrow \varphi(n).$$

Therefore, (Σ^0_{k+1}-CA) holds. □

The following theorem states that some properties of real numbers are not only unprovable in RCA_0, but also equivalent to ACA_0.

Theorem 3.4

The followings are pairwise equivalent over RCA_0.

(1) ACA_0,

(2) The Bolzano-Weierstrass theorem: Every bounded sequence of real numbers has a convergent subsequence,

(3) Every Cauchy sequence converges,

(4) Every bounded sequence of real numbers has a supremum,

(5) The monotone convergence theorem: Every bounded increasing sequence converges.

Proof. $(1) \Rightarrow (2), (3), (4), (5)$ can be deduced by the usual proofs found in calculus textbooks. We here only notice that in the bisection method, conditions like an interval containing at least one element or infinitely many elements from the sequence can be written in arithmetical formulas. Also, (5) immediately follows from (2), (3), or (4), so it suffices to show (5) implies (1).

Monotone Convergence Theorem $\Rightarrow$ ACA_0

Assume (5). To show (1) by using Lemma 3.3(3), take any one-to-one function $f : \mathbb{N} \to \mathbb{N}$. We want to show the range of f exists.
Define a bounded increasing sequence of rational numbers $\{c_n\}$ by

$$c_n = \sum_{i=0}^{n} 2^{-f(i)}.$$

Then, by the Monotone Convergence Theorem, the limit

$$c = \lim_{n\to\infty} c_n = \sum_{i=0}^{\infty} 2^{-f(i)}$$

exists. Since $|c_k - c| = \sum_{i=k+1} 2^{-f(i)} < 2^{-n} \to \forall i > k f(i) > n$, we have, for any n,

$$n \in \text{range} f \leftrightarrow \exists m\, f(m) = n \leftrightarrow \forall k\, (|c_k - c| < 2^{-n} \to \exists m \le k\, (f(m) = n))$$

Thus, by (Δ^0_1-CA), the range of f exists. Thus, by the above lemma, we have done. $\square$

Binary Trees

We will see many important theorems in analysis are equivalent to the system WKL_0, which is properly weaker than ACA_0. To define WKL_0, we need some concepts of trees.

For each $n \in \mathbb{N}$, a code s $(\in \mathbb{N})$ for functions with domain $\{i \in \mathbb{N} : i < n\}$ is called a **finite sequence** with length $n = \mathrm{leng}(s)$.

Particularly, finite sequences that only take values 0 or 1 are called **binary sequences**, and the set of (codes for) finite sequences of 0's and 1's is denoted by Seq_2.

A subset T of Seq_2 which is closed under initial segment, is called a **tree**, i.e.,

$$\forall s, t(s \subseteq t \wedge t \in T \implies s \in T),$$

where $s \subseteq t$ means that s is an **initial segment** of t.

A subset of a tree T which is a tree with no branching is called a **path** through T. A path through T is also expressed as a function $g : \mathbb{N} \to \mathrm{Seq}_2$ such that $g(0) = \varnothing$, $g(n) \subseteq g(n+1) \in T$ and $\neg\exists t\ (g(n) \subsetneq t \subsetneq g(n+1))$ for all $n \in \mathbb{N}$.

WKL_0

Definition 3.5

Weak König's lemma is the statement that every infinite tree $T \subset \mathrm{Seq}_2$ has an infinite path. The system WKL_0 is RCA_0 plus weak König's lemma.

To show the existence of a specific path through a given infinite tree such as the left-most path, we need ACA_0, which is properly stronger than WKL_0, as we will see later. We first show that WKL_0 is strictly stronger than RCA_0.

Lemma 3.6

In RCA_0, WKL_0 is equivalent to the following statement:

$$(\Sigma^0_1\text{-}\mathsf{SP}) : \forall n(\varphi(n) \to \psi(n)) \to \exists X \forall n\{(\varphi(n) \to n \in X) \land (n \in X \to \psi(n))\},$$

where $\varphi(n)$ is a Σ^0_1 formula and $\psi(n)$ is a Π^0_1 formula, neither containing X as a free variable. stands for the **Separation Principle**.

Note that in the Separation Principle, if we replace the premise $\forall n(\varphi(n) \to \psi(n))$ with $\forall n(\psi(n) \leftrightarrow \psi(n))$, it becomes the Δ^0_1 comprehension axiom (definition 1.2(2)). That is, WKL_0 is equivalent to RCA_0 with (Δ^0_1-CA) replaced by (Σ^0_1-SP).

Proof of Lemma 3.6 First, we show that the Σ^0_1 Separation Principle holds in WKL_0. Given two Σ^0_1 formulas $\varphi_0(n)$ and $\varphi_1(n)$ with

$$\forall n(\varphi_0(n) \to \neg\varphi_1(n)), \text{ or equivalently } \forall n\neg(\varphi_0(n) \land \varphi_1(n)).$$

We suppose $\varphi_i(n) \equiv \exists m\theta_i(m,n)$, where $\theta_i(m,n) \in \Sigma^0_0$. Then, define a set $T \subseteq \mathrm{Seq}_2$ in RCA_0 as follows:

$$t \in T \Leftrightarrow \forall m,n < \mathrm{leng}(t)[(\theta_0(m,n) \to t(n) = 0) \land (\theta_1(m,n) \to t(n) = 1)].$$

It is easy to see that T forms an infinite tree. Then by weak König's lemma, there exists an infinite path f. If we set

$$X = \{n : f(n) = 0\},$$

then obviously

$$\forall n\{(\varphi_0(n) \to n \in X) \land (n \in X \to \neg\varphi_1(n))\}.$$

Conversely, we prove weak König's lemma from Σ^0_1 Separation Principle. Fix any infinite tree $T \subseteq \mathrm{Seq}_2$. For $i = 0, 1$, let $\varphi_i(s)$ denote the Σ^0_1 formula expressing

$$\text{“}s \in \mathrm{Seq}_2 \text{ and the set } \{t \in T : s^\frown i \subseteq t\} \text{ is finite”.}$$

Here, $s^\frown i$ denotes the binary sequence s followed by i, that is, $s \cup \{(\mathrm{leng}(s), i)\}$. “Tree T' is finite” can be expressed as $T' \cap \{0,1\}^n \neq \emptyset$ for some sufficiently large n, which can be written as a Σ^0_1 formula.

Now, let $\varphi_i(s) \equiv \exists m \theta_i(m, s)$, with $\theta_i(m, s) \in \Sigma^0_0$, and modify them for the Σ^0_1 Separation Principle as

$$\varphi'_0(s) \equiv \exists m(\theta_0(m, s) \wedge \forall k < m \neg\theta_1(k, s)),$$
$$\varphi'_1(s) \equiv \exists m(\theta_1(m, s) \wedge \forall k \leq m \neg\theta_0(k, s)).$$

Since $\forall n \neg(\varphi'_0(n) \wedge \varphi'_1(n))$, or $\forall n(\varphi'_0(n) \to \neg\varphi'_1(n))$, the Σ^0_1 Separation Principle ensures the existence of X such that $\forall n\{(\varphi'_0(n) \to n \in X) \wedge (n \in X \to \neg\varphi'_1(n))\}$.

Using X, we inductively define an infinite sequence of binary sequences $s_0 \subset s_1 \subset \cdots$ as follows: Let $s_0 = \varnothing$. If $s_n \in X$, then $s_{n+1} = s_n{}^\frown\{1\}$. Otherwise $s_{n+1} = s_n{}^\frown\{0\}$.

To show that $f = \{s_n\}$ forms an infinite path through T, we show by induction that for all n, $T_n = \{t \in T : s_n \subseteq t\}$ is infinite. Now, assume T_n is infinite, i.e., $\neg\varphi_0(s_n) \vee \neg\varphi_1(s_n)$. Consider the case $s_n \in X$. Then, $\neg\varphi'_1(s_n)$ and $s_{n+1} = s_n{}^\frown\{1\}$. Thus, T_{n+1} is infinite or $\exists m_0, m_1(m_0 \leq m_1 \wedge \theta_0(m_0, s_n) \wedge \theta_1(m_1, s_n))$. However, since $\neg\varphi_0(s_n) \vee \neg\varphi_1(s_n)$, the latter disjunctive condition does not hold. The case $s_n \notin X$ can be treated similarly. □

By the above lemma, the following is straightforward.

Corollary 3.7

WKL_0 is strictly stronger than RCA_0.

Proof It suffices to show that the minimal model of RCA_0, (ω, Rec), cannot be a model of WKL_0. Here, Rec denotes the set of recursive subsets of ω.

We need to show the existence of two disjoint Σ^0_1 sets A and B that are recursively inseparable.
Let A and B be defined as follows:

$$A = \{\ulcorner\sigma\urcorner \mid \mathsf{R} \vdash \sigma\} \quad \text{(the set of Gödel numbers of the theorems of R)}$$
$$B = \{\ulcorner\sigma\urcorner \mid \mathsf{R} \vdash \neg\sigma\} \quad \text{(the set of Gödel numbers of the negations of the theorems of R)}$$

Then, we derive a contradiction by assuming the existence of recursive set C $(A \subset C \subset B^c)$.

Since C is recursive, by Representation Theorem in Part 4, there exists Σ^0_1 formula $\varphi(x)$ such that

$$n \in C \to \mathsf{R} \vdash \varphi(\bar{n})$$
$$n \notin C \to \mathsf{R} \vdash \neg\varphi(\bar{n})$$

By Diagonalization Lemma in Part 4, there exists σ such that

$$\ulcorner\sigma\urcorner \notin C \to \mathsf{R} \vdash \neg\varphi(\overline{\ulcorner\sigma\urcorner}) \to \mathsf{R} \vdash \neg\sigma \to \ulcorner\sigma\urcorner \in A.$$

$$\ulcorner\sigma\urcorner \in C \to \mathsf{R} \vdash \varphi(\overline{\ulcorner\sigma\urcorner}) \to \mathsf{R} \vdash \neg\sigma \to \ulcorner\sigma\urcorner \in B,$$

Then, C does not separate A and B (Not $A \subset C \subset B^c$).

Therefore, (Σ^0_1-SP) does not hold in (ω, Rec).

□

There are various ways to show that ACA_0 is strictly stronger than WKL_0.

One is the fact that WKL_0 has no minimal model, and for any model (M, S) of WKL_0, there exists $\langle A_n \mid n \in M \rangle \in S$ such that $(M, \{A_n\})$ is a model of WKL_0. This fact can be proved from the compactness shown in the next chapter (via the strong Π^0_1 dependent choice axiom [Simpson, 1999, Lemma VIII. 2. 5]).

While this might seem to contradict the incompleteness theorem, defining the satisfaction relation $\models$ on $(M, \{A_n\})$ requires ACA_0; thus, the existence of a model of WKL_0 within WKL_0 itself is not asserted. However, if ACA_0 is assumed, the existence of a model of WKL_0 and hence the consistency of WKL_0 can be proved, we can show that ACA_0 is strictly stronger than WKL_0.

Moreover, as shown in the next part, WKL_0 and RCA_0 are equivalent over arithmetic formulas, i.e., both are conservative extensions of $\mathsf{I}\Sigma_1$. Whereas, ACA_0 is a conservative extension of PA. Thus, these three systems have different strength.