Logic and Foundation II

Part 7. Theory of reals and Reverse Mathematics

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April 16, 2024

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Logic and Foundations II

- Part 5. Models of first-order arithmetic (continued)
- Part 6. Real-closed ordered fields: completeness and decidability
- Part 7. Theory of reals and reverse mathematics
- Part 8. Second order arithmetic and non-standard methods

Part 7. Schedule

- Apr. 16, (1) Introduction and the base system RCA_0
- Apr. 18, (2) Defining reals in RCA_0
- Apr. 23, (3) Completeness and compactness of reals
- Apr. 25, (4) ...
- to be continued

From Hilbert's program to reverse mathematics

- **Second-order arithmetic** is a formal theory targeting natural numbers and sets of natural numbers. It was D. Hilbert who first drew attention to its importance as foundations of mathematics. He formulated a deductive system of second-order arithmetic Z_2 around 1920, which can also encompass real numbers, sequences of real numbers, continuous functions and much more.
- The second problem of Hilbert's 23 problems was to show the consistency of basic arithmetic of reals. This problem was then conceived as **Hilbert's program** whose aim is to establish the consistency of Z_2 finitistically. As known well, Gödel's second incompleteness theorem blocked its progress.
- However, it is also known that a considerable breadth of mathematics can be developed within weak subsystems of Z_2, whose consistency can be shown finitistically. From the mid-1970's, H. Friedman, S. Simpson, and others started research to investigate which subsystem is needed to prove a popular theorem of mathematics in the framework of second order arithmetic. This research program has evolved into a significant field known as **reverse mathematics**.

Reverse Mathematics Program

Reverse Mathematics

Which axioms are needed to prove a theorem?

Big Five subsystems in order of increasing strength: RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0

- RCA_0 stands for the Recursive Comprehension Axiom, and it only guarantees the existence of recursive (computable) sets. The subscript 0 indicates a restriction on induction, which will be discussed later.

- $\mathsf{WKL}_0 = \mathsf{RCA}_0 + \overbrace{\text{any infinite binary tree has an infinite path}}^{\text{Weak König Lemma}}$
$= \mathsf{RCA}_0 + \Sigma^0_1$-$\mathsf{SP}$

Σ^0_1-SP (Σ^0_1 *separation*):

$$\neg\exists x(\varphi_0(x) \wedge \varphi_1(x)) \to \exists X \forall x((\varphi_0(x) \to x \in X) \wedge (\varphi_1(x) \to x \notin X)),$$

where $\varphi_0(x)$ and $\varphi_1(x)$ are Σ^0_1 formulas.

- $\mathsf{ACA}_0 = \mathsf{RCA}_0 + \overbrace{\exists X \forall n \big(n \in X \leftrightarrow \varphi(n)\big) \text{ for all arithmetical } \varphi(n)}^{\text{Arithmetical Comprehension}}$
  $= \mathsf{RCA}_0 + \Sigma^0_1\text{-}\mathsf{CA}$

- $\mathsf{ATR}_0 = \mathsf{RCA}_0 + \overbrace{\text{the existence of a transfinite hierarchy produced}}^{\text{Arithmetical Transfinite Recursion}}$
  by interating arithemetic comprehension along a given well order

- $\Pi^1_1\text{-}\mathsf{CA}_0 = \mathsf{RCA}_0 + \overbrace{\exists X \forall n \big(n \in X \leftrightarrow \varphi(n)\big) \text{ for all } \Pi^1_1\ \varphi(n)}^{\Pi^1_1 \text{ Comprehension}}$
  A formula in the form $\forall X \psi$ with ψ arithmetical is called a Π^1_1 formula.

The Reverse Mathematics Phenomenon

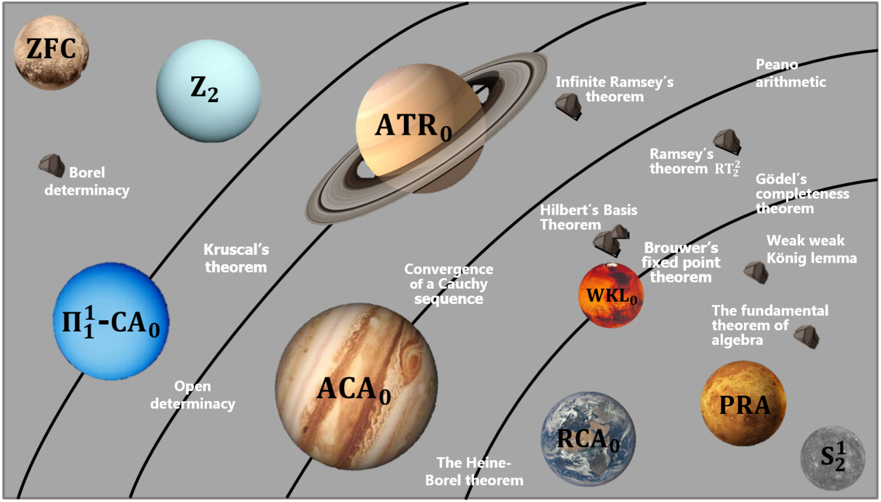
Many theorems of mathematics are either provable in RCA_0*, or logically equivalent (over* RCA_0*) to one of the other four systems mentioned above.*

$$
\begin{aligned}
\mathsf{RCA}_0 &\Rightarrow \text{the intermediate value theorem} \\
&\Rightarrow \text{fundamental theorem of algebra} \\
\mathsf{WKL}_0 &\leftrightarrow \text{the maximum principle} \quad \leftrightarrow \quad \text{the Cauchy-Peano theorem} \\
&\leftrightarrow \text{Brouwer's fixed point theorem} \\
\mathsf{ACA}_0 &\leftrightarrow \text{the Bolzano-Weierstrass theorem} \quad \leftrightarrow \quad \text{the Ascoli-Arzela lemma} \\
\mathsf{ATR}_0 &\leftrightarrow \text{the Luzin separation theorem} \quad \leftrightarrow \quad \text{Open-determinacy} \\
\Pi^1_1\text{-}\mathsf{CA}_0 &\leftrightarrow \text{the Cantor-Bendixson theorem} \quad \leftrightarrow \quad \text{(Open} \land \text{Closed)-determinacy}
\end{aligned}
$$

Friedman's conservation result

$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad$ Primitive Recursive Arithmetic $\vdash \sigma \quad$ for $\sigma \in \Pi^0_2$.

Planets and Reverse Mathematics



Formulas of second-order arithmetic

- The language $\mathcal{L}^2_{\mathrm{OR}}$ of second-order arithmetic is the language of first-order arithmetic $\mathcal{L}_{\mathrm{OR}} = \{+, \cdot, 0, 1, <\}$ plus a symbol $\in$ for the membership relation.
- The **formulas** of second-order arithmetic are constructed from atomic formulas ($t_1 = t_2$, $t_1 < t_2$, $t \in X$) by propositional connectives such as $\neg$, $\vee$, etc., and quantifiers over arithmetic $\forall x$, $\exists x$, as well as over sets $\forall X$, $\exists X$.
- A formula can be rewritten in the prenex normal form by shifting quantifiers to the head of formula. Moreover, all second-order quantifiers can be placed outside of the scopes of any first-order quantifier. The following transformation is possible even in a very weak theory,

$$\forall x \exists Y \varphi(x, Y) \Leftrightarrow \forall X \exists Y (\exists! x(x \in X) \to \forall x(x \in X \to \varphi(x, Y))).$$

  If the axiom of choice is available, the places of quantifiers are exchanged as follows:

$$\forall x \exists Y \varphi(x, Y) \Leftrightarrow \exists Y' \forall x \varphi(x, Y'_x),$$

  where Y' is a set-valued choice function, that is, $Y'(x) = Y'_x = \{y : (x, y) \in Y'\}$.

Hierarchy of formulas

We inductively define the hierarchy of $\mathcal{L}^2_{\mathrm{OR}}$-formulas, Σ^i_j and Π^i_j $(i = 0, 1,\ j \in \mathbb{N})$.

Definition 1.1

- The **bounded** formulas are constructed from atomic formulas $t_1 = t_2, t_1 < t_2, t \in X$ by propositional connectives and bounded quantifiers $\forall x < t$, $\exists x < t$.
  The class of such formulas is written as Π^0_0 or Σ^0_0.
- For each $j \geq 0$, if $\varphi \in \Sigma^0_j$, then $\forall x_1 \cdots \forall x_k \varphi \in \Pi^0_{j+1}$;
  if $\varphi \in \Pi^0_j$, then $\exists x_1 \cdots \exists x_k \varphi \in \Sigma^0_{j+1}$.
  All formulas in Σ^0_j and Π^0_j are called **arithmetical**.
  The class of arithmetical formulas is also denoted as Π^1_0 or Σ^1_0.
- For each $j \geq 0$, if $\varphi \in \Sigma^1_j$, then $\forall X_1 \cdots \forall X_k \varphi \in \Pi^1_{j+1}$;
  if $\varphi \in \Pi^1_j$ then $\exists X_1 \cdots \exists X_k \varphi \in \Sigma^1_{j+1}$.
  All formulas in Σ^1_j and Π^1_j are called **analytical**.

- Formulas belonging to Σ^i_j or Π^i_j are referred to as Σ^i_j or Π^i_j formulas, respectively.
- Σ^0_i (or Π^0_i) formulas without set variables are nothing but Σ_i (or Π_i) formulas of first-order arithmetic.
- A formula that is equivalent to a Σ^i_j (or Π^i_j) formula on a given base system is also called Σ^i_j (or Π^i_j).
- Furthermore, if a Σ^i_j formula is equivalent to a Π^i_j formula, each of them is called a Δ^i_j formula. Since the equivalence of formulas depends on a base theory T, Δ^i_j is strictly expressed as $(\Delta^i_j)^T$.
- When dealing with arithmetical hierarchies Σ^0_i Π^0_i, a system of second-order arithmetic RCA_0 is often assumed as a base theory. While dealing with analytical hierarchies, a stronger system ACA_0 is often needed.

Examples:

- "X is an infinite set" is represented by a Π^0_2 formula $\forall x \exists y(x < y \land y \in X)$.
- "A linear order $\preceq$ is a well-ordering", that is, "every non-empty set has the least element", can be represented by the following Π^1_1 formula
  $\forall X(\exists z(z \in X) \to \exists x(x \in X \land \forall y \in X(x \preceq y)))$,
  or rewritten as $\forall X \forall z \exists x(z \notin X \lor (x \in X \land \forall y \in X(x \preceq y)))$.

The system of recursive comprehension axioms (RCA_0) is a weak base system of second-order arithmetic, which serves as the foundation for our subsequent observation.

Definition 1.2

The system of **recursive comprehension axioms** (RCA_0) consists of the following axioms:

(0) Axioms and inference rules of first-order logic with axioms of equality for numbers. Equality between sets $X = Y$ is defined as $\forall n(n \in X \leftrightarrow n \in Y)$.

(1) Basic arithmetic axioms: Same as $\mathsf{Q}_<$ (Chapter 4).

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0):

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is a Σ^0_1 formula, $\psi(n)$ is a Π^0_1 formula, and neither includes X as a free variable. This axiom ensures the existence of set $X = \{n : \varphi(n)\}$.

(3) Σ^0_1 induction: $\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

- Since the Δ^0_1 comprehension axiom asserts the existence of recursive sets (=computable sets) in the standard model $\mathbb{N}$, it is called the **recursive comprehension axiom**.
- More precisely, since $\psi(x)$ and $\varphi(x)$ in the axiom may include set variables (other than X) as parameters, this axiom indeed asserts that there exists a set that can be computed in a parameter set as an oracle. But notice that it does not assert the non-existence of a non-recursive set.
- RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$. That is, a sentence of $\mathcal{L}_{\mathrm{OR}}$ that is provable in RCA_0 is already provable in $\mathsf{I}\Sigma_1$, as shown in the next lemma.

Definition (preliminary). The system of **arithmetical comprehension axioms** ACA_0 is obtained from RCA_0 by replacing the Δ^0_1 comprehension with the Σ^0_1 comprehension [1].

- ACA_0 is a conservative extension of first-order arithmetic PA.

[1] Arithmetical (Σ^1_0) comprehension can be achieved by repeatedly applying the Σ^0_1 comprehension axiom to the parameters.

Lemma 1.3

RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$, that is, any theorem of $\mathsf{I}\Sigma_1$ is provable in RCA_0, and any sentence in $\mathcal{L}_{\mathrm{OR}}$ provable in RCA_0 is already provable in $\mathsf{I}\Sigma_1$.

Proof: It is obvious that any theorem of $\mathsf{I}\Sigma_1$ can be proved in RCA_0, since all axioms of $\mathsf{I}\Sigma_1$ are included in RCA_0.

To prove the converse, consider a sentence σ in $\mathcal{L}_{\mathrm{OR}}$ such that $\mathsf{I}\Sigma_1 \nvdash \sigma$. By the completeness theorem, there exists a model $\mathfrak{M} = (M, +, \cdot, 0, 1, <)$ of $\mathsf{I}\Sigma_1$ where $\mathfrak{M} \models \neg\sigma$.

For a Σ_1 formula $\varphi(x, y_1, \ldots, y_k)$, a Π_1 formula $\psi(x, y_1, \ldots, y_k)$ and $b_1, \ldots, b_k \in M$, if

$$\mathfrak{M} \models \forall x(\varphi(x, b_1, \ldots, b_k) \leftrightarrow \psi(x, b_1, \ldots, b_k))$$

holds, then we put

$$A_{\varphi,\psi,b_1,\ldots,b_k} = \{a \in M : \mathfrak{M} \models \varphi(a, b_1, \ldots, b_k)\}.$$

Otherwise, we let $A_{\varphi,\psi,b_1,\ldots,b_k} = \emptyset$. Finally, let S be the set of Δ_1 definable subsets of M,

$$S = \{A_{\varphi,\psi,b_1,\ldots,b_k} : \varphi \in \Sigma_1, \psi \in \Pi_1, \text{ and } b_1, \ldots, b_k \in M\}.$$

- To show that $(\mathfrak{M}, S) = (M \cup S, +, \cdot, 0, 1, <, \in)$ forms a model of RCA_0, it suffices to prove that any Σ^0_1 formula with set parameters from S can be rewritten as an equivalent Σ^0_1 formula without set parameters. If so, Σ^0_1 induction of $(\mathfrak{M}, S)$ can be derived from Σ_1 induction of $\mathfrak{M}$. Also, $(\mathfrak{M}, S)$ satisfies Δ^0_1 comprehension, since any set Δ^0_1 (i.e., Σ^0_1 and Π^0_1) definable with set parameters can be Δ^0_1 definable without set parameters, and so already belongs to S.

- Now, consider a Σ^0_1 formula $\theta(x, b_1, \ldots, b_k, A_{\varphi_1,\psi_1,\bar{c}}, \ldots, A_{\varphi_l,\psi_l,\bar{c}})$ with $b_i \in M$ and $A_{\varphi_j,\psi_j,\bar{c}} \in S$. In the formula, replace $t \in A_{\varphi_j,\psi_j,\bar{c}}$ with either $\varphi_i(t, \bar{c})$ or $\psi_i(t, \bar{c})$ so that the whole formula keeps in Σ^0_1. Thus, we obtain a Σ^0_1 formula $\theta'(x, b_1, \ldots, b_k, \bar{c})$, which is equivalent to $\theta(x, b_1, \ldots, b_k, A_{\varphi_1,\psi_1,\bar{c}}, \ldots, A_{\varphi_l,\psi_l,\bar{c}})$. The same for Π^0_1 formulas. Thus, $(\mathfrak{M}, S)$ is a model of RCA_0.

- Finally, since σ does not contain set variables, its truth value is independent of S, and hence $(\mathfrak{M}, S) \models \neg\sigma$. Therefore, $\mathsf{RCA}_0 + \neg\sigma$ is consistent, which implies $\mathsf{RCA}_0 \nvdash \sigma$. This completes the proof. $\square$

The various properties of $\mathrm{I}\Sigma_1$ demonstrated in Chapter 4 also hold true in RCA_0. In particular, the following fact is frequently used.

Lemma 1.4

In RCA_0, the following holds:
(1) Π^0_1 induction.
(2) The class of Σ^0_1 formulas is closed under bounded quantification.

Proof ideas. (1) Let $\varphi(x)$ be a Π^0_1 formula and assume $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1))$. By way of contradiction, we assume $\neg\varphi(c)$. Use induction for a Σ^0_1 formula $\neg\varphi(c-x)$. Then, $\neg\varphi(c-0)$ and $\neg\, \varphi(c-x) \to \neg\varphi(c-(x+1))$ imply $\neg\varphi(0)$, a contradiction.

(2) Suppose $\forall x < u \exists y \varphi(x,y)$ with $\varphi(x,y)$ bounded. Let $\psi(w)$ be a Σ^0_1 formula $\exists v \forall x < w \exists y < v \varphi(x,y) \vee u < w$. By Σ^0_1 induction, we have $\forall w \psi(w)$, in particular, $\exists v \forall x < u \exists y < v \varphi(x,y)$.

Let X, Y be sets of natural numbers. $X \subseteq Y$ is an abbreviation for $\forall n(n \in X \to n \in Y)$, and $X = Y$ is defined as $X \subseteq Y \wedge Y \subseteq X$. The equality of terms $t_1 = t_2$ is a Π^0_0 formula, but the equality of sets $X = Y$ is a Π^0_1 formula.

The **pair** of natural numbers (m, n) is coded by a natural number $\frac{(m+n)(m+n+1)}{2} + m$.

The **product** $X \times Y$ is the set of pairs (codes) of one from X and the other from Y. Thus,

$$n \in X \times Y \leftrightarrow \exists x \leq n \exists y \leq n(x \in X \wedge y \in Y \wedge (x, y) = n).$$

Since the above formula is Σ^0_0, the existence of $X \times Y$ is guaranteed in RCA_0.

A **function** $f : X \to Y$ is a subset $F \subseteq X \times Y$ such that

$$\forall x \forall y_0 \forall y_1((x, y_0) \in F \wedge (x, y_1) \in F \to y_0 = y_1) \text{ and } \forall x \in X \exists y \in Y (x, y) \in F.$$

When $(x, y) \in F$, we write $f(x) = y$. The set X of $f : X \to Y$ is called the domain of f.

A function whose domain is $\mathbb{N}$ or $\mathbb{N}^k$ is called a **total function**.

Furthermore, a function f whose domain is $X = \{i : i < n\}$ is called a **finite sequence** with **length** n. In RCA_0, a finite sequence can be coded by a natural number, and this code (Gödel number) is often identified with the sequence itself.

Lemma 1.5

In RCA_0, it is provable that the set of total functions is closed under primitive recursion.

Proof In the last semester (in Part 4), we proved that a function defined by primitive recursion is Δ_1 definable in $\mathsf{I}\Sigma_1$, thus by Δ^0_1 comprehension, it exists as a set. □

Moreover, we have

Lemma 1.6

In RCA_0, it is provable that the set of (partial) functions is closed under minimization μ.

Proof Expressing $g(x_1, \cdots, x_n) = \mu y(f(x_1, \cdots, x_n, y) = 0)$ in a formula, we have

$$((x_1, \cdots, x_n), y) \in g \Leftrightarrow ((x_1, \cdots, x_n, y), 0) \in f \wedge \forall z < y((x_1, \cdots, x_n, z), 0) \notin f.$$

The right side is a Σ^0_0 formula, so the existence of g and its totality can be shown in RCA_0. □

Note For a recursive function defined using μ-operator, if its totality is provable in RCA_0, it can be defined by primitive recursion without using μ (by Friedman's Theorem in the next chapter).

Lemma 1.7

In RCA_0, for any Σ^0_1 formula $\varphi(x)$, there exists a finite set X such that $\forall x(x \in X \leftrightarrow \varphi(x))$, or there exists a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall y(\exists x f(x) = y \leftrightarrow \varphi(y))$.

Proof

- Let $\varphi(x)$ be a Σ^0_1 formula. So, there exists a Σ^0_0 formula $\theta(x, y)$ such that $\varphi(x) \leftrightarrow \exists y \theta(x, y)$. By ($\Sigma^0_0$-CA),

$$Y = \{(x, y) : \theta(x, y) \land \forall y' < y \neg\theta(x, y')\}$$

  exists. Note that $\exists y \theta(x, y) \leftrightarrow \exists y(x, y) \in Y \leftrightarrow \exists! y(x, y) \in Y$ for all x.
- If Y is bounded, there exist u, v such that $\varphi(x) \leftrightarrow (x < u \land \exists y < v \theta(x, y))$. Then, by ($\Sigma^0_0$-CA), there exists a finite set X such that $\forall x(x \in X \leftrightarrow \varphi(x))$.
- Next, suppose that Y is unbounded. By Lemmas1.5 and 1.6, we can define a function which enumerates the elements of Y, and a function which extracts the first component x from (x, y). Combining them, we can create a one-to-one function f such that $\forall y(\exists x f(x) = y \leftrightarrow \varphi(y))$. This proves the lemma. □

Lemma 1.8

In RCA_0, the following form of set existence axiom is provable:

$$(\text{Bounded } \Sigma^0_1\text{-CA}) : \forall x \exists X \forall y (y \in X \leftrightarrow (y < x \land \varphi(y))),$$

where $\varphi(y)$ is a Σ^0_1 formula, not containing X as a free variable.

Proof For a fixed x, if there exist no finite set X such that

$$\forall y (y \in X \leftrightarrow (y < x \land \varphi(y))),$$

then by the previous lemma, there must exist a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall y (\exists z f(z) = y \leftrightarrow (y < x \land \varphi(y))),$$

which is absurd. □

Problem 1 (Strong Σ^0_1 Collection Axiom)

Prove in RCA_0: for a Σ^0_1 formula $\varphi(i, j)$ (not containing n as a free variable),

$$(\mathsf{S}\Sigma^0_1) : \forall m \exists n \forall i < m (\exists j \varphi(i, j) \to \exists j < n \varphi(i, j)).$$

Thank you for your attention!