Logic and Foundation II

Part 5. Models of first-order arithmetic

Kazuyuki Tanaka

BIMSA

March 21, 2024

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Foundations II (Spring 2024)

- Part 5. Models of first-order arithmetic (continued)
- Part 6. Real-closed ordered fields: completeness and decidability
- Part 7. Theory of reals and reverse mathematics
- Part 8. Second order arithmetic and non-standard methods

Part 5. Models of first-order arithmetic

- Jan. 04, Non-standard models and the omitting type theorem
- Jan. 11, Recursively saturated models

—

- Mar. 12, Reviews
- Mar. 14, Friedman's theorem
- Mar. 19, Friedman's theorem (continued)
- Mar. 21, Resplendency and applications
- Mar. 26, Resplendency and applications (continued)

Recap

- A **type** $\Phi(\vec{x})$ is a set of formulas in free variables $\vec{x} = (x_1, \cdots, x_n)$.
- $\mathfrak{A}$ **realizes** $\Phi(\vec{x})$ by $\vec{a}$, if $\mathfrak{A}_A \models \varphi(\vec{a})$ for all formulas $\varphi(\vec{x})$ in $\Phi(\vec{x})$.
- A type $\Phi(\vec{x})$ is a **type of a theory T** if $T \cup \Phi(\vec{c})$ ($\vec{c}$ new constants) is consistent. That is, there exists a model of T that realizes $\Phi(\vec{x})$.
- For a subset C of the universe of $\mathfrak{A}$, a **type on C in $\mathfrak{A}$** is a type of theory $\mathrm{Th}(\mathfrak{A}_C)$.
- An $\mathcal{L}$-structure $\mathfrak{A}$ is **recursively saturated**, if any recursive 1-type on $\{\vec{a}\} \subseteq A$ is realized in $\mathfrak{A}$, that is, for any recursive type $\{\varphi_i(x, \vec{x}) \mid i \in \mathbb{N}\}$ and any $\{\vec{a}\} \subseteq A$,

$$\forall j \exists a \in A \forall i < j\, \mathfrak{A}_A \models \varphi_i(a, \vec{a}) \Rightarrow \exists a \in A \forall i\, \mathfrak{A}_A \models \varphi_i(a, \vec{a}).$$

- An $\mathcal{L}$-structure $\mathfrak{A}$ is **resplendent**, if for a sentence φ in a language $\mathcal{L}^+ \supseteq \mathcal{L}_A$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ such that $\mathfrak{A}^+ \models \varphi$.

Definition

An $\mathcal{L}$-structure $\mathfrak{A}$ is **strongly resplendent**, if for any recursive type $\Phi(\vec{x})$ in a language $\mathcal{L}^+ = \mathcal{L} \cup \{\text{finitely many additional symbols}\}$ and $\vec{a} \in A^{<\omega}$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ which is a model of $\Phi(\vec{a})$.

- In the definition of **strongly resplendent**, if we restrict the type $\Phi(\vec{x})$ to be a single formula, we obtain the definition of **resplendent**, and if we let $\mathcal{L}^+ = \mathcal{L} \cup \{c\}$, it becomes the definition of **recursive saturation**. Hence, strongly resplendent structures are both resplendent and recursively saturated.
- Furthermore, similar to the case of resplendent structures, it is worth noting that the consistency of $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ coincides with the consistency of $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \Phi(\vec{a})$.
- We will now demonstrate that under certain natural assumptions, the above three properties coincide.

Theorem (Barwise-Ressayre)

Countable recursively saturated structures are strongly resplendent.

Proof

- Let $\mathfrak{A}$ be a countable recursively saturated structure in a countable language $\mathcal{L}$.
- To show that $\mathfrak{A}$ is strongly resplendent, let $\Phi(\vec{x})$ be a recursive type in a finitely extended language $\mathcal{L}^+$ of $\mathcal{L}$ and $\vec{a} \in A^\omega$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent. Then, we want to construct a model $\mathfrak{A}^+$ of $\Phi(\vec{a})$ with the same domain $|\mathfrak{A}|$.
- $\mathfrak{A}^+$ will be constructed by Henkin's method, in which Henkin constants are selected as elements of A by the recursively saturated nature of $\mathfrak{A}$.

Now, let's look into the details of construction of $\mathfrak{A}^+$.

- First, let $\{\varphi_n(x) : n \in \omega\}$ enumerate the formulas in $\mathcal{L}_A$ with only one free variable x.

- We construct a sequence of finite subsets of A and that of recursive theories in $\mathcal{L}_A^+$,

$$A_0 = \{\vec{a}\} \subseteq A_1 \subseteq A_2 \subseteq \cdots , \quad T_0 = \Phi(\vec{a}) \subseteq T_1 \subseteq T_2 \subseteq \cdots ,$$

  satisfying the following: for each n
  (1) T_n is a recursive set of sentences in $\mathcal{L}_{A_n}^+$, and $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent.
  (2) either $\varphi_n(a) \in T_{n+1}$ for some $a \in A$ or $\neg\exists x \varphi_n(x) \in T_{n+1}$.
- Now, suppose the construction is completed, and let $T_\omega = \bigcup_n T_n$.
- To show T_ω is complete, take a sentence σ in $\mathcal{L}_A^+$ such that $T_\omega \nvdash \sigma$. Then σ is φ_k (with no free variable) for some k. Obviously, $\sigma \notin T_{k+1}$, since $T_\omega \nvdash \sigma$. Thus, by condition (2), we have $\neg\exists x \sigma \in T_{k+1}$, and so $T_\omega \vdash \neg\sigma$. Therefore, T_ω is complete. Hence, we also have $\mathrm{Th}(\mathfrak{A}_A) \subseteq T_\omega$ since $T_\omega \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent by condition (1).
- If $T_\omega \vdash \exists x \varphi_n(x, \vec{a})$, then by (2), there exists some $a \in A$ such that $\varphi_n(a) \in T_\omega$.
- Then T_ω is a complete Henkin theory. By Henkin method, we can construct a structure $\mathfrak{A}^+$ over the domain A, such that $T_\omega = \mathrm{Th}(\mathfrak{A}_A^+)$, and therefore $\mathfrak{A}^+ \models \Phi(\vec{a})$.

Finally, we will construct the sequences $\{A_n\}$ and $\{T_n\}$ by induction.

- Assume that the construction up to n has been done.
- Take $\varphi_n(x)$ and let $B = A_n \cup \{$elements of A occurring in $\varphi_n(x)\}$, and define

$$\Psi(x) = \{\psi(x) : \psi(x) \text{ is a one-variable formula in } \mathcal{L}_B, \text{ and } T_n \vdash \varphi_n(x) \to \psi(x)\}.$$

- Although $\Psi(x)$ is Σ_1 as it is, it can be treated as a recursive type by Craig's method.
- Since the structure $\mathfrak{A}$ is recursively saturated, we can either find an $a \in A$ realizing $\Psi(x)$ or find a finite subset $\{\psi_i(x) : i \leq j\}$ of $\Psi(x)$ such that

$$\mathfrak{A}_A \models \neg\exists x \bigwedge_{i \leq j} \psi_i(x).$$

- In the former case, we let $A_{n+1} = B \cup \{a\}, \quad T_{n+1} = T_n \cup \{\varphi_n(a)\}$.
- To check the consistency of $T_{n+1} \cup \mathrm{Th}(\mathfrak{A}_A)$, we show that any $\mathcal{L}_{A_{n+1}}$ sentence provable in T_{n+1} is true in $\mathfrak{A}_A$. So, let $\psi(x)$ be a formula in $\mathcal{L}_B$ such that $T_{n+1} \vdash \psi(a)$. If $a \notin B$, $T_n \vdash \varphi_n(a) \to \psi(a)$ implies $T_n \vdash \varphi_n(x) \to \psi(x)$ and so $\psi(x) \in \Psi(x)$. Since a realizes $\Psi(x)$, $\psi(a)$ holds in $\mathfrak{A}_A$. On the other hand, if $a \in B$, then by $T_n \vdash \varphi_n(x) \to (x = a \to \psi(x))$, we get $(x = a \to \psi(x)) \in \Psi(x)$, which implies $(a = a \to \psi(a)) \in \mathrm{Th}(\mathfrak{A}_A)$. Thus, $\psi(a)$ holds in $\mathfrak{A}_A$.

- Next, we consider the case that $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\leq j} \psi_i(x)$. In this case, we simply set

$$A_{n+1} = A_n, \quad T_{n+1} = T_n \cup \{\neg\exists x \varphi_n(x)\}.$$

- Since $T_n \vdash \varphi_n(x) \to \bigwedge_{i\leq j} \psi_i(x)$, we have $T_n \vdash \neg\exists x \bigwedge_{i\leq j} \psi_i(x) \to \neg\exists x \varphi_n(x)$. So, to show that $T_{n+1} \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent, we may show the consistency of

$$T_n \cup \{\neg\exists x \bigwedge_{i\leq j} \psi_i(x)\} \cup \mathrm{Th}(\mathfrak{A}_A).$$

- So, take a sentence ψ in $\mathcal{L}_B$ such that $T_n \vdash \neg\exists x \bigwedge_{i\leq j} \psi_i(x) \to \psi$, and we will show ψ holds in $\mathfrak{A}_A$.
- By the induction hypothesis, $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent, so $\neg\exists x \bigwedge_{i\leq j} \psi_i(x) \to \psi$ holds in $\mathfrak{A}_A$. Moreover, we have the premise $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\leq j} \psi_i(x)$. Therefore, ψ also holds in $\mathfrak{A}_A$. This completes the proof. □

Recall Problem 5 of Lec05-02

Let $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ be a non-standard model of $\mathrm{I}\Sigma_1$. Show that $\mathfrak{A}' = (A, +, 0, 1, <)$ is recursively saturated.

Example 5

- In the above problem 5, it was shown that if $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ is a nonstandard model of $\mathrm{I}\Sigma_1$, then $\mathfrak{A}' = (A, +, 0, 1, <)$ becomes recursively saturated.
- Conversely, suppose $\mathfrak{A}' = (A, +, 0, 1, <)$ is a recursively saturated model of Presburger arithmetic and is countable. Then, by the previous theorem, $\mathfrak{A}'$ is strongly resplendent.
- On the other hand, Presburger arithmetic is complete, and the set of its theorems coincides with $\mathrm{Th}(\mathfrak{A}')$. Therefore, $\mathrm{Th}(\mathfrak{A}') \cup$ PA is nothing but PA, which is a recursive consistent set.
- Hence, there exists a suitable interpretation of $\bullet$ such that $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ becomes a model of PA. In summary, a countable model $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ of $\mathrm{I}\Sigma_1$ can be turned into a model $\mathfrak{A}' = (A, +, \bullet', 0, 1, <)$ of PA by changing the interpretation of multiplication (the "misbuttoning theorem").

Next, when $\mathcal{L}$ is finite, the equivalence of resplendency and strong resplendency can be derived from the following Kleene's theorem.

Theorem (Kleene)

Let $\mathcal{L}$ be finite, and $\Phi(\vec{v})$ be a recursive type in $\mathcal{L}$. Then, there exists a formula $\varphi(\vec{v})$ in some finite extension language $\mathcal{L}^+ \supseteq \mathcal{L}$ such that,
(1) If a structure $\mathfrak{A}^+$ in $\mathcal{L}^+$ satisfies $\varphi(\vec{a})$, then its reduct $\mathfrak{A}$ to $\mathcal{L}$ satisfies $\Phi(\vec{a})$.
(2) If an infinite structure $\mathfrak{A}$ in $\mathcal{L}$ satisfies $\Phi(\vec{a})$, then there exists an expansion $\mathfrak{A}^+$ in $\mathcal{L}^+$ that satisfies $\varphi(\vec{a})$.

Proof. The basic idea is to transform meta-mathematical arguments about $\mathcal{L}$-structures into mathematical arguments by extending the language to include $\mathsf{Q}_<$ so that recursive types of $\mathcal{L}$-structures can be represented by a single formula. In other words, we will incorporate the arithmetical structure with part of the domain.

- Let $\mathcal{L}^+$ be an extended language of $\mathcal{L}$ obtained by adding the following symbols:

$$\mathrm{N}(x),\ +,\ \bullet,\ 0,\ 1,\ <,\ \mathrm{Eval}(n,x),\ \mathrm{Sat}(n,x),\ \pi(a,i).$$

  Here, $\mathrm{N}(x)$ represents the domain of arithmetic. $\mathrm{Eval}(n,a)$ is a function to evaluate terms in $\mathcal{L}$ and $\mathrm{Sat}(n,a)$ the satisfaction relation of $\mathcal{L}$-structures, where n is the Gödel number of a term or formula, and a represents an assignment to variables. Finally, $\pi(a,i) = a_i$ is the projection function extracting the i-th component a_i from a code a intending to express an infinite sequence $(a_0, a_1, \cdots)$.
- We want to express the recursive type $\Phi(\vec{v})$ in $\mathcal{L}$ as a formula $\varphi(\vec{v})$ in $\mathcal{L}^+$, which consists of six components σ_i $(i = 1, \cdots, 6)$. Each σ_i $(i = 1, \cdots, 5)$ is a sentence, and σ_6 is a formula with free variables $\vec{v}$, and $\varphi(\vec{v})$ is defined by

$$\varphi(\vec{v}) \equiv \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_6.$$

1. σ_1 expresses the basic properties of $\mathrm{N}(x)$ as follows:

$$\mathrm{N}(0) \wedge \mathrm{N}(1) \wedge \forall x \forall y (\mathrm{N}(x) \wedge \mathrm{N}(y) \to \mathrm{N}(x+y) \wedge \mathrm{N}(x \bullet y)).$$

2. σ_2 represents $(N, +, \bullet, 0, 1, <) \models \mathsf{Q}_<$, i.e., σ_2 is the conjunction of the eight axioms of $\mathsf{Q}_<$ with quantifiers restricted to N. For example, A10 (predecessor) is expressed as

$$\forall x(\mathrm{N}(x) \to (x \neq 0 \to \exists y(\mathrm{N}(y) \land y + 1 = x))).$$

3. σ_3 is the following sentence stipulating a projection $\pi(x, i)$: for i, j ranging over N,

$$\forall x \forall i \forall z \exists y(\forall j \neq i(\pi(y, j) = \pi(x, j)) \land \pi(y, i) = z).$$

Here, y is the code of a sequence obtained by replacing the i-th element of x with z, denoted $x[z/i]$.
Note that σ_3 does not assert the existence of infinite sequences in general, but it says that finite parts can be specified arbitrarily. In fact, we will treat a finite sequence $\vec{u} = (u_0, u_1, \dots, u_{k-1})$ as $0[u_0/\overline{0}][u_1/\overline{1}] \cdots [u_{k-1}/\overline{k-1}]$, where $0 = (0, 0, 0, \dots)$.

Since any primitive recursive function over N is representable in $\mathsf{Q}_<$, Gödel numbers $\ulcorner\ \urcorner$ of terms and formulas in $\mathcal{L}$ can be handled as elements of N.

4. σ_4 describes the function $\mathrm{Eval}(n, x)$ that evaluates terms in $\mathcal{L}$. It is defined as the conjunction of the following sentences: For variables $v_0, v_1, \cdots$,

$$\forall i (\in N) \forall a (\mathrm{Eval}(\ulcorner v_i \urcorner, a) = \pi(a, i)).$$

For each m-ary function symbol $\mathtt{f}$ in $\mathcal{L}$,

$$\forall t_0, \cdots, t_{m-1} (\in N) \forall a (\mathrm{Eval}(\ulcorner \mathtt{f}(t_0, \cdots, t_{m-1}) \urcorner, a)$$

$$= \mathtt{f}(\mathrm{Eval}(\ulcorner t_0 \urcorner, a), \cdots, \mathrm{Eval}(\ulcorner t_{m-1} \urcorner, a))).$$

5. σ_5 describes the satisfaction relation $\mathrm{Sat}(n, x)$ of $\mathcal{L}$-structures. It consists of the following sentences. For each n-ary relation symbol $\mathtt{R}$ of $\mathcal{L}$ (including equality),

$$\forall t_0, \cdots, t_{n-1} \forall a (\mathrm{Sat}(\ulcorner \mathtt{R}(t_0, \cdots, t_{n-1}) \urcorner, a) \leftrightarrow \mathtt{R}(\mathrm{Eval}(\ulcorner t_0 \urcorner, a), \cdots, \mathrm{Eval}(\ulcorner t_{m-1} \urcorner, a))).$$

For each logical symbol, we have

$$\forall a (\mathrm{Sat}(\ulcorner \psi_0 \wedge \psi_1 \urcorner, a) \leftrightarrow (\mathrm{Sat}(\ulcorner \psi_0 \urcorner, a) \wedge \mathrm{Sat}(\ulcorner \psi_1 \urcorner, a))),$$

$$\forall a (\mathrm{Sat}(\ulcorner \exists x_i \psi \urcorner, a) \leftrightarrow \exists b \, \mathrm{Sat}(\ulcorner \psi \urcorner, a[b/i]))$$

and so on.

6. σ_6 is a formula expressing $\Phi(\vec{v})$ using Sat. Let $\gamma(n)$ be a formula expressing the recursive set of the Gödel numbers of formulas in $\Phi(\vec{v})$ in $\mathsf{Q}_<$, and define σ_6 as follows:

$$\forall n \in N(((N,+,\bullet,0,1,<) \models \gamma(\overline{n})) \to \mathrm{Sat}(n,\vec{v})).$$

In this way, we have defined $\varphi(\vec{x})$, and we will now verify that it satisfies the theorem.

First, to prove condition (1), suppose that in a structure $\mathfrak{A}^+$ in $\mathcal{L}^+$, $a = (a_0, \cdots, a_{l-1})$ realizes $\varphi(\vec{v})$. Take any $\psi(\vec{v})$ in $\Phi(\vec{v})$. Then, $\mathrm{Q}_< \vdash \gamma(\overline{\ulcorner \psi(\vec{v}) \urcorner})$, and by σ_2 and σ_6, we have:

$$\mathfrak{A}^+ \models \mathrm{Sat}(\ulcorner \psi \urcorner, a).$$

By meta-induction on the construction of the formula ψ, we can prove by σ_4 and σ_5 that

$$\mathfrak{A}^+ \models \mathrm{Sat}(\ulcorner \psi \urcorner, a) \leftrightarrow \psi(a_0, \cdots, a_{l-1})$$

Therefore, we have

$$\mathfrak{A}^+ \models \psi(a_0, \cdots, a_{l-1}),$$

which implies that $\psi(a_0, \cdots, a_{l-1})$ holds in its reduct $\mathfrak{A}$ to $\mathcal{L}$. Since $\psi(\vec{v}) \in \Phi(\vec{v})$ is arbitrary, $\mathfrak{A}$ realizes $\Phi(\vec{v})$ by $\vec{a}$, which proves condition (1).

Next, to prove (2), suppose conversely that an infinite structure $\mathfrak{A}$ in $\mathcal{L}$ realizes $\Phi(\vec{v})$ by $\vec{a}$.

- Choose a countably infinite subset N of $|\mathfrak{A}|$ and define $+, \bullet, 0, 1, <$ on N so that $(N, +, \bullet, 0, 1, <)$ is isomorphic to the standard structure of arithmetic. And extend $+, \bullet$ to total functions on A in an arbitrary way. Then, σ_1 and σ_2 clearly hold.
- Since A is infinite, there exists a bijection between A and $A^{<\omega}$. Let $B \subset A^\omega$ be the set of infinite sequences with all but finitely many elements being 0. Then, we can take a surjection $h : A \to B$. Now, define $\pi(a, i)$ to be the i-th element b_i of $h(a) = (b_0, b_1, \cdots)$. Then, σ_3 holds.
- Furthermore, by defining $\mathrm{Eval}(\ulcorner t \urcorner, a)$ as the value of a term t at a, and the satisfaction relation $\mathrm{Sat}(n, x)$ as

$$\mathrm{Sat}(\ulcorner \psi \urcorner, a) \Leftrightarrow \mathfrak{A} \models \psi(a_0, \cdots, a_{l-1}),$$

  we establish σ_4 and σ_5.
- Finally, for σ_6, we have:

$$(N, +, \bullet, 0, 1, <) \models \gamma(\overline{\ulcorner \psi \urcorner}) \Leftrightarrow \psi(\vec{v}) \in \Phi(\vec{v}) \Rightarrow \psi(a_0, \cdots, a_{l-1}) \Leftrightarrow \mathrm{Sat}(\overline{\ulcorner \psi \urcorner}, a)).$$

Thus, $\varphi(\vec{a})$ holds in $\mathfrak{A}^+$, and so condition (2) is satisfied. □

Corollary (Barwise)

A resplendent structure in a finite language $\mathcal{L}$ is strongly resplendent, and so recursively saturated.

Proof.

- Let $\mathcal{L}$ be a finite language, and $\mathfrak{A}$ be a resplendent structure in $\mathcal{L}$. If $\mathfrak{A}$ is finite, then it is already recursively saturated and so strongly resplendent (by Barwise-Ressayre). Thus, we may assume that $\mathfrak{A}$ is infinite.
- To show that $\mathfrak{A}$ is strongly resplendent, suppose a recursive type $\Phi(\vec{v})$ in $\mathcal{L}'(\supset \mathcal{L})$ is given so that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent.
- Then, we can construct $\varphi(\vec{v})$ in $\mathcal{L}'^{+}$ to satisfy Kleene's Theorem.
- Let $\mathfrak{A}'$ be an $\mathcal{L}'$-expansion of an elementary extension of $\mathfrak{A}$ which satisfies $\Phi(\vec{a})$. Then, by Kleene's Theorem (2), $\mathfrak{A}'$ has an $\mathcal{L}'^{+}$-expansion $\mathfrak{A}'^{+}$ which satisfies $\varphi(\vec{a})$. Thus by the resplendency of $\mathfrak{A}$, $\mathfrak{A}$ also has an $\mathcal{L}'^{+}$-expansion which satisfies $\varphi(\vec{a})$.
- Finally, by Kleene's Theorem (1), $\Phi(\vec{a})$ holds in $\mathfrak{A}$. This proves that $\mathfrak{A}$ is strongly resplendent. □

Let us consider Kleene's Theorem for an arithmetic structure $\mathfrak{A}$.

- If $\mathcal{L}$ already includes the language of arithmetic $\mathcal{L}_{\mathrm{OR}}$, and a $\mathcal{L}$-structure $\mathfrak{A}$ is already a model of $\mathrm{Q}_{<}$, there is no need to introduce $+$, $\bullet$, 0, 1, $<$, $\mathrm{Eval}(n,x)$, $\pi(x,i)$ separately. To prove Kleene's theorem, it suffices to use $\mathrm{N}(x)$ and $\mathrm{Sat}(n,x)$.
- If $\mathfrak{A}$ is resplendent, we can introduce $\mathrm{N}(x)$ and $\mathrm{Sat}(n,x)$ as relations in $\mathfrak{A}$. Then, we can derive various properties of $\mathfrak{A}$.

Theorem

For any countable resplendent model $\mathfrak{A}$ of Peano Arithmetic PA, there exists a (proper) initial segment that is isomorphic to $\mathfrak{A}$, and $\mathfrak{A}$ is an elementary extension of this initial segment.

Proof. To the language of arithmetic $\mathcal{L}_{\mathrm{OR}}$, add $\mathrm{N}(x)$, $\mathrm{Sat}(n,x)$, as well as $\mathrm{Sat}_N(n,x)$ to represent the satisfaction relation for N, and $\mathtt{f}(x)$ to represent an isomorphism between the whole structure and its restriction to N.
Now, consider a recursive type claiming that N is an initial segment isomorphic to the whole $\mathfrak{A}$, and is also an elementary substructure. This type is consistent with Th$(\mathfrak{A}_A)$ by Friedman's theorem. By resplendency, N can be realized as an initial segment of $\mathfrak{A}$. □

Theorem

For a resplendent model $\mathfrak{A}$ of Peano Arithmetic PA, there exists a satisfaction relation Sat, such that for any $\mathcal{L}_{\mathrm{OR}}$ formula ψ,

$$(\mathfrak{A}, Sat) \models \forall a(\mathrm{Sat}(\ulcorner\psi\urcorner, a) \leftrightarrow \psi(a_0, \cdots, a_{l-1}))$$

and $(\mathfrak{A}, Sat)$ satisfies induction for formulas in $\mathcal{L}_{\mathrm{OR}} \cup \{\mathrm{Sat}\}$. Conversely, if a model $\mathfrak{A}$ of Peano Arithmetic PA has such a relation Sat, then $\mathfrak{A}$ is recursively saturated. Hence, if countable, it is resplendent.

Proof. The existence of Sat follows from the resplendency and the definition of Sat in Kleene's theorem. To show that $(\mathfrak{A}, Sat)$ satisfies induction, it is enough to see that the recursive set of sentences representing the induction for $\mathcal{L}_{\mathrm{OR}} \cup \{\mathrm{Sat}\}$ is consistent with $\mathrm{Th}(\mathfrak{A}_A)$. The second part is obvious from the proof of the following lemma.

Lemma (revisit)

For each $n > 0$, a non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$ is Σ_n-recursively saturated.

□

Theorem (Robinson's Joint Consistency Theorem)

Let $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$, and T be a complete theory in the language $\mathcal{L}$, with T_1 and T_2 as extensions of T in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then, $T_1 \cup T_2$ is consistent if and only if T_1 and T_2 are separately consistent.

Proof. The necessity is clear, so we will prove the sufficiency. Assume T_1 and T_2 are consistent, but $T_1 \cup T_2$ is inconsistent.

- Since $T_1 \cup T_2$ is inconsistent, there exist finite subsets $S_1 \subseteq T_1$ and $S_2 \subseteq T_2$ such that $S_1 \cup S_2$ also leads to a contradiction.
- Suppose S_1 and S_2 are theories in finite languages $\mathcal{L}'_1$ and $\mathcal{L}'_2$, respectively. Define $\mathcal{L}' = \mathcal{L}'_1 \cap \mathcal{L}'_2$, and let T' be the set of $\mathcal{L}'$-sentences that can be deduced from T. Then, T' is a complete and consistent set in the language $\mathcal{L}'$, since T is a complete and consistent set in $\mathcal{L}$
- Moreover, let $S'_1 = S_1 \cup T'$ and $S'_2 = S_2 \cup T'$. Since S'_1 and S'_2 are subsets of T_1 and T_2, respectively, they are separately consistent.

- Consider a countable saturated model $\mathfrak{A}$ of T'. Since T' is complete, $T' = \mathrm{Th}(\mathfrak{A})$.
- Since $S_1' = S_1 \cup \mathrm{Th}(\mathfrak{A})$ is consistent, by resplendency of $\mathfrak{A}$, $\mathfrak{A}$ can be extended to a model $\mathfrak{A}_1$ of S_1 in $\mathcal{L}_1'$.
- Similarly, $\mathfrak{A}$ can be extended to a model $\mathfrak{A}_2$ of S_2 in $\mathcal{L}_2'$. Therefore, by defining the interpretation of symbols in $\mathcal{L}_1' - \mathcal{L}'$ to be the same as in $\mathfrak{A}_1$ and in $\mathcal{L}_2' - \mathcal{L}'$ to be the same as in $\mathfrak{A}_2$, we extend $\mathfrak{A}$ to a structure $\mathfrak{A}'$ in $\mathcal{L}_1' \cup \mathcal{L}_2'$.
- Then, $\mathfrak{A}'$ is a model of $S_1 \cup S_2$, which contradicts our assumption. Thus, we complete the proof.

$\square$

Corollary (Craig's Interpolation Theorem)

If a formula $\varphi \to \psi$ is provable ($\vdash \varphi \to \psi$), then there exists a formula θ consisting of mathematical symbols appearing in φ and ψ commonly, besides logical symbols and $=$, such that $\vdash \varphi \to \theta$ and $\vdash \theta \to \psi$.

The formula θ satisfying the above theorem is called an **interpolant** for φ and ψ.

Proof

- Assume $\vdash \varphi \to \psi$ with no interpolant θ. Let $\mathcal{L}$ be the language consisting of symbols common to φ and ψ. Let T_0 be the set of formulas ξ in $\mathcal{L}$ such that $\vdash \varphi \to \xi$.
- Since no finite subset of T_0 implies ψ, $T_0 \cup \{\neg\psi\}$ is consistent.
- Consider a model $\mathfrak{A}$ of $T_0 \cup \{\neg\psi\}$, and let T be the set of all $\mathcal{L}$ formulas contained in $\mathrm{Th}(\mathfrak{A})$. Clearly, $T \cup \{\neg\psi\}$ is consistent.
- To show that $T \cup \{\varphi\}$ is also consistent, assume otherwise. Then there exists a formula σ in T such that $\vdash \varphi \to \neg\sigma$. Thus, $\neg\sigma \in T_0 \subseteq T$, which implies the inconsistency of T.
- By Robinson's joint consistency theorem, $T \cup \{\varphi, \neg\psi\}$ is also consistent, contradicting the assumption $\vdash \varphi \to \psi$. □

Thank you for your attention!