Logic and Foundation II

Part 5. Models of first-order arithmetic

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Logic and Foundations II (Spring 2024)

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PA and its subsystems

- PA$^-$ is the theory of discrete ordered semirings. Example: $\mathbb{Z}[X]^+ = \{p \in \mathbb{Z}[X] :$ the highest-order coefficient of p is positive$\}$ is a non-standard model of PA$^-$
- **Peano arithmetic** PA is PA$^-$+ full induction.
- IΣ_n is PA$^-$+ induction for Σ_n formulas. PA $\supset$ I$\Sigma_1 \supset$ I$\Sigma_0 \supset$ IOpen $\supset$ PA$^-$.
- Robinson's system Q (or Q$_<$) is a weak subsystem of PA$^-$.

Theorem (Overspill principle)

Let $\mathfrak{A}$ be any non-standard model of $\mathrm{I}\Sigma_n$, and $\varphi(x)$ be any Σ_n formula. If $\mathfrak{A}_A \models \varphi(i)$ holds for infinitely many $i \in \mathbb{N}$, then there exists a non-standard element a such that $\mathfrak{A}_A \models \varphi(a)$ holds.

Types

- A **type** $\Phi(\vec{x})$ is a set of formulas in free variables $\vec{x} = (x_1, \cdots, x_n)$.
- $\mathfrak{A}$ **realizes** $\Phi(\vec{x})$ by $\vec{a}$, if $\mathfrak{A}_A \models \varphi(\vec{a})$ for all formulas $\varphi(\vec{x})$ in $\Phi(\vec{x})$.
- $\mathfrak{A}$ **omits** $\Phi(\vec{x})$, if $\mathfrak{A}$ does not realize $\Phi(\vec{x})$ by any $\vec{a}$.
- A type $\Phi(\vec{x})$ is a **type of a theory** T if $T \cup \Phi(\vec{c})$ ($\vec{c}$ new constants) is consistent. That is, there exists a model of T that realizes $\Phi(\vec{x})$.
- For a subset C of the universe of $\mathfrak{A}$, a **type on** C **in** $\mathfrak{A}$ is a type of theory $\mathrm{Th}(\mathfrak{A}_C)$.
- A type $\Phi(\vec{x})$ is a **principal** type of theory T, if there exists a formula $\psi(\vec{x})$ such that $T \cup \{\exists \vec{x} \psi(\vec{x})\}$ is consistent, and for any $\varphi(\vec{x}) \in \Phi(\vec{x})$, $T \vdash \forall \vec{x}(\psi(\vec{x}) \to \varphi(\vec{x}))$. In this case, we say that $\psi(\vec{x})$ **generates** $\Phi(\vec{x})$ in T.
- A type $\Phi(\vec{x})$ on $C (\subseteq A)$ in $\mathfrak{A}$ is a **principal** type, if it is a principal type of $\mathrm{Th}(\mathfrak{A}_A)$.
- **Remark**. Any structure $\mathfrak{A}$ realizes each of its principal types $\Phi(\vec{x})$. $\because$ If $\psi(\vec{x})$ generates $\Phi(\vec{x})$, then by definition $\mathrm{Th}(\mathfrak{A}_A) \cup \{\exists \vec{x} \psi(\vec{x})\}$ is consistent. Since $\mathrm{Th}(\mathfrak{A}_A)$ is a complete theory, it includes $\exists \vec{x} \psi(\vec{x})$ and so $\mathfrak{A}_A \models \exists \vec{x} \psi(\vec{x})$. Therefore, $\psi(\vec{x})$ and $\Phi(\vec{x})$ are realized by $\mathfrak{A}$.

Theorem (The omitting type theorem)

Let T be a consistent theory in a countable language $\mathcal{L}$. Given countably many non-principal types $\Phi_i(\vec{x}_i)$ of T, then there is a countable model of T that omits all Φ_i.

Definition

$\mathfrak{A}$ is an **end-extension** of $\mathfrak{B}$, denoted as $\mathfrak{B} \subseteq_e \mathfrak{A}$, if $(b \in |\mathfrak{B}| \wedge \mathfrak{A} \models a < b) \Rightarrow a \in |\mathfrak{B}|$.
$\mathfrak{A}$ is an **elementary end-extension** of $\mathfrak{B}$, if $\mathfrak{B} \subseteq_e \mathfrak{A}$ and $\mathfrak{B} \prec \mathfrak{A}$.

Definition

In a language $\mathcal{L}$ with $<$, the following schema is called **collection principle**:

$$\forall x < u \exists y_1 \cdots \exists y_k \varphi(x, y_1, \cdots, y_k) \to \exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \cdots, y_k).$$

Theorem

A countably infinite structure that satisfies the collection principle and the transitivity law has a proper elementary end-extension. In particular, a countable model of Peano arithmetic PA has a proper elementary end-extension.

Introduction to recursively saturated models

Definition

Let $\mathcal{L}$ be a countable language. An $\mathcal{L}$-structure $\mathfrak{A}$ is **recursively saturated** if any recursive 1-type on $\{a_1, \cdots, a_n\} \subseteq A$ is realized in $\mathfrak{A}$, i.e., for any recursive type $\{\varphi_i(x, x_1, \cdots, x_n) \mid i \in \mathbb{N}\}$ and any $a_1, \cdots, a_n \in A$,

$$\forall j \exists a \in A \forall i < j\, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n) \Rightarrow \exists a \in A \forall i\, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n).$$

Lemma

A countable structure in a countable language has a countable elementary extension which is recursively saturated.

Proof.

- Let $\mathfrak{A}$ be a countable structure in a countable language. By the compactness theorem and the downward Löwenheim–Skolem Theorem, $\mathfrak{A}$ has a countable elementary extension $\mathfrak{A}_1$ which realizes all recursive 1-types on any finite subset of A (in $\mathfrak{A}_1$).
- Similarly, we create $\mathfrak{A}_1 \prec \mathfrak{A}_2 \prec \mathfrak{A}_3 \prec \cdots$, and set $\mathfrak{A}_\infty = \bigcup_k \mathfrak{A}_k$. By the elementary chain theorem, $\mathfrak{A} \prec \mathfrak{A}_\infty$ and is also recursively saturated.

Lemma (Tarski's "undefinability of truth")

Let T be a consistent extension of $\mathsf{Q}_<$. There is no formula $\mathrm{Sat}(x,y)$ such that: for any $\mathcal{L}_{\mathrm{OR}}$ formula $\varphi(v_1,\cdots,v_k)$ (with only free variables $v_1,\cdots,v_k$),

$$T \vdash \forall s(\mathrm{Sat}(\overline{\ulcorner\varphi\urcorner},s) \leftrightarrow \varphi(s_1,\cdots,s_k)),$$

where s is the code of a sequence $(s_1,\cdots,s_k)$.

Lemma

For each $n>0$, there exist formulas $\mathrm{Sat}_{\Sigma_n}(x,y)$ and $\mathrm{Sat}_{\Pi_n}(x,y)$ in language $\mathcal{L}_{\mathrm{OR}}$ such that for any Σ_n formula $\varphi(v_1,\cdots,v_k)$ and Π_n formula $\psi(v_1,\cdots,v_k)$ (neither includes free variables other than $v_1,\cdots,v_k$),

$$\mathrm{I\Sigma_1} \vdash \forall s(\mathrm{Sat}_{\Sigma_n}(\overline{\ulcorner\varphi\urcorner},s) \leftrightarrow \varphi(s_1,\cdots,s_k)),$$
$$\mathrm{I\Sigma_1} \vdash \forall s(\mathrm{Sat}_{\Pi_n}(\overline{\ulcorner\psi\urcorner},s) \leftrightarrow \psi(s_1,\cdots,s_k)),$$

where s is the code of $(s_1,\cdots,s_k)$. When $n>0$, $\mathrm{Sat}_{\Sigma_n}\in\Sigma_n$ and $\mathrm{Sat}_{\Pi_n}\in\Pi_n$.

Proof.

- To start with, consider the case where $n = 0$. Roughly speaking, the truth of a Σ_0 sentence is defined primitive-recursively, and so Sat_{Σ_0} (which is the same as Sat_{Π_0}) can be expressed by either Σ_1 or Π_1 in $\mathrm{I}\Sigma_1$.
- Next, by meta-induction on n, we construct $\mathrm{Sat}_{\Sigma_{n+1}}$ assuming Sat_{Π_n} is already obtained. For a Σ_{n+1} formula $\exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k)$ (with $\varphi \in \Pi_n$), $\mathrm{Sat}_{\Sigma_{n+1}}$ is defined as follows.

$$\begin{aligned}&\mathrm{Sat}_{\Sigma_{n+1}}(\ulcorner \exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k)\urcorner, (s_1, \cdots, s_k))\\ &\quad \leftrightarrow \exists y \mathrm{Sat}_{\Pi_n}(\ulcorner \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k)\urcorner, (y_1, \cdots, y_j, s_1, \cdots, s_k)).\end{aligned}$$

- Then the following is provable in $\mathrm{I}\Sigma_1$.

$$\begin{aligned}&\mathrm{Sat}_{\Sigma_{n+1}}(\overline{\ulcorner \exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k)\urcorner}, (s_1, \cdots, s_k))\\ &\leftrightarrow \exists y \mathrm{Sat}_{\Pi_n}(\overline{\ulcorner \varphi(u_1, \cdots, u_j, v_1, \cdots, v_k)\urcorner}, (y_1, \cdots, y_j, s_1, \cdots, s_k))\\ &\leftrightarrow \exists y \varphi(y_1, \cdots, y_j, s_1, \cdots, s_k)\\ &\leftrightarrow \exists u_1 \cdots \exists u_j \varphi(u_1, \cdots, u_j, s_1, \cdots, s_k).\end{aligned}$$

- Finally, $\mathrm{Sat}_{\Pi_{n+1}}$ can be defined in the same way. □

Lemma

For $n > 0$, a non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$ is **Σ_n-recursively saturated**, i.e., it realizes any (finitely satisfiable) recursive 1-type on a finite subset of A consisting of only Σ_n formulas.

Proof. Let $\Phi(x, \vec{x})$ be a recursive type consisting only of Σ_n formulas. Then, the Gödel numbers of formulas in Φ can be expressed by a Δ_1 formula $\theta(i)$. Thereby,

- The finite satisfiability of $\Phi(x, \vec{a})$ is expressed as: for each natural number j,

$$\exists x \forall i < \bar{j} (\theta(i) \to \mathrm{Sat}_{\Sigma_n}(i, (x, \vec{a}))),$$

  which is shown to be Σ_n in $\mathrm{B}\Sigma_n (\subseteq \mathrm{I}\Sigma_n)$.
- Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. By the overspill principle, the above formula holds for some infinite element j'. Suppose $x = a$ satisfies the formula for this j'.
- Then, we have $\theta(\bar{i}) \to \mathrm{Sat}_{\Sigma_n}(\bar{i}, (a, \vec{a}))$ for any natural number i. Namely, all Σ_n formulas in $\Phi(x, \vec{a})$ are realized by a in $\mathfrak{A}_A$. □

By the above lemma, any non-standard model of PA is Σ_n-recursively saturated for each $n > 0$, but in the next problem, we show there is a non-standard model of PA which is not recursively saturated.

If the satisfaction relation $\mathrm{Sat}(x, y)$ were defined in PA, any non-standard model of PA would be recursively saturated in the same way as in the above lemmma. So, this is another proof that the satisfaction relation is not definable in PA.

Problem 4

Let $\mathfrak{A}$ be a non-standard model of PA, and $a \in A$ be an arbitrary non-standard element. Then, in $\mathfrak{A}$, let $\mathrm{K}(\mathfrak{A}\,;a)$ denote the set of all element $b \in A$ that can be defined by the formula $\varphi(x, a)$ (does not include parameters other than a). That is, $\mathrm{K}(\mathfrak{A}\,;a)$ denote the set of b's such that $\mathfrak{A}_{\{a,b\}} \models \forall x(x = b \leftrightarrow \varphi(x, a))$. Then prove the following.
(1) By restricting functions and relations of $\mathfrak{A}$ to that of $\mathrm{K}(\mathfrak{A}\,;a)$, $\mathrm{K}(\mathfrak{A}\,;a)$ can be seen as a substructure of $\mathfrak{A}$. $\mathrm{K}(\mathfrak{A}\,;a)$ is an elementary substructure of $\mathfrak{A}$.
(2) $\Phi(x, a) = \{\exists v \varphi(v, a) \to \exists v < x \ \varphi(v, a) : \varphi(v, u)$ contains no free variables or parameters other than $u, v\}$ is recursive and finitely satisfiable, but it cannot be realized by $\mathrm{K}(\mathfrak{A}\,;a)$.

Problem 5

Let $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ be a non-standard model of $\mathrm{I}\Sigma_1$. Show that $\mathfrak{A}' = (A, +, 0, 1, <)$ is recursively saturated.

Solutions.

- Problem 4 (1) Use the Tarski-Vaught test. If $\mathfrak{A}_{\{a\}} \models \exists x \varphi(x, a)$, then a formula $\varphi(x, a) \wedge \forall y < x \neg\varphi(y, a)$ defines an element $b \in \mathrm{K}(\mathfrak{A}\,; a)$ such that $\mathfrak{A}_{\{a,b\}} \models \varphi(b, a)$.

- Problem 4 (2) Obviously, $\Phi(x, a)$ is recursive and finitely satisfiable. Since it includes $\exists v \varphi(v, a) \to \exists v < x\ \varphi(v, a)$ for all formulas $\varphi(v, a)$ defining an element of $\mathrm{K}(\mathfrak{A}\,; a)$, it can not be realized in $\mathrm{K}(\mathfrak{A}\,; a)$.

- Problem 5 Any arithmetical formula without the multiplication symbol can be expressed as a Σ_0 formula (recall Presburger arithmetic). Since $\mathfrak{A}$ is Σ_1-recursively saturated, $\mathfrak{A}'$ is also, and then it is recursively saturated.

In the above lemma, we will extend a recursive type to a little more general class. To this end, we introduce the following concept.

Definition

Let $\mathfrak{A}$ be a model of $\mathrm{I}\Sigma_1$, and $a \in A$. The set

$$\{n \in \mathbb{N} : \mathfrak{A} \models \overline{p(n)}|a\}$$

is called the set **coded by a** in $\mathfrak{A}$, where $p(n)$ is a primitive recursive function representing the $n+1$-th prime number, and $u|v \equiv \exists w \leq v(u \bullet w = v)$. The collection of all the sets encoded by an element in $\mathfrak{A}$ is called the **standard system** of $\mathfrak{A}$, denoted as $\mathrm{SSy}(\mathfrak{A})$.

Lemma (D. Scott)

Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_1$. Given two disjoint Σ_1 sets, there exists a set in $\mathrm{SSy}(\mathfrak{A})$ which separates them. In particular, any recursive set belongs to $\mathrm{SSy}(\mathfrak{A})$.

Proof.

- Let $\exists y \theta_i(x, y)$ (θ_i is a Σ_0 formula, $i = 0, 1$) represent two disjoint Σ_1 sets.
- Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_1$. Then consider the following Σ_1 formula:

$$\exists v \forall x, y < \bar{j}((\theta_0(x, y) \to p(x)|v) \land (\theta_1(x, y) \to p(x) \not| v)).$$

  This holds for any standard natural number j in $\mathfrak{A}$. Then by the overspill principle, it also holds for a non-standard element $j = b$.
- Let c be such that $v = c$ satisfies the above formula with $j = b$. Then, the set coded by c separates the two initially given Σ_1 sets as follows.

$$\mathfrak{N} \models \exists y\ \theta_0(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{b\}} \models \exists y < b\ \theta_0(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{c\}} \models \overline{p(n)}|c,$$
$$\mathfrak{N} \models \exists y\ \theta_1(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{b\}} \models \exists y < b\ \theta_1(\overline{n}, y) \Rightarrow \mathfrak{A}_{\{c\}} \models \overline{p(n)} \not| c. \quad \square$$

Note that in general, a set that separates two Σ_1 sets cannot be obtained recursively. That is, $\mathrm{SSy}(\mathfrak{A})$ is properly larger than the class of recursive sets.

Lemma

Let $n > 0$ and $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. If a type $\Phi(\vec{x})$ of Σ_n formulas on a finite subset of A is coded in $\mathfrak{A}$, then $\mathfrak{A}$ realizes $\Phi(\vec{x})$.

The proof is exactly the same as that of lemma in Page 9. The converse holds as follows.

Lemma

Let $n > 0$ and $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. Fix $\vec{a} \in A^{<\omega}$ arbitrarily. Then the following k types can be coded.

$$\Phi(\vec{x}) = \{\varphi(\vec{x}) : \varphi(\vec{x}) \in \Sigma_n \wedge \mathfrak{A} \models \varphi(\vec{a})\},$$
$$\Psi(\vec{x}) = \{\psi(\vec{x}) : \psi(\vec{x}) \in \Pi_n \wedge \mathfrak{A} \models \psi(\vec{a})\}$$

Proof. In $\mathrm{I}\Sigma_1$, $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$ can be defined. So, there exist Σ_n formula $\varphi_1(k, \vec{a})$ and Π_n formula $\psi_1(k, \vec{a})$ s.t. $\varphi \in \Phi \leftrightarrow \varphi_1(\ulcorner\varphi\urcorner, \vec{a})$ and $\psi \in \Psi \leftrightarrow \psi_1(\ulcorner\psi\urcorner, \vec{a})$ hold. Then, letting c be a non-standard element of $\mathfrak{A}$, by Σ_n induction, we can define a code $\Pi_{b\in U} p(b)$ for $U = \{b < c : \varphi_1(b, \vec{a})\}$ and a code $\Pi_{b\in V} p(b)$ for $V = \{b < c : \psi_1(b, \vec{a})\}$. It is clear that these code $\Phi(\vec{x})$ and $\Psi(\vec{x})$, respectively. □

With the above preparations, we will prove Friedman's self-embedding theorem. The following is a key lemma, and also used in several variations of the theorem.

Lemma

Assuming $n > 0$, let $\mathfrak{A}$, $\mathfrak{B}$ be countable non-standard models of $\mathrm{I}\Sigma_n$. Take $a_0 \in A$ and $b_0, c \in B$ arbitrarily. Then the following two conditions are equivalent.

(1) There exists $\mathfrak{B}' \subseteq_e \mathfrak{B}$ such that $c \notin B'$. There is an isomorphism h between $\mathfrak{A}$ and $\mathfrak{B}'$ such that $h(a_0) = b_0$. For any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{b} \in B'^{<\omega}$,

$$\mathfrak{B}'_{\{\vec{b}\}} \models \varphi(\vec{b}) \Leftrightarrow \mathfrak{B}_{\{\vec{b}\}} \models \varphi(\vec{b}).$$

(2) $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$, and for any Π_{n-1} formula $\varphi(\vec{v}, u)$,

$$\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0) \Rightarrow \mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0),$$

where $\vec{v} = (v_1, \ldots, v_k)$ and $\exists \vec{v} < c$ means $\exists v_1 < c \cdots \exists v_k < c$.

Proof. Assume (1) and we show the first half of (2).

- By $\mathfrak{A} \cong \mathfrak{B}'$, $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B}')$ is obvious.
- Since $\mathfrak{B}' \subseteq_e \mathfrak{B}$, it is also clear that $\mathrm{SSy}(\mathfrak{B}') \subseteq \mathrm{SSy}(\mathfrak{B})$.
- Assume that $R \in \mathrm{SSy}(\mathfrak{B})$ and R is coded by r in $\mathfrak{B}$. We will show that R is also coded in $\mathfrak{B}'$.
- Take any non-standard element l of B'. Since $\mathfrak{B}'$ is also a model of $\mathrm{I}\Sigma_n$ $(n > 0)$, the $l + 1$-th prime $p(l)$ belongs to B', and $p(l)! \in B'$.
- Therefore, letting m be the greatest common divisor of r and $p(l)!$ in $\mathfrak{B}$, we have $m \in B'$ since $\mathfrak{B}'$ is an initial segment of $\mathfrak{B}$. Then, it is clear that m also encodes R.
- From the above, we obtain $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$.

Next we show the second half of (2).

- Let $\varphi(\vec{v}, u)$ be a Π_{n-1} formula, and $\mathfrak{A}_A \models \exists\vec{v}\varphi(\vec{v}, a_0)$.
- By the isomorphism between $\mathfrak{A}$ and $\mathfrak{B}'$, $\mathfrak{B}'_{B'} \models \exists\vec{v}\varphi(\vec{v}, b_0)$.
- Then, since there exists $\vec{d} \in B'$ such that $\mathfrak{B}'_{B'} \models \varphi(\vec{d}, b_0)$, from the assumption (1), $\mathfrak{B}_B \models \varphi(\vec{d}, b_0)$. Therefore, $\mathfrak{B}_B \models \exists\vec{v} < c\,\varphi(\vec{v}, b_0)$.

Next, assuming (2), we show (1).

- This is an application of the so-called **back-and-forth argument**. We alternately produce a list $a_0, a_1, \cdots$ of the elements of A and a list $b_0, b_1, \cdots$ of the elements of B', and an isomorphism h between $\mathfrak{A}$ and $\mathfrak{B}'$ defiend by $h(a_i) = b_i$.
- Now, suppose $a_0, a_1, \cdots, a_{2k}$ and $b_0, b_1, \cdots, b_{2k}$ have been chosen, and for any Π_{n-1} formula $\varphi(\vec{v}, \vec{u})$,

$$\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0, \cdots, a_{2k}) \Rightarrow \mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0, \cdots, b_{2k}) \qquad (\sharp)$$

  holds.
- We next choose a_{2k+1}, a_{2k+2} and b_{2k+1}, b_{2k+2} such that this condition is preserved. We will explain later that (1) can be obtained by this.
- Since A is countable, each member can be assigned by a natural number uniquely. Then choose one with the smallest number among the elements that do not appear in $a_0, a_1, \cdots, a_{2k}$ and denote it as a_{2k+1}. This process guarantees that $\{a_i : i \in \mathbb{N}\}$ lists all the members of A.

- Now we will search for b_{2k+1} such that ($\sharp$) holds.
- Let $\Phi(\vec{x})$ be the set of Σ_n formulas $\exists\vec{v}\varphi(\vec{v}, x_0, \cdots, x_{2k+1})$ ($\varphi \in \Pi_{n-1}$) which holds for $a_0, \cdots, a_{2k}, a_{2k+1}$ in $\mathfrak{A}$. By the second lemma in page 14, $\Phi(\vec{x})$ is coded in $\mathfrak{A}$. Since $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$, so it is also coded in $\mathfrak{B}$.
- Furthermore, we let

$$\begin{aligned}&\Phi'(x_0, \cdots, x_{2k+1}, x_{2k+2})\\&= \{\exists\vec{v} < x_{2k+2}\, \varphi(\vec{v}, x_0, \cdots, x_{2k+1}) : \exists\vec{v}\varphi(\vec{v}, x_0, \cdots, x_{2k+1}) \in \Phi\}.\end{aligned}$$

  Since there is a primitive recursive transformation between Φ and Φ', Φ' is also coded in $\mathfrak{B}$.
- Then, if $\Phi'(b_0, \cdots, b_{2k}, x, c)$ is shown to be finitely satisfiable in $\mathfrak{B}$, then by the first lemma in page 14, we can find an element $x = b$ that realizes $\Phi'(b_0, \cdots, b_{2k}, x, c)$, and letting b_{2k+1} be such a b, ($\sharp$) holds.
- Now, let $\exists\vec{v} < c\, \varphi_i(\vec{v}, b_0, \cdots, b_{2k}, x)$ $(i \leq j)$ be any finite set of formulas from $\Phi'(b_0, \cdots, b_{2k}, x, c)$.

- From the definition of Φ', for each $i \leq j$, $\exists \vec{v} \varphi_i(\vec{v}, a_0, \cdots, a_{2k}, a_{2k+1})$ holds in $\mathfrak{A}$, so

$$\mathfrak{A}_A \models \exists \vec{v}_0 \cdots \exists \vec{v}_j \exists x \bigwedge_{i \leq j} \varphi_i(\vec{v}_i, a_0, \cdots, a_{2k}, x).$$

- On the other hand, using ($\sharp$),

$$\mathfrak{B}_B \models \exists \vec{v}_0 < c \cdots \exists \vec{v}_j < c \exists x < c \bigwedge_{i \leq j} \varphi_i(\vec{v}_i, b_0, \cdots, b_{2k}, x).$$

- Therefore, by simple transformation,

$$\mathfrak{B}_B \models \exists x \bigwedge_{i \leq j} \exists \vec{v} < c \, \varphi_i(\vec{v}, b_0, \cdots, b_{2k}, x).$$

- In other words, $\Phi'(b_0, \cdots, b_{2k}, x, c)$ is finitely satisfiable, and b_{2k+1} is obtained.

- Next, we first select b_{2k+2} and we search for a corresponding a_{2k+2}. If $\{b_0, \cdots, b_{2k}, b_{2k+1}\}$ is an initial segment of $\mathfrak{B}$, then $b_{2k+2} = b_{2k+1}$, $a_{2k+2} = a_{2k+1}$, and $(\sharp)$ holds.
- Otherwise, there exists a $b < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\}$ such that b does not appear in $b_0, \cdots, b_{2k}, b_{2k+1}$. Then among such, let b_{2k+2} be one with the minimal number assigned in advance to the members of B. This finally produces $\{b_i : i \in \mathbb{N}\}$ as an initial segment of $\mathfrak{B}$.
- Then we will find a_{2k+2} corresponding to b_{2k+2}.
- Let $\Psi(\vec{x})$ be the set of Σ_n formulas $\forall \vec{v} < x_{2k+3}\, \psi(\vec{v}, x_0, \cdots, x_{2k+2})$ holds for $b_0, \cdots, b_{2k+1}, b_{2k+2}, c$ in $\mathfrak{B}$. This can be coded in $\mathfrak{B}$.
- Therefore, if we define

$$\begin{aligned}&\Psi'(x_0, \cdots, x_{2k+1}, x_{2k+2})\\&= \{\forall \vec{v} \psi(\vec{v}, x_0, \cdots, x_{2k+2}) : \forall \vec{v} < x_{2k+3}\, \psi(\vec{v}, x_0, \cdots, x_{2k+2}) \in \Psi\}\end{aligned}$$

  then Ψ' is coded in $\mathfrak{A}$ by the same argument as above.
- All that remains is to show $\Psi'(a_0, \cdots, a_{2k+1}, x)$ is finitely satisfiable in $\mathfrak{A}$. So, let $\forall \vec{v} \psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x)$ $(i \leq j)$ be a finite subset of $\Psi'(a_0, \cdots, a_{2k+1}, x)$.

- We will show that these formulas are realized by $x = a$ such that $a < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\}$.
- By way of contradiction, assume

$$\mathfrak{A}_A \models \forall x < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\} \exists \vec{v} \bigvee_{i \le j} \neg \psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x).$$

- By the Σ_n collection principle that follows from Σ_n induction,

$$\mathfrak{A}_A \models \exists y \forall x < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\} \exists \vec{v} < y \bigvee_{i \le j} \neg \psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x).$$

- On the other hand, using ($\sharp$),

$$\mathfrak{B}_B \models \exists y < c \forall x < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\} \exists \vec{v} < y \bigvee_{i \le j} \neg \psi_i(\vec{v}, b_0, \cdots, b_{2k+1}, x).$$

- Therefore, by simple transformation,

$$\mathfrak{B}_B \models \forall x < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\} \exists \vec{v} < c \bigvee_{i \le j} \neg \psi_i(\vec{v}, b_0, \cdots, b_{2k+1}, x)$$

  This is contradicts with the assumption that $b_0, \cdots, b_{2k+1}, b_{2k+2}, c$ realize $\Psi(\vec{x})$.
- Thus, $\Psi'(a_0, \cdots, a_{2k+1}, x)$ is finitely satisfiable, and so the desired a_{2k+2} exists.

- Suppose that we have completed the construction of a list $a_0, a_1, \cdots$, and a list $b_0, b_1, \cdots$. As described above, $A = \{a_i : i \in \mathbb{N}\}$ and $B' = \{b_i : i \in \mathbb{N}\}$ is an initial segment of $\mathfrak{B}$. It is also obvious that $c \notin B'$.
- Next, we define a function h between $\mathfrak{A}$ and $\mathfrak{B}'$ by $h(a_i) = b_i$. Then, h is an isomorphism, since by ($\sharp$), for an atomic formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Rightarrow \mathfrak{B}_B \models \varphi(b_0, \cdots, b_k),$$

  which implies h preserves operations and $<$.
- Moreover, by ($\sharp$), we can show that for any Π_{n-1} formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Leftrightarrow \mathfrak{B}_B \models \varphi(b_0, \cdots, b_k).$$

  $\Rightarrow$ is clear. For $\Leftarrow$, let $\mathfrak{A}_A \not\models \varphi(a_0, \cdots, a_k)$. Then $\mathfrak{A}_A \models \neg\varphi(a_0, \cdots, a_k)$, and $\neg\varphi(a_0, \cdots, a_k)$ is Σ_{n-1}, so by ($\sharp$), $\mathfrak{B}_B \models \neg\varphi(b_0, \cdots, b_k)$, and $\mathfrak{B}_B \not\models \varphi(b_0, \cdots, b_k)$.
- On the other hand, since h is isomorphic, for any formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Leftrightarrow \mathfrak{B}'_{B'} \models \varphi(b_0, \cdots, b_k).$$

  So for any Π_{n-1} formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{B}'_{B'} \models \varphi(b_0, \cdots, b_k) \Leftrightarrow \mathfrak{B}_{B'} \models \varphi(b_0, \cdots, b_k),$$

  and thus (1) is obtained. □

Theorem (Friedman's self-embedding theorem)

Let $n > 0$, $\mathfrak{A}$ be a countable non-standard model of $\mathrm{I}\Sigma_n$, and take $a \in A$ arbitrarily. Then there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $a \in A'$ but $A' \subsetneq A$, and $\mathfrak{A} \cong \mathfrak{A}'$ and for any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{a'} \in A'^{<\omega}$,

$$\mathfrak{A}'_{A'} \models \varphi(\vec{a'}) \Leftrightarrow \mathfrak{A}_{A'} \models \varphi(\vec{a'}).$$

Proof.

- In last lemma, we consider the case $\mathfrak{A} = \mathfrak{B}$. In order to satisfy the condition (2) of the last lemma, for any Π_{n-1} formula $\varphi(\vec{v}, u)$, it is sufficient to find c such that

$$\mathfrak{A}_{\{a\}} \models \exists \vec{v} \varphi(\vec{v}, a) \Rightarrow \mathfrak{A}_{\{a,c\}} \models \exists \vec{v} < c\, \varphi(\vec{v}, a).$$

- Now, let

$$\Phi(x) = \{\exists \vec{v} \varphi(\vec{v}, a) \rightarrow \exists \vec{v} < x\, \varphi(\vec{v}, a) : \varphi(\vec{v}, u) \in \Pi_{n-1}\}.$$

  This is a recursive type consisting only of Π_n formulas, and is clearly finitely satisfiable.

- Therefore, there exists c that realizes $\Phi(x)$. Therefore, by the last lemma, there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ which satisfies the conditions of the theorem. $\square$

- The essence of this theorem is that a countable non-standard model of $\mathrm{I}\Sigma_1$ has an initial segment that is isomorphic to itself.
- Friedman first proved this theorem for a countable non-standard model of Peano arithmetic, and several researchers sophisticated it to the above form.
- The same theorem does not hold for non-countable models, and also it does not hold in general for countable non-standard models of $\mathrm{I}\Sigma_0$.
- Furthermore, an important result related to this is McAloon's theorem, which states that a countable non-standard model of $\mathrm{I}\Sigma_0$ has an initial segment that is a model of Peano arithmetic PA.

Thank you for your attention!