Logic and Foundation II

Part 5. Models of first-order arithmetic

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Theorem (Friedman's self-embedding theorem)

Let $n > 0$, $\mathfrak{A}$ be a countable non-standard model of $\mathrm{I}\Sigma_n$, and take $a \in A$ arbitrarily. Then there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $a \in A'$ but $A' \subsetneq A$, and $\mathfrak{A} \cong \mathfrak{A}'$ and for any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{a'} \in A'^{<\omega}$,

$$\mathfrak{A}'_{A'} \models \varphi(\vec{a'}) \Leftrightarrow \mathfrak{A}_{A'} \models \varphi(\vec{a'}).$$

- The essence of this theorem is that a countable non-standard model of $\mathrm{I}\Sigma_1$ has an initial segment that is isomorphic to itself.
- Friedman first proved this theorem for a countable non-standard model of Peano arithmetic, and several researchers sophisticated it to the above form.
- The same theorem does not hold for non-countable models, and also it does not hold in general for countable non-standard models of $\mathrm{I}\Sigma_0$.
- Furthermore, an important result related to this is McAloon's theorem, which states that a countable non-standard model of $\mathrm{I}\Sigma_0$ has an initial segment that is a model of Peano arithmetic PA.

Introduction to Resplendency

- Recursive saturation of a structure means that it contains many "elements" that satisfy recursive conditions, but by generalizing this property to relations and functions, we introduce a new concept.
- By saying that a structure $\mathfrak{A}$ in the language $\mathcal{L}$ has "resplendency", we mean that if a formula $\varphi(\vec{\mathrm{R}})$ with new relation symbols $\vec{\mathrm{R}} \notin \mathcal{L}$ consistent with $\mathrm{Th}(\mathfrak{A}_A)$, $\varphi(\vec{\mathrm{R}})$ can hold in $\mathfrak{A}$ by appropriate interpretation of $\vec{\mathrm{R}}$.
- In a resplendent model of arithmetic, hidden properties of the structure can be found by using new relation symbols for an initial segment and satisfaction relation.

Definition

The $\mathcal{L}$-structure $\mathfrak{A}$ is said to be **resplendent**, if for a sentence φ in a language $\mathcal{L}^+ \supseteq \mathcal{L}_A$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ such that $\mathfrak{A}^+ \models \varphi$.

- The statement that $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is consistent is equivalent to that φ is true in the $\mathcal{L}^+$-extension of an elementary extension of $\mathfrak{A}$.

  In other words, resplendent structures are considered to potentially possess the properties of relations and functions manifested in their elementary extensions.

- The consistency of $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is that of $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \varphi$ where $\vec{a}$ denotes the elements of A contained in φ.

  $\because$ Suppose $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is inconsistent. Then there exists a formula $\psi(\vec{a}, \vec{b})$ in $\mathrm{Th}(\mathfrak{A}_A)$ such that $\vdash \psi(\vec{a}, \vec{b}) \to \neg\varphi$. Thus we also have $\vdash \exists y \psi(\vec{a}, \vec{y}) \to \neg\varphi$. Since $\exists y \psi(\vec{a}, \vec{y}) \in \mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}})$, it follows that $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \{\varphi\}$ is inconsistent. The reverse implication is trivial.

- Every finite structure is resplendent because its elementary extension is only itself.

Since "resplendency" does not imply "recursive saturation" in general, we introduce the following stronger notion which implies both.

Definition

An $\mathcal{L}$-structure $\mathfrak{A}$ is **strongly resplendent**, if for any recursive type $\Phi(\vec{x})$ in a language $\mathcal{L}^+ = \mathcal{L} \cup \{\text{finitely many additional symbols}\}$ and $\vec{a} \in A^{<\omega}$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ which is a model of $\Phi(\vec{a})$.

- In the definition of **strongly resplendent**, if we restrict the type $\Phi(\vec{x})$ to be a single formula, we obtain the definition of **resplendent**, and if we let $\mathcal{L}^+ = \mathcal{L} \cup \{c\}$, it becomes the definition of **recursive saturation**. Hence, strongly resplendent structures are both resplendent and recursively saturated.
- Furthermore, similar to the case of resplendent structures, it is worth noting that the consistency of $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ coincides with the consistency of $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \Phi(\vec{a})$.

We will now demonstrate that under certain natural assumptions, the above three properties coincide.

Theorem (Barwise-Ressayre)

Countable recursively saturated structures are strongly resplendent.

Proof

- Let $\mathfrak{A}$ be a countable structure in a countable language $\mathcal{L}$ and assume it is recursively saturated. Furthermore, suppose we are given a recursive type $\Phi(\vec{x})$ in a finitely extended language $\mathcal{L}^+$ of $\mathcal{L}$ and $\vec{a} \in A^\omega$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent.
- Then, we want to construct a model $\mathfrak{A}^+$ of this theory without expanding the domain $|\mathfrak{A}|$. The key idea of the construction is that by utilizing the recursively saturated nature of $\mathfrak{A}$, we can select Henkin constants from elements of A.

Now, let's look into the details of construction of $\mathfrak{A}^+$.

- First, we enumerate the formulas in $\mathcal{L}_A$ with only one free variable x, denoted by $\{\varphi_n(x) : n \in \omega\}$.

- We construct a sequence of finite subsets of A and that of recursive theories in $\mathcal{L}_A^+$,

$$A_0 = \{\vec{a}\} \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad T_0 = \Phi(\vec{a}) \subseteq T_1 \subseteq T_2 \subseteq \cdots,$$

  satisfying the following conditions: for each n
  (1) T_n is a recursive set of sentences in $\mathcal{L}_{A_n}^+$, and $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent.
  (2) either $\varphi_n(a) \in T_{n+1}$ for some $a \in A$ or $\neg\exists x \varphi_n(x) \in T_{n+1}$.
- Once the construction is completed, letting $T_\omega = \bigcup_n T_n$, we will show T_ω is a complete Henkin theory.
- Let σ be a sentence in $\mathcal{L}_A^+$ such that $T_\omega \nvdash \sigma$. Suppose σ is φ_k (with no free variable) for some k. Then we have $\sigma \notin T_{k+1}$, since $T_\omega \nvdash \sigma$. Thus, by condition (2), we have $\neg\exists x \sigma \in T_{k+1}$, and so $T_\omega \vdash \neg\sigma$. Therefore, T_ω is complete, and so $\mathrm{Th}(\mathfrak{A}_A) \subseteq T_\omega$ since $T_\omega \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent by condition (1).
- If $T_\omega \vdash \exists x \varphi_n(x, \vec{a})$, then by (2), there exists some $a \in A$ such that $\varphi_n(a) \in T_\omega$.
- Then T_ω is a complete Henkin theory. By Henkin method, we can construct a structure $\mathfrak{A}^+$ over the domain A, such that $T_\omega = \mathrm{Th}(\mathfrak{A}_A^+)$, and therefore $\mathfrak{A}^+ \models \Phi(\vec{a})$.

Finally, we will construct the sequences $\{A_n\}$ and $\{T_n\}$ by induction.

- Assuming that the constructions up to A_n and T_n have been done. Take $\varphi_n(x)$.
- Let $B = A_n \cup \{$elements of A occurring in $\varphi_n(x)\}$, and define

$$\Psi(x) = \{\psi(x) : \psi(x) \text{ is a one-variable formula in } \mathcal{L}_B, \text{ and } T_n \vdash \varphi_n(x) \to \psi(x)\}.$$

- Although $\Psi(x)$ is Σ_1 as it is, it can be treated as a recursive type by Craig's method.
- Since the structure $\mathfrak{A}$ is recursively saturated, we can either find an $a \in A$ realizing $\Psi(x)$ or find a finite subset $\{\psi_i(x) : i \le j\}$ of $\Psi(x)$ such that

$$\mathfrak{A}_A \models \neg \exists x \bigwedge_{i \le j} \psi_i(x).$$

- In the former case, we let $A_{n+1} = B \cup \{a\}, \quad T_{n+1} = T_n \cup \{\varphi_n(a)\}$.
- To check the consistency of $T_{n+1} \cup \mathrm{Th}(\mathfrak{A}_A)$, we show that any $\mathcal{L}_{A_{n+1}}$ sentence provable in T_{n+1} is true in $\mathfrak{A}_A$. So, let $\psi(x)$ be a formula in $\mathcal{L}_B$ such that $T_{n+1} \vdash \psi(a)$. If $a \notin B$, $T_n \vdash \varphi_n(a) \to \psi(a)$ implies $T_n \vdash \varphi_n(x) \to \psi(x)$ and so $\psi(x) \in \Psi(x)$. Since a realizes $\Psi(x)$, $\psi(a)$ holds in $\mathfrak{A}_A$. On the other hand, if $a \in B$, then by $T_n \vdash \varphi_n(x) \to (x = a \to \psi(x))$, we get $(x = a \to \psi(x)) \in \Psi(x)$, which implies $(a = a \to \psi(a)) \in \mathrm{Th}(\mathfrak{A}_A)$. Thus, $\psi(a)$ holds in $\mathfrak{A}_A$.

- Next, we consider the case that $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\le j} \psi_i(x)$. In this case, we can simply set

$$A_{n+1} = A_n, \quad T_{n+1} = T_n \cup \{\neg\exists x \varphi_n(x)\}.$$

- Since $T_n \vdash \neg\exists x \bigwedge_{i\le j} \psi_i(x) \to \neg\exists x \varphi_n(x)$, we may show the consistency of

$$T_n \cup \{\neg\exists x \bigwedge_{i\le j} \psi_i(x)\} \cup \mathrm{Th}(\mathfrak{A}_A).$$

- Let ψ be a sentence in $\mathcal{L}_B$ such that $T_n \vdash \neg\exists x \bigwedge_{i\le j} \psi_i(x) \to \psi$. By the induction hypothesis, $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent, so $\neg\exists x \bigwedge_{i\le j} \psi_i(x) \to \psi$ holds in $\mathfrak{A}_A$.
- Moreover, since we have the premise $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\le j} \psi_i(x)$, it follows that ψ also holds in $\mathfrak{A}_A$. This completes the proof. □

Recall Problem 5 of Lec05-02

Let $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ be a non-standard model of $\mathrm{I}\Sigma_1$. Show that $\mathfrak{A}' = (A, +, 0, 1, <)$ is recursively saturated.

Example 5

- In the above problem 5, it was shown that if $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ is a nonstandard model of $\mathrm{I}\Sigma_1$, then $\mathfrak{A}' = (A, +, 0, 1, <)$ becomes recursively saturated.
- Conversely, suppose $\mathfrak{A}' = (A, +, 0, 1, <)$ is a recursively saturated model of Presburger arithmetic and is countable. Then, by the previous theorem, $\mathfrak{A}'$ is strongly resplendent.
- On the other hand, Presburger arithmetic is complete, and the set of its theorems coincides with $\mathrm{Th}(\mathfrak{A}')$. Therefore, $\mathrm{Th}(\mathfrak{A}') \cup$ PA is nothing but PA, which is a recursive consistent set.
- Hence, there exists a suitable interpretation of $\bullet$ such that $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ becomes a model of PA. In summary, a countable model $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ of $\mathrm{I}\Sigma_1$ can be turned into a model $\mathfrak{A}' = (A, +, \bullet', 0, 1, <)$ of PA by changing the interpretation of multiplication (the "misbuttoning theorem").

Now, when $\mathcal{L}$ is finite, the equivalence of resplendency and strong resplendency can be derived from the following Kleene's theorem.

Theorem (Kleene)

Let $\mathcal{L}$ be finite, and $\Phi(\vec{v})$ be its recursive type. Then, there exists a formula $\varphi(\vec{v})$ in some finite extension language $\mathcal{L}^+ \supseteq \mathcal{L}$ such that,
(1) If a structure $\mathfrak{A}^+$ in $\mathcal{L}^+$ satisfies $\varphi(\vec{a})$, then its reduct $\mathfrak{A}$ to $\mathcal{L}$ satisfies $\Phi(\vec{a})$.
(2) If an infinite structure $\mathfrak{A}$ in $\mathcal{L}$ satisfies $\Phi(\vec{a})$, then there exists an expansion $\mathfrak{A}^+$ in $\mathcal{L}^+$ that satisfies $\varphi(\vec{a})$.

In part 4 of last semester, we show that in weak arithmetic such as $\mathsf{Q}_<$ (or Q), all recursive sets are representable, and hence ample meta-mathematical arguments of arithmetic can be developed. Here, we aim to formalize meta-mathematics of general $\mathcal{L}$-structures, and this can also be done in $\mathsf{Q}_<$, so by extending the language to include $\mathsf{Q}_<$, recursive types of $\mathcal{L}$-structures can be represented by a single formula.

Proof. The basic idea is to transform meta-mathematical arguments about $\mathcal{L}$-structures into mathematical (object-language) arguments by utilizing the language of $\mathsf{Q}_<$. The crucial point is that instead of creating the natural numbers outside of $\mathcal{L}$-structures, we will incorporate the arithmetical structure with part of the domain.

- Let $\mathcal{L}^+$ be an extended language of $\mathcal{L}$ obtained by adding the following symbols:

$$\mathrm{N}(x),\ +,\ \bullet,\ 0,\ 1,\ <,\ \mathrm{Eval}(n,x),\ \mathrm{Sat}(n,x),\ \pi(x,i).$$

  Here, $\mathrm{N}(x)$ represents the domain of arithmetic, $\mathrm{Eval}(n,x)$ is a function to evaluate terms in $\mathcal{L}$, $\mathrm{Sat}(n,x)$ the satisfaction relation of $\mathcal{L}$-structures, and $\pi(x,i) = x_i$ the projection function extracting the i-th component x_i from the code x of an infinite sequence $(x_0, x_1, \cdots)$.
- We want to express the recursive type $\Phi(\vec{v})$ in $\mathcal{L}^+$ as a formula $\varphi(\vec{v})$, which we will define in six components σ_i $(i = 1, \cdots, 6)$. Each σ_i $(i = 1, \cdots, 5)$ is a sentence, and σ_6 is a formula with free variables $\vec{v}$, and $\varphi(\vec{v})$ is defined by

$$\varphi(\vec{v}) \equiv \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_6.$$

1. σ_1 expresses the basic properties of $\mathrm{N}(x)$ as follows:

$$\mathrm{N}(0) \wedge \mathrm{N}(1) \wedge \forall x \forall y (\mathrm{N}(x) \wedge \mathrm{N}(y) \to \mathrm{N}(x+y) \wedge \mathrm{N}(x \bullet y)).$$

2. σ_2 represents $(N, +, \bullet, 0, 1, <) \models \mathsf{Q}_<$, i.e., σ_2 is the conjunction of the eight axioms of $\mathsf{Q}_<$ with quantifiers restricted to N. For example, A10 (predecessor) is expressed as

$$\forall x(\mathrm{N}(x) \to (x \neq 0 \to \exists y(\mathrm{N}(y) \land y + 1 = x))).$$

   Since all primitive recursive functions over N are representable in $\mathsf{Q}_<$, Gödel numbers of terms and formulas in $\mathcal{L}$ can be handled as elements of N.

3. σ_3 is the following sentence which stipulates a projection function $\pi(x, i)$: assuming variables i, j ranges over N for simplicity,

$$\forall x \forall i \forall z \exists y(\forall j \neq i(\pi(y, j) = \pi(x, j)) \land \pi(y, i) = z).$$

   Here, y is the code of a sequence obtained by replacing the i-th element of $x = (x_0, x_1, \cdots)$ with z. We write this y as $x[z/i]$. Note that σ_3 does not assert the existence of infinite sequences in general, but it says that finite parts can be specified arbitrarily.
   In fact, we will treat an infinite sequence as a finite sequence followed by infinitely many 0's. More strictly, letting $0 = (0, 0, 0, \dots)$, $0[u_0/\overline{0}][u_1/\overline{1}] \cdots [u_{k-1}/\overline{k-1}]$ denotes $\vec{u} = (u_0, u_1, \dots, u_{k-1})$.

4. σ_4 describes the function $\mathrm{Eval}(n, x)$ that evaluates terms in $\mathcal{L}$. It is defined as the conjunction of the following sentences: For variables $v_0, v_1, \cdots$,

$$\forall i(\in N)\forall a(\mathrm{Eval}(\ulcorner v_i \urcorner, a) = \pi(a, i)).$$

For each m-ary function symbol $\mathtt{f}$ in $\mathcal{L}$,

$$\forall t_0, \cdots, t_{m-1}(\in N)\forall a(\mathrm{Eval}(\ulcorner \mathtt{f}(t_0, \cdots, t_{m-1}) \urcorner, a)$$

$$= \mathtt{f}(\mathrm{Eval}(\ulcorner t_0 \urcorner, a), \cdots, \mathrm{Eval}(\ulcorner t_{m-1} \urcorner, a))).$$

5. σ_5 describes the satisfaction relation $\mathrm{Sat}(n, x)$ of $\mathcal{L}$-structures. It consists of the following sentences. For each n-ary relation symbol $\mathtt{R}$ of $\mathcal{L}$ (including equality), we have

$$\forall t_0, \cdots, t_{n-1}\forall a(\mathrm{Sat}(\ulcorner \mathtt{R}(t_0, \cdots, t_{n-1}) \urcorner, a) \leftrightarrow \mathtt{R}(\mathrm{Eval}(\ulcorner t_0 \urcorner, a), \cdots, \mathrm{Eval}(\ulcorner t_{m-1} \urcorner, a))).$$

For each logical symbol, we have

$$\forall a(\mathrm{Sat}(\ulcorner \psi_0 \land \psi_1 \urcorner, a) \leftrightarrow (\mathrm{Sat}(\ulcorner \psi_0 \urcorner, a) \land \mathrm{Sat}(\ulcorner \psi_1 \urcorner, a))),$$

$$\forall a(\mathrm{Sat}(\ulcorner \exists x_i \psi \urcorner, a) \leftrightarrow \exists b\, \mathrm{Sat}(\ulcorner \psi \urcorner, a[b/i]))$$

and so on.

6. σ_6 is a formula expressing $\Phi(\vec{v})$ using Sat. For a recursive type $\Phi(\vec{v})$, let $\gamma(n)$ be a formula expressing the set of Gödel numbers of $\Phi(\vec{v})$ in $\mathsf{Q}_<$, and define σ_6 as follows:

$$\forall n \in N(((N, +, \bullet, 0, 1, <) \models \gamma(\overline{n})) \to \mathrm{Sat}(n, \vec{v})).$$

In this way, we have obtained $\varphi(\vec{x})$, and we will now verify that it satisfies the conditions of the theorem. First, to prove condition (1), suppose that in a structure $\mathfrak{A}^+$ in $\mathcal{L}^+$, $a = (a_0, \cdots, a_{l-1})$ realizes $\varphi(\vec{v})$. Let $\mathfrak{A}$ be its reduct to $\mathcal{L}$. For each $\psi(\vec{v})$ in $\Phi(\vec{v})$, we have $\mathsf{Q}_< \vdash \gamma(\ulcorner \psi(\vec{v}) \urcorner)$, and then by σ_2 and σ_6, we have:

$$\mathfrak{A}^+ \models \mathrm{Sat}(\ulcorner \psi \urcorner, a)$$

Furthermore, by meta-induction on the construction of the formula ψ, we can prove by σ_4 and σ_5 that

$$\mathfrak{A}^+ \models \mathrm{Sat}(\ulcorner \psi \urcorner, a) \leftrightarrow \psi(a_0, \cdots, a_{l-1})$$

Therefore, we have

$$\mathfrak{A}^+ \models \psi(a_0, \cdots, a_{l-1}),$$

which implies that $\psi(a_0, \cdots, a_{l-1})$ holds in $\mathfrak{A}$. Since $\psi(\vec{v}) \in \Phi(\vec{v})$ is arbitrary, $\mathfrak{A}$ realizes $\Phi(\vec{v})$ by $\vec{a}$, which proves condition (1).

Next, to prove (2), suppose conversely that an infinite structure $\mathfrak{A}$ in $\mathcal{L}$ realizes $\Phi(\vec{v})$ by $\vec{a}$.

- Choose a countably infinite subset N of $|\mathfrak{A}|$ and define $+, \bullet, 0, 1, <$ on N so that $(N, +, \bullet, 0, 1, <)$ is isomorphic to the standard structure of arithmetic. And extend $+, \bullet$ to total functions on A in an arbitrary way. Then, σ_1 and σ_2 clearly hold.
- Since A is infinite, there exists a bijection between A and $A^{<\omega}$. Let $B \subset A^\omega$ be the set of infinite sequences with all but finitely many elements being 0. Then, we can take a surjection $h : A \to B$. Now, define $\pi(a, i)$ to be the i-th element b_i of $h(a) = (b_0, b_1, \cdots)$. Then, σ_3 holds.
- Furthermore, by defining $\mathrm{Eval}(\ulcorner t \urcorner, a)$ as the value of a term t at a, and the satisfaction relation $\mathrm{Sat}(n, x)$ as

$$\mathrm{Sat}(\ulcorner \psi \urcorner, a) \Leftrightarrow \mathfrak{A} \models \psi(a_0, \cdots, a_{l-1}),$$

  we establish σ_4 and σ_5.
- Finally, for σ_6, we have:

$$(N, +, \bullet, 0, 1, <) \models \gamma(\overline{\ulcorner \psi \urcorner}) \Leftrightarrow \psi(\vec{v}) \in \Phi(\vec{v}) \Leftrightarrow \psi(a_0, \cdots, a_{l-1}) \Leftrightarrow \mathrm{Sat}(\overline{\ulcorner \psi \urcorner}, a)).$$

Thus, condition (2) is also satisfied. □

Corollary (Barwise)

A resplendent structure in a finite language $\mathcal{L}$ is strongly resplendent, and so recursively saturated.

Proof.

- Let $\mathcal{L}$ be a finite language, and $\mathfrak{A}$ be a resplendent structure in $\mathcal{L}$. If $\mathfrak{A}$ is finite, then it is already recursively saturated and so strongly resplendent (by Barwise-Ressayre). Thus, we may assume that $\mathfrak{A}$ is infinite.
- To show that $\mathfrak{A}$ is strongly resplendent, suppose a recursive type $\Phi(\vec{v})$ in $\mathcal{L}'(\supset \mathcal{L})$ is given so that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent.
- Then, we can construct $\varphi(\vec{v})$ in $\mathcal{L}'^{+}$ to satisfy Kleene's Theorem.
- Let $\mathfrak{A}'$ be an $\mathcal{L}'$-expansion of an elementary extension of $\mathfrak{A}$ which satisfies $\Phi(\vec{a})$. Then, by Kleene's Theorem (2), $\mathfrak{A}'$ has an $\mathcal{L}'^{+}$-expansion $\mathfrak{A}'^{+}$ which satisfies $\phi(\vec{a})$. Thus by the resplendency of $\mathfrak{A}$, $\mathfrak{A}$ also has an $\mathcal{L}'^{+}$-expansion which satisfies $\phi(\vec{a})$.
- Finally, by Kleene's Theorem (1), $\Phi(\vec{a})$ holds in $\mathfrak{A}$. This proves that $\mathfrak{A}$ is strongly resplendent. □

Next we consider Kleene's Theorem for an arithmetic structure $\mathfrak{A}$.

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- If $\mathcal{L}$ already includes the language of arithmetic $\mathcal{L}_{\mathrm{OR}}$, and a $\mathcal{L}$-structure $\mathfrak{A}$ is an expansion of a model of $\mathrm{Q}_{<}$, there is no need to introduce $+$, $\bullet$, 0, 1, $<$, $\mathrm{Eval}(n, x)$, $\pi(x, i)$ separately. To prove Kleene's theorem, it suffices to use $\mathrm{N}(x)$ and $\mathrm{Sat}(n, x)$.
- If $\mathfrak{A}$ is resplendent, we can introduce $\mathrm{N}(x)$ and $\mathrm{Sat}(n, x)$ as relations in $\mathfrak{A}$, and then we can derive various properties of $\mathfrak{A}$ by adding various conditions to them.
- We start with a representative application.

Theorem

For any countable resplendent model $\mathfrak{A}$ of Peano Arithmetic PA, there exists a (proper) initial segment that is isomorphic to $\mathfrak{A}$, and $\mathfrak{A}$ is an elementary extension of this initial segment.

Proof. To the language of arithmetic $\mathcal{L}_{\mathrm{OR}}$, add $\mathrm{N}(x)$, $\mathrm{Sat}(n, x)$, as well as $\mathrm{Sat}_N(n, x)$ to represent the satisfaction relation for N, and $\mathtt{f}(x)$ to represent an isomorphism.
Now, consider a recursive type claiming that N is an initial segment isomorphic to the whole $\mathfrak{A}$, and is also an elementary substructure. This type is consistent with Th($\mathfrak{A}_A$) by Friedman's theorem. By resplendency, N can be realized as an initial segment of $\mathfrak{A}$. □

Theorem

For a resplendent model $\mathfrak{A}$ of Peano Arithmetic PA, there exists a satisfaction relation Sat, such that for any $\mathcal{L}_{\mathrm{OR}}$ formula ψ,

$$(\mathfrak{A}, Sat) \models \forall a(\mathrm{Sat}(\ulcorner\psi\urcorner, a) \leftrightarrow \psi(a_0, \cdots, a_{l-1}))$$

and $(\mathfrak{A}, Sat)$ satisfies induction for formulas in $\mathcal{L}_{\mathrm{OR}} \cup \{\mathrm{Sat}\}$. Conversely, if a model $\mathfrak{A}$ of Peano Arithmetic PA has such a relation Sat, then $\mathfrak{A}$ is recursively saturated, and hence, if countable, it is resplendent.

Proof. The existence of Sat follows from the resplendency and the definition of Sat in Kleene's theorem. To show that $(\mathfrak{A}, Sat)$ satisfies induction, it is enough to see that the recursive set of sentences representing the induction for $\mathcal{L}_{\mathrm{OR}} \cup \{\mathrm{Sat}\}$ is consistent with $\mathrm{Th}(\mathfrak{A}_A)$. The second part is obvious from the following lemma.

Lemma (revisit)

For each $n > 0$, a non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$ is **Σ_n-recursively saturated**.

□

Theorem (Robinson's Joint Consistency Theorem)

Let $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$, and let T be a complete theory in the language $\mathcal{L}$, with T_1 and T_2 being extensions of T in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then, the necessary and sufficient condition for $T_1 \cup T_2$ to be consistent is that T_1 and T_2 are separately consistent.

Proof. The necessity is clear, so we will prove the sufficiency. Assume T_1 and T_2 are consistent, but $T_1 \cup T_2$ is inconsistent.

- Since $T_1 \cup T_2$ is inconsistent, there exist finite subsets $S_1 \subseteq T_1$ and $S_2 \subseteq T_2$ such that $S_1 \cup S_2$ also leads to a contradiction.
- Suppose S_1 and S_2 are theories in finite languages $\mathcal{L}'_1$ and $\mathcal{L}'_2$, respectively. Define $\mathcal{L}' = \mathcal{L}'_1 \cap \mathcal{L}'_2$, and let T' be the set of $\mathcal{L}'$-sentences that can be deduced from T. Then, T' is a complete and consistent set in the language $\mathcal{L}'$, since T is a complete and consistent set in $\mathcal{L}$
- Moreover, let $S'_1 = S_1 \cup T'$ and $S'_2 = S_2 \cup T'$. Since S'_1 and S'_2 are subsets of T_1 and T_2, respectively, they are separately consistent.

- Consider a countable saturated model $\mathfrak{A}$ of T'. Since T' is complete, $T' = \mathrm{Th}(\mathfrak{A})$.
- Since $S_1' = S_1 \cup \mathrm{Th}(\mathfrak{A})$ is consistent, by resplendency of $\mathfrak{A}$, $\mathfrak{A}$ can be extended to a model $\mathfrak{A}_1$ of S_1 in $\mathcal{L}_1'$.
- Similarly, $\mathfrak{A}$ can be extended to a model $\mathfrak{A}_2$ of S_2 in $\mathcal{L}_2'$. Therefore, by defining the interpretation of symbols in $\mathcal{L}_1' - \mathcal{L}'$ to be the same as in $\mathfrak{A}_1$ and in $\mathcal{L}_2' - \mathcal{L}'$ to be the same as in $\mathfrak{A}_2$, we extend $\mathfrak{A}$ to a structure $\mathfrak{A}'$ in $\mathcal{L}_1' \cup \mathcal{L}_2'$.
- Then, $\mathfrak{A}'$ is a model of $S_1 \cup S_2$, which contradicts our assumption. Thus, we complete the proof.

Corollary (Craig's Interpolation Theorem)

If a formula $\varphi \to \psi$ is provable from logical axioms ($\vdash \varphi \to \psi$), then there exists a formula θ consisting of mathematical symbols commonly appearing both in φ and ψ besides logical symbols and $=$, such that $\vdash \varphi \to \theta$ and $\vdash \theta \to \psi$.

The formula θ satisfying the above theorem is called an **interpolant** for φ and ψ.

Proof

- Assume $\vdash \varphi \to \psi$ with no interpolant θ. Let $\mathcal{L}$ be the language consisting of symbols common to φ and ψ. Let T_0 be the set of formulas ξ in $\mathcal{L}$ such that $\vdash \varphi \to \xi$.
- Since no finite subset of T_0 implies ψ, $T_0 \cup \{\neg\psi\}$ is consistent.
- Consider a model $\mathfrak{A}$ of $T_0 \cup \{\neg\psi\}$, and let T be the set of all $\mathcal{L}$ formulas contained in $\mathrm{Th}(\mathfrak{A})$. Clearly, $T \cup \{\neg\psi\}$ is consistent.
- To show that $T \cup \{\varphi\}$ is also consistent, assume otherwise. Then there exists a formula σ in T such that $\vdash \varphi \to \neg\sigma$. Thus, $\neg\sigma \in T_0 \subseteq T$, which implies the inconsistency of T.
- By Robinson's joint consistency theorem, $T \cup \{\varphi, \neg\psi\}$ is also consistent, contradicting the assumption $\vdash \varphi \to \psi$. □

Thank you for your attention!