Topics in Applied Math: Logic and Foundations of Mathematics

Part 7. Models of second order arithmetic

Kazuyuki Tanaka

BIMSA

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北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

Logic and Foundations

- **Part 1. Equational theory**
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Part 7. Schedule

- Dec. 3, (1) A self-embedding theorem I
- Dec. 5, (2) A self-embedding thm II and Harrington's conservation thm
- **Dec. 10, (3) STY theorem**

§2. Forcing and Harrington's Theorem

Let $\mathfrak{M} = (M, S)$ be a countable model of RCA_0. Here, M is the first-order part (the domain corresponding to the natural numbers), and S is the second-order part consisting of subsets of M, that is, $S \subseteq \mathcal{P}(M)$. Then, the set of forcing conditions is defined as

$$\mathbb{P} = \{T \in S \mid \mathfrak{M} \models \text{“}T(\subseteq \mathrm{Seq}_2) \text{ is an infinite binary tree”}\},$$

and a partial order on $\mathbb{P}$ is given by

$$T_1 \leq T_2 \Leftrightarrow T_1 \subseteq T_2.$$

For each $T \in \mathbb{P}$, we want to generate an infinite path and put it into S. But if we bring in an arbitrary path of T from outside, it might break the condition of $\mathfrak{M}$, e.g., induction axiom. Instead, we approximate an infinite path by $T' \leq T$, and for this purpose, the concept of density is important, namely

$$D \subseteq \mathbb{P} \text{ is } \textbf{dense} \Leftrightarrow \forall T \in \mathbb{P}\ \exists T' \in D\ T' \leq T.$$

A set $F \subseteq \mathbb{P}$ is called a **filter**, if it satisfies the following conditions:

$$p \in F \land p < q \to q \in F, \quad \forall p, q \in F \ \ [p] \cap [q] \cap F \neq \varnothing.$$

A filter G is called a **$\mathcal{D}$-generic filter** if it intersects every dense set $D \subseteq \mathbb{P}$ belonging to $\mathcal{D}$.

$E \subseteq \mathbb{P}$ is said to be **definable in $\mathfrak{M}$** if there exists a formula $\varphi(X)$ (with parameters from $M \cup S$) such that $E = \{T \in \mathbb{P} \mid \mathfrak{M} \models \varphi(T)\}$. The totality of such sets is denoted by $\mathrm{Def}(\mathfrak{M})$. Since we only consider a countable model $\mathfrak{M} = (M, S)$ in a countable language, $\mathrm{Def}(\mathfrak{M})$ is a countable set. By Lemma 2.3, any $T \in \mathbb{P}$ is contained in some $\mathrm{Def}(\mathfrak{M})$-generic filter. Such a filter is simply referred to as an $\mathfrak{M}$-generic filter.

Lemma 2.4

If $F \subseteq \mathbb{P}$ is an $\mathfrak{M}$-generic filter, then there exists a unique infinite path $G = \cap F = \cap_{T \in F} T$ common to all $T \in F$. That is, F is contained in the principal filter generated by G.

Proof For each $k \in M$, let $E_k = \{T \in \mathbb{P} \mid \exists! s \in \{0,1\}^k \ s \in T\}$ be dense and definable in $\mathfrak{M}$. If F is an $\mathfrak{M}$-generic filter, then for each k, there exists some $s_k \in \{0,1\}^k$ such that there is $T_k \in F$ with $T_k \cap \{0,1\}^k = \{s_k\}$. Moreover, if $k < k'$, then s_k is an initial segment of $s_{k'}$, and $s_{k'} \in T_k$. If not, $[T_k] \cap [T_{k'}] = \varnothing$[1], which would contradict the filter condition of F. Thus, let $G = \bigcup_{k \in M} s_k$; then $G = \bigcap_k T_k$ as well. Finally, to show $G = \cap F$, if $G \not\subseteq T \in F$, then there exists some k such that $s_k \notin T$, and $[T] \cap [T_k] = \varnothing$, which contradicts the filter condition of F. □

[1]Here, $[T]$ denotes $\{T' \in \mathbb{P} \mid T' \subset T\}$. In the latter half of part 8, the same notation $[T]$ represents the set of infinite paths of T. Since both are conventional, we would use both as they are.

Definition 2.5

$G(\subseteq M)$ is called an **$\mathfrak{M}$-generic path**, if for every dense set $D \in \mathrm{Def}(\mathfrak{M})$, there exists a tree $T \in D$ such that G is an infinite path through T.

Lemma 2.6

Every $T \in \mathbb{P}$ has an $\mathfrak{M}$-generic path G.

Proof By Lemma 2.3, every T is contained in some $\mathfrak{M}$-generic filter F. Then, by Lemma 2.4, there is a common infinite path G in the trees of F. It is clear from the definition that this G is an $\mathfrak{M}$-generic path. □

From now on, an $\mathfrak{M}$-generic path will simply be referred to as a generic path.

Lemma 2.7

If G is a generic path, then $(M, S \cup \{G\}) \models \Sigma^0_1$-induction.

Proof Let $\varphi(i, X)$ be any Σ^0_1 formula, and choose any $b \in M$, and we will show that $A = \{a \leq_M b \mid \varphi(a, G)\} \in S$ [2]. If $A \in S$, induction on $\varphi(n, G)$ can be shown as follows.

[2] See Lemma 1.8 of part 7 for $\mathrm{RCA}_0 \vdash$ (bounded Σ^0_1-CA). We show (bounded Σ^0_1-CA) $\to \Sigma^0_1$ induction.

Suppose $A \in S$. Then, $B = \{a \mid a \in A \vee a >_M b\} \in S$ since $\mathfrak{M} \models (\Delta^0_1\text{-CA})$. Now, assume $\varphi(0, G)$ and $\forall n(\varphi(n, G) \to \varphi(n+1, G))$. Then, we have $0 \in B$ and $\forall m(m \in B \to m+1 \in B)$. Since $\mathfrak{M} \models \Sigma^0_1$-induction, by induction on B, we have $B = M$. Therefore, $b \in A$, that is, $\varphi(b, G)$. Since $b \in M$ is arbitrary, we get $\forall n \varphi(n, G)$.

Now we show $A \in S$. Let $\varphi(i, X) \equiv \exists j \theta(i, X\lceil j)$ (where $\theta \in \Sigma^0_0$)[3], and set

$$D_b = \{T \in \mathbb{P} \mid \mathfrak{M} \models \forall a \leq b \ (1)\ \forall t \in T \neg\theta(a, t) \vee \\ (2)\ \exists k \forall t \in T \cap \{0,1\}^k \exists s \subseteq t \theta(a, s)\}.$$

Of course, D_b is definable in $\mathfrak{M}$. Here, note that if $T \in D_b$ and $T' \subseteq T$, then $T' \in D_b$. And as shown below, D_b is dense, so there exists a tree T_0 in D_b that has G as an infinite path. Fix such a T_0. For simplicity, we write $(1)_{T_0}$ for above condition (1) with $T = T_0$, and $(2)_{T_0}$ for condition (2) with $T = T_0$.

[3] $X \restriction j$ represents the code of the initial segment $(f(0), \cdots, f(j-1))$ of the characteristic function f of X. The truth value of the Σ^0_0 formula $\theta(X)$ depends only on a finite part of X, so for sufficiently large j, X can be replaced by $X \restriction j$. See [Simpson, Theorem II.2.7] for details.

Then, for each $a \leq_M b$,

$$\mathfrak{M} \models (1)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \neg\varphi(a, G),$$
$$\mathfrak{M} \models (2)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \varphi(a, G).$$

Since $\mathfrak{M} \models (1)_{T_0} \vee (2)_{T_0}$, we have

$$\mathfrak{M} \models (2)_{T_0} \Leftrightarrow (M, S \cup \{G\}) \models \varphi(a, G)$$

Since (2) is a Σ^0_1 formula, and $\mathfrak{M} \models (\text{bounded}\Sigma^0_1\text{-CA})$ (Lemma 1.8, Chapter 7), $A = \{a \leq_M b \mid \mathfrak{M} \models (2)_{T_0}\} \in S$.

Finally, we show that D_b is dense. Choose any $\tilde{T} \in \mathbb{P}$. For each $\sigma \in \{0,1\}^{\leq b}$, define a tree T_σ inductively as follows:

$T_\varnothing = \tilde{T}$,
$T_{\sigma^\frown 0} = \{t \in T_\sigma \mid \forall s \subseteq t \ \neg\theta(a, s)\}$, where $a = \text{leng}(\sigma)$,
$T_{\sigma^\frown 1} = T_\sigma$.

Here, $\varnothing$ is the empty sequence, and $\sigma^\frown i$ denotes the sequence σ followed by $i (= 0, 1)$.

Next, let $S_b = \{\sigma \in \{0,1\}^{b+1} \mid T_\sigma$ is an infinite tree$\}$. Then, since "T_σ is an infinite tree" is expressed by a Π^0_1 formula $\forall n \exists \tau \in \{0,1\}^n\ \tau \in T_\sigma$, by (bounded Σ^0_1-CA), we have

$S_b \in S$. Also, since $\overbrace{< 1, 1, \cdots, 1 >}^{b+1} \in S_b$, we get $S_b \neq \varnothing$.
Thus, let σ_b be the lexicographically first element in S_b.

Take any $a \leq_M b$. $\sigma_b(a) = 0$, then $(\sigma_b \lceil a)^\frown 0 \subset \sigma_b$, so

$$T_{\sigma_b} \subseteq T_{(\sigma_b \lceil a)^\frown 0} \subseteq \{t \mid \neg\theta(a,t)\},$$

from which we have $(1)_{T_{\sigma_b}}$.

If $\sigma_b(a) = 1$, then $T_{(\sigma_b \lceil a)^\frown 0}$ is finite, and thus $(2)_{T_{\sigma_b \lceil a}}$ and also $(2)_{T_{\sigma_b}}$ holds.

From all the above, $T_{\sigma_b} \in D_b$, which proves that D_b is dense. □

Fix a generic path G for $T \in \mathbb{P}$, and let

$$S^T = \{X \subseteq M \mid X \text{ is definable in } (M, S \cup \{G\}) \text{ by a } \Delta^0_1 \text{ formula}\}.$$

Lemma 2.8

$(M, S^T) \models \mathsf{RCA}_0 + T$ has an infinite path.

Proof For a Σ^0_1 formula φ with parameters from S^T, there exists an equivalent Σ^0_1 formula ψ with parameters only from $S \cup \{G\}$, which is obtained from the former by replacing a parameter X of S^T with a Δ^0_1 formula defining it. Recall that the same argument was used to show that RCA_0 is a conservative extension of $\mathsf{I}\Sigma_1$ (in part 5, Lemma 1.3). Then, by Lemma 2.7, $(M, S^T) \models \mathsf{RCA}_0$. Also, in (M, S^T), T has an infinite path G. □

Notice that if (M, S) is countable, then S^T is also countable. In the following lemma, this process is repeated to construct a model (M, S_∞) of WKL_0, which is also countable.

Lemma 2.9

For any countable model (M,S) of RCA_0, there exists a countable set S_∞ such that $S \subseteq S_\infty \subseteq \mathcal{P}(M)$ and $(M,S_\infty) \models \mathsf{WKL}_0$.

Proof Construct $S_0 \subseteq S_1 \subseteq \cdots$ as follows: $S_0 = S$, and

$$S_{(n,m)+1} = S^T_{(n,m)}, \text{ where } T \text{ is the } m\text{-th infinite tree in } S_n(\subseteq S_{(n,m)}).$$

Here, $(n,m) = \frac{(n+m)(n+m+1)}{2} + n$, and so $(n,m) \geq n$. Finally, let $S_\infty = \bigcup_{i\in\omega} S_i$.
It is clear from the definition that this is the desired set. □

Theorem 2.10 (Harrington)

For any Π^1_1 sentence σ, $\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{RCA}_0 \vdash \sigma$.

Proof Suppose σ is a Π^1_1 sentence that is not provable in RCA_0. By Gödel's completeness theorem, there exists a countable model $(M,S) \models \mathsf{RCA}_0 + \neg\sigma$. Now, $\neg\sigma$ can be expressed as $\exists X \varphi(X)$ with $\varphi \in \Pi^1_0$. Then there exists $A \in S$ such that $(M,S) \models \mathsf{RCA}_0 + \varphi(A)$. By constructing S_∞ by Lemma 2.9, we have $(M,S_\infty) \models \mathsf{WKL}_0 + \varphi(A)$. Note that since $\varphi(X)$ is arithmetical, the truth value of $\varphi(A)$ depends only on M and A. Therefore, $(M,S_\infty) \models \mathsf{WKL}_0 + \neg\sigma$, which implies $\mathsf{WKL}_0 \nvdash \sigma$. □

Summary

Main Lemma 2.9 (for Harrington Thm)

Let $\mathfrak{M} = (M, S)$ be a countable model of RCA_0. Then $\exists\, S_1 \supset S$ s.t. $(M, S_1) \models \mathsf{WKL}_0$.

- Let $\mathcal{T}_{\mathfrak{M}} = \{T \in S : \mathfrak{M} \models T \text{ is an infinite binary tree } \}$.
- Take any $T \in \mathcal{T}_{\mathfrak{M}}$ and find an $\mathfrak{M}$-generic path G (i.e., any $\mathfrak{M}$-definable dense $D \subset \mathcal{T}_{\mathfrak{M}}$ includes a tree with path G).
- Let $M[G] = (M, S_G)$, where S_G is the set of sets Δ^0_1-definable with parameters from $M \cup S \cup \{G\}$.
- Then $M[G] \models \mathsf{RCA}_0 \wedge T$ has an infinite path. (Note that a generic path does not break Σ^0_1 induction.)
- Repeating this process until we add all the necessary paths to S, we finally get a model of WKL_0.

Harrington (unpublished)

$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \sigma$ for $\sigma \in \Pi^1_1$.

Simpson-T.-Yamazaki (2002)

$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \sigma$ for $\sigma \equiv \forall X \exists ! Y \varphi(X, Y)$ with φ arithmetic.

Lemma for STY

Let $\mathfrak{M} = (M, S)$ be a countable model of RCA_0 and $A \in S$. Then $\exists\ S_1, S_2$ s.t.

1. $S_1 \cap S_2 = (A) = \{X \subseteq M \mid \mathfrak{M} \models X \leq_{\mathrm{T}} A\}$,
2. $(M, S_i) \models \mathsf{WKL}_0$, $i = 1, 2$,
3. (M, S_1) and (M, S_2) satisfy the same sentences of $\mathcal{L}_2(M \cup \{A\})$.

§3. STY Theorem

Theorem 3.1 (STY theorem)

For any sentence σ in the form $\forall X \exists! Y \varphi(X, Y)$ (where $\varphi(X, Y)$ is arithmetic),

$$\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{RCA}_0 \vdash \sigma,$$

where $\exists! Y \varphi(X, Y)$ means $\exists Y \varphi(X, Y) \wedge \forall Y_1 \forall Y_2 (\varphi(X, Y_1) \wedge \varphi(X, Y_2) \to Y_1 = Y_2)$.

A key to the proof of this theorem is the following lemma.

Lemma 3.2

Let $\mathfrak{M} = (M, S)$ be a countable nonstandard model of RCA_0 with $A \in S$. Then, there exist sets S_1 and S_2 satisfying the following conditions:

1. $S_1 \cap S_2 = \mathrm{Rec}^{\mathfrak{M}}(A) = \{X \subseteq M \mid \mathfrak{M} \models X \leq_T A\}$
2. $(M, S_i) \models \mathsf{WKL}_0$, for $i = 1, 2$.
3. (M, S_1) and (M, S_2) satisfies the same sentences in $\mathcal{L}_2(M \cup \{A\})$.

In the above lemma, it is not necessary that S contains $S_1 \cup S_2$. Also, since elements of S other than A are not essentially used, it is sufficient for the lemma that $(M, \{A\})$ is a countable model of Σ^0_1 induction. We first assume the lemma to prove the main theorem.

Proof of Theorem 3.1 Suppose $\mathsf{WKL}_0 \vdash \forall X \exists! Y \varphi(X, Y)$ with an arithmetic formula $\varphi(X, Y)$. For contradiction, assume $\mathsf{RCA}_0 \nvdash \forall X \exists! Y \varphi(X, Y)$. By the completeness theorem, there exists a countable model $\mathfrak{M} = (M, S)$ of RCA_0 such that

$$(M, S) \models \neg \forall X \exists! Y \varphi(X, Y).$$

Consequently, there exists some $A \in S$ such that either

(i) $(M, S) \models \exists Y_1 \exists Y_2 (\varphi(A, Y_1) \wedge \varphi(A, Y_2) \wedge Y_1 \neq Y_2)$, or

(ii) $(M, S) \models \forall Y \neg \varphi(A, Y)$.

Case (i) There exist $B_1, B_2 \in S$ such that $(M, S) \models \varphi(A, B_1) \wedge \varphi(A, B_2) \wedge B_1 \neq B_2$. By Lemma 2.9 (Harrington's lemma), there exists $S' \supseteq S$ such that $(M, S') \models \mathsf{WKL}_0$. Since (M, S) and (M, S') agree on first-order parts, they validate the same arithmetic formulas. Hence, $(M, S') \models \varphi(A, B_1) \wedge \varphi(A, B_2) \wedge B_1 \neq B_2$. However, since $\mathsf{WKL}_0 \vdash \forall X \exists! Y \varphi(X, Y)$, we have $(M, S') \models \forall X \exists! Y \varphi(X, Y)$, a contradiction.

Case (ii) By Lemma 3.2, there exist sets S_1 and S_2 such that

(a) $S_1 \cap S_2 = \mathsf{Rec}^{\mathfrak{M}}(A)$,

(b) $(M, S_i) \models \mathsf{WKL}_0$,

(c) (M, S_1) and (M, S_2) satisfy the same sentences of $\mathcal{L}_2(M \cup \{A\})$.

From (b) and $\mathsf{WKL}_0 \vdash \forall X \exists! Y \varphi(X, Y)$, there exists a unique $B_i \in S_i$ such that $(M, S_i) \models \varphi(A, B_i)$ for each $i = 1, 2$. By (c), for any $n \in M$,

$$\begin{aligned} n \in B_1 &\Leftrightarrow (M, S_1) \models \exists Y(\varphi(A, Y) \land n \in Y) \\ &\Leftrightarrow (M, S_2) \models \exists Y(\varphi(A, Y) \land n \in Y) \\ &\Leftrightarrow n \in B_2 \end{aligned}$$

Therefore, $B_1 = B_2$ and thus $B_1 \in S_1 \cap S_2$. From (a), $B_1 \in \mathsf{Rec}^{\mathfrak{M}}(A)$. Since (M, S) is a model of RCA_0 and $B_1 \in S$, $(M, S) \models \exists Y \varphi(A, Y)$, a contradiction. □

In the following, we will introduce several new concepts such as a generic sequence, to proceed with the proof of Lemma 3.2.

First, let us consider $\mathfrak{M} = (M, S)$ as a countable nonstandard model of WKL_0 [4]. Take any $A \in S$ and consider the formulas involving it. If $\varphi(X, A)$ is a Π^0_1 formula with a unique free variable X and a parameter A, the set $\{X \in S \mid \mathfrak{M} \models \varphi(X, A)\}$ is called a $\Pi^{0,A}_1$ **class** in $\mathfrak{M}$. Note that a set $P \subseteq S$ is a $\Pi^{0,A}_1$ class iff there exists a binary tree $T \subseteq 2^{<M}$ recursive in A such that $P = [T]$. Here, $[T]$ represents the set of all infinite paths through a tree T.

From now on, the display of parameter A is omitted due to complexity in description. By $\langle P_e \mid e \in M \rangle$, we denote a computable enumeration of all Π^0_1 classes. Formally, using the Π^0_1 satisfaction predicate $\mathrm{Sat}_{\Pi^0_1}(x, X)$, we define it as: for any $e \in M, X \in S$,

$$X \in P_e \Leftrightarrow \mathfrak{M} \models \mathrm{Sat}_{\Pi^0_1}(e, X).$$

We also write $P_e(X)$ for $X \in P_e$.

[4] Note that in the claim of Lemma 3.2, $\mathfrak{M} = (M, S)$ was a countable nonstandard model of RCA_0.

Definition 3.3

For an M-finite subset $p \subseteq M \times M^{<M}$ (denoted as $p \subseteq_{\mathrm{fin}} M \times M^{<M}$) [5], a sequence of sets $\langle X_n \mid n \in M \rangle$ **meets** p, if for every $(e, \langle n_1, \cdots, n_k \rangle) \in p$,

$$X_{n_1} \oplus \cdots \oplus X_{n_k} \in P_e,$$

where $X_{n_1} \oplus \cdots \oplus X_{n_k} = \{(x,1) \mid x \in X_{n_1}\} \cup \{(x,2) \mid x \in X_{n_2}\} \cdots \cup \{(x,k) \mid x \in X_{n_k}\}$. The condition $X_{n_1} \oplus \cdots \oplus X_{n_k} \in P_e$ is also expressed as $P_e(X_{n_1}, \cdots, X_{n_k})$.

Definition 3.4

Define a p.o. set $(\mathbb{P}^{\mathfrak{M}}, \leq)$ as follows:

$$\mathbb{P}^{\mathfrak{M}} = \{p \subseteq_{\mathrm{fin}} M \times M^{<M} \mid \text{there exists } \langle X_n \mid n \in M \rangle \in S^M \text{ that meets } p\},$$

and the order $p \leq q$ on $\mathbb{P}^{\mathfrak{M}}$ is defined as $p \supseteq q$. [6]

[5] $M^{<M}$, i.e., $\mathrm{Seq}^{\mathfrak{M}}$, includes all M-finite sequences from M.
[6] The reason why the order is the reverse inclusion is that when $q \subseteq p$, p has more conditions, hence fewer sequences meet it.

In WKL_0, the condition that "there exists $\langle X_n \mid n \in M\rangle \in S^M$ that meets p" can be rephrased as the existence of an infinite path in an infinite tree, since the part "(something) meets p" is a Π^0_1 condition. Thus by compactness, the whole condition can be expressed by a Π^0_1 formula.

Furthermore, $p(\subseteq_{\mathrm{fin}} M \times M^{<M})$ can be considered an element of M, so $\mathbb{P}^{\mathfrak{M}}$ can be regarded as a Π^0_1 subset of M.

Henceforth, unless otherwise stated, $\mathbb{P}^{\mathfrak{M}}$ will simply be referred to as $\mathbb{P}$.

A sequence $\langle G_n \mid n \in M\rangle$ is said to be a **generic sequence** if for any dense subset $D \in \mathrm{Def}(\mathfrak{M})$ of $\mathbb{P}$, there exists a $p \in D$ that $\langle G_n \mid n \in M\rangle$ meets. [7]

[7] Even if some G_n does not belong to S, the definition remains valid as long as their existence does not violate the Σ^0_1 induction.

Lemma 3.5

For any $p_0 \in \mathbb{P}$, there exists a generic sequence $\langle G_n \mid n \in M \rangle$ that meets p_0. [8]

Proof For each $p \in \mathbb{P}$, let $[[p]]$ be the collection of sequences $\langle X_n \mid n \in M \rangle \in S^M$ that meet p, which is a Π^0_1 class. By Lemma 2.3, there exists a $\mathrm{Def}(\mathfrak{M})$-generic filter G in $\mathbb{P}$ containing p_0. Obviously, G is closed under finite unions by noticing the order $\leq$ of $\mathbb{P}$. Thus, for any finite set $\{p_1, \ldots, p_k\} \subseteq G$, $[[p_1]] \cap \cdots \cap [[p_k]] = [[p_1 \cup \cdots \cup p_k]] \neq \varnothing$. Then, by compactness (Lemma 1.2(2)), $\bigcap_{p \in G}[[p]] \neq \varnothing$. (In fact, $\bigcap_{p \in G}[[p]]$ has a unique element, which can be shown in a way similar to the proof of Lemma 2.4.)

Now, take an element $\langle G_n \mid n \in M \rangle$ from $\bigcap_{p \in G}[[p]]$. Then, it meets p_0, since $p_0 \in G$. Moreover, for any dense set $D \in \mathrm{Def}(\mathfrak{M})$, there exists $p \in D \cap G$, and thus $\langle G_n \rangle$ meets p. Therefore, $\langle G_n \mid n \in M \rangle$ is a demanded generic sequence. □

[8] This does not necessarily exist within $\mathfrak{M} = (M, S)$.

For any generic sequence $\langle G_n \mid n \in M \rangle$, we will show that any finite subsequence $\langle G_{n_1}, \cdots, G_{n_k} \rangle$ is generic in the sense of Jockusch-Soare[9].

First of all, let $\mathbb{P}_0 = \{P_e \mid e \in M, P_e \neq \varnothing\}$. Identifying a set with the binary sequence expressing its characteristic function, $\mathbb{P}_0$ can be seen as a collection of infinite trees. [10]

Then, define the **order** by inclusion, as well as **density**, etc., in the same way as in §1.

$G = \langle G_{n_1}, \cdots, G_{n_k} \rangle$ is a **generic path**, if for any dense set $D \subseteq \mathbb{P}_0$ definable in $(M, \mathrm{Rec}^{\mathfrak{M}}(A))$, there exists $P_e \in D$ such that $P_e(G)$ holds.

We now prove the following lemma:

Lemma 3.6

For any generic sequence $\langle G_n \mid n \in M \rangle$ in $\mathbb{P}$, any finite subsequence G is a generic path with respect to $\mathbb{P}_0$.

[9] referred to as a generic path when distinction is needed
[10] $P_e \neq \varnothing$ implies that the tree corresponding to P_e has an infinite path.

Proof Take any generic sequence $\langle G_n \mid n \in M \rangle$ in $\mathbb{P}$. Let $G = \langle G_{n_1}, \cdots, G_{n_k} \rangle$ be a finite subsequence of $\langle G_n \rangle$.

Now, define a function $\Psi : \mathbb{P} \to \mathbb{P}_0$ as follows: For $p \in \mathbb{P}$, consider the set of subsequences $X = \langle X_{n_1}, \cdots, X_{n_k} \rangle$ made from sequences $\langle X_n \mid n \in M \rangle$ that meet p.
Within a model $\mathfrak{M} = (M, S)$ of WKL$_0$, this can be described as $\{\langle X_{n_1}, \cdots, X_{n_k} \rangle \mid \exists \langle X_n \mid n \in M \rangle$ that meets $p\}$, which forms a Π^0_1 class; thus, it is P_f for some $f \in M$. Define $\Psi(p)$ to be such P_f.

To show that G is a generic path, take any dense set $D_0 \subset \mathbb{P}_0$. It suffices to show that there exists $P_{f_0} \in D_0$ such that $G \in P_{f_0}$. First, define $D \subset \mathbb{P}$ as follows:

$$p \in D \Leftrightarrow \Psi(p) \in D_0$$

Now, take any $p \in \mathbb{P}$ and suppose $\Psi(p) = P_f$. Due to the density of D_0, there exists $P_e \leq P_f$ such that $P_e \in D_0$. So, if $q = p \cup \{(e, \langle n_1, \cdots, n_k \rangle)\}$, then $q \in D$ since $\Psi(q) = P_e \in D_0$. Therefore, D is also dense. Consequently, there exists $p \in D$ that the generic sequence $\langle G_n \mid n \in M \rangle$ meets. Thus, letting $P_{f_0} = \Psi(p)$, we have $P_{f_0} \in D_0$. Also, $G \in P_{f_0}$ by the definition of Ψ. Therefore, G is a generic path with respect to $\mathbb{P}_0$. □

Lemma 3.7

For any generic sequence $\langle G_n \mid n \in M \rangle$, $(M, \{G_n \mid n \in M\})$ becomes a model of WKL_0.

Proof First, $(M, \{G_n \mid n \in M\})$ satisfies Σ^0_1 induction by Lemma 1.7 since each finite subsequence G is a generic path (Lemma 3.6). Consequently, according to several lemmas for Harrington's theorem (particularly Lemma 1.9), there exists a countable set $S \subseteq \mathcal{P}(M)$ such that $\{G_n \mid n \in M\} \subseteq S$ and $\mathfrak{M} = (M, S)$ becomes a model of WKL_0. [11] It suffices to show that the following (Σ^0_1-SP) holds in $(M, \{G_n \mid n \in M\})$[12]

$$\forall x \neg(\varphi(x) \land \psi(x)) \to \exists X \forall x ((\varphi(x) \to x \in X) \land (\psi(x) \to x \notin X)),$$

where $\varphi(x), \psi(x)$ are Σ^0_1 formulas.

[11] Finally, we want to show $(M, \{G_n \mid n \in M\})$ is a model of WKL_0, but it is easier to argue with a countable set S between $(M, \{G_n \mid n \in M\})$ and $\mathcal{P}(M)$.

[12] See Lemma 3.6 in Part 6

Let $G_{n_1}, \cdots, G_{n_k}$ be parameters appearing in at least one of the Σ^0_1 formulas $\varphi(x)$, $\psi(x)$. We assume $(M, \{G_n \mid n \in M\}) \models \forall x\, \neg(\varphi(x) \wedge \psi(x))$. Then it is also true in $\mathfrak{M} = (M, S)$, since $\forall x\, \neg(\varphi(x) \wedge \psi(x))$ is an arithmetical formula, and hence independent of any sets other than the parameters.

By the way, since $\forall x\, ((\varphi(x) \to x \in X) \wedge (\psi(x) \to x \notin X))$ is a Π^0_1 formula, there exists $e_1 \in M$ such that:

$$X \oplus G_{n_1} \oplus \cdots \oplus G_{n_k} \in P_{e_1} \Leftrightarrow \forall x\, ((\varphi(x) \to x \in X) \wedge (\psi(x) \to x \notin X)).$$

Then, $(M, S) \models \exists X\, (X \oplus G_{n_1} \oplus \cdots \oplus G_{n_k} \in P_{e_1})$, since $(M, S) \models \Sigma^0_1$-SP. Due to the compactness (Lemma 3.2 (2)), this is also a Π^0_1 formula, and there exists $e_2 \in M$ such that:

$$(M, S) \models G_{n_1} \oplus \cdots \oplus G_{n_k} \in P_{e_2} \leftrightarrow \exists X\, (X \oplus G_{n_1} \oplus \cdots \oplus G_{n_k} \in P_{e_1})$$

Define $D \subseteq \mathbb{P}$ as follows:

$$D = \{p \in \mathbb{P} \mid p \cup \{(e_2, \langle n_1, \cdots, n_k \rangle)\} \notin \mathbb{P} \vee \exists m\, (e_1, \langle m, n_1, \cdots, n_k \rangle) \in p\}.$$

To show D is dense, take any $p \in \mathbb{P}$. If $p \cup \{(e_2, \langle n_1, \cdots, n_k \rangle)\} \notin \mathbb{P}$, then p is already in D. Otherwise, there exists a sequence that meets $p \cup \{(e_2, \langle n_1, \cdots, n_k \rangle)\}$. This implies that there is some m such that a sequence meeting $p \cup \{(e_1, \langle m, n_1, \cdots, n_k \rangle)\}$ also exists. Let $q = p \cup \{(e_1, \langle m, n_1, \cdots, n_k \rangle)\}$. Then $q \leq p$ and q is in D.

The generic sequence $\langle G_n \mid n \in M \rangle$ meets some $p \in D$. If we assume that $p \cup \{(e_2, \langle n_1, \cdots, n_k \rangle)\} \notin \mathbb{P}$, then $G_{n_1} \oplus \cdots \oplus G_{n_k} \notin P_{e_2}$, which also denies $\exists X\, (X \oplus G_{n_1} \oplus \cdots \oplus G_{n_k} \in P_{e_1})$, a contraction. Thus, it must hold that $\exists m\, (e_1, \langle m, n_1, \cdots, n_k \rangle) \in p$. Then, $\exists m\, G_m \oplus G_{n_1} \oplus \cdots \oplus G_{n_k} \in P_{e_1}$, which leads to $(M, S) \models \forall x\, ((\varphi(x) \to x \in G_m) \land (\psi(x) \to x \notin G_m))$. Since $\forall x\, ((\varphi(x) \to x \in G_m) \land (\psi(x) \to x \notin G_m))$ is an arithmetical formula, it follows that

$$(M, \{G_n \mid n \in M\}) \models \forall x\, ((\varphi(x) \to x \in G_m) \land (\psi(x) \to x \notin G_m)).$$

Thus, (Σ^0_1-SP) holds in $(M, \{G_n \mid n \in M\})$. □

Lemma 3.8

Let $p \in \mathbb{P}$ and $m \in M$. Suppose that $C = X_m$ for any sequence $\langle X_n \mid n \in M \rangle$ that meets p. Then, $C \in \mathsf{Rec}^{\mathfrak{M}}(\varnothing)$. [13]

Proof Assume C satisfies the conditions of the lemma. For any $l \in M$, it holds that:

$$\begin{aligned} l \in C &\Leftrightarrow \forall \langle X_n \mid n \in M \rangle \, (\text{if } \langle X_n \mid n \in M \rangle \text{ meets } p, \text{ then } l \in X_m) \\ &\Leftrightarrow \exists \langle X_n \mid n \in M \rangle \, (\langle X_n \mid n \in M \rangle \text{ meets } p \text{ and } l \in X_m) \end{aligned}$$

Since the upper/lower expressions can be written as Σ^0_1/Π^0_1 formulas by compactness, $C \in \mathsf{Rec}^{\mathfrak{M}}(\varnothing)$ is established. □

[13] We have Assumed $p \in \mathbb{P}^A$ and incorporated the parameter A into all discussions, and hence $C \in \mathsf{Rec}^{\mathfrak{M}}(A)$ can be established.

Lemma 3.9

Assume (M, S) is a countable model of WKL_0. For any generic sequence $\langle G_n \mid n \in M\rangle$, $\{G_n \mid n \in M\} \cap S = \mathrm{Rec}^{\mathfrak{M}}(\varnothing)$. [14]

Proof For $B \in S - \mathrm{Rec}^{\mathfrak{M}}(\varnothing)$ and $m \in M$, we set

$$D_{B,m} = \{p \in \mathbb{P} \mid \forall \langle X_n \mid n \in M\rangle (\text{if } \langle X_n \mid n \in M\rangle \text{ meets } p, \text{ then } X_m \neq B)\}.$$

To show that $D_{B,m}$ is dense, take any $p \in \mathbb{P}$. Since $B \notin \mathrm{Rec}^{\mathfrak{M}}(\varnothing)$, by Lemma 3.8 there exists $\langle Z_n \mid n \in M\rangle$ that meets p with $Z_m \neq B$. Therefore, there exists l_1 such that $l_1 \in Z_m - B$ or $l_1 \in B - Z_m$. Assume $l_1 \notin B$ and take a Π^0_1 class $P_e = \{\langle X_n\rangle \mid l_1 \in X_m\}$, and set $q = p \cup \{(e, \langle m\rangle)\}$. (If $l_1 \in B$, then use $P_{e'} = \{\langle X_n\rangle \mid l_1 \notin X_m\}$ and set $q = p \cup \{(e', \langle m\rangle)\}$.) Since $\langle Z_n \mid n \in M\rangle$ meets q, it follows that $q \in \mathbb{P}$. Thus, $q \in D_{B,m}$ and $q \leq p$, which implies that $D_{B,m}$ is dense.

Next, take any generic sequence $\langle G_n \mid n \in M\rangle$. Since $D_{B,m}$ is dense, there exists a $p \in D_{B,m}$ that $\langle G_n\rangle$ meets, and so by the definition of $D_{B,m}$, we have $G_m \neq B$. Thus, $\{G_n \mid n \in M\} \cap S \subseteq \mathrm{Rec}^{\mathfrak{M}}(\varnothing)$. Finally, $\{G_n \mid n \in M\} \cap S \supseteq \mathrm{Rec}^{\mathfrak{M}}(\varnothing)$ follows from Lemma 3.7. □

[14] Incorporating the parameter A into discussions, $\mathrm{Rec}^{\mathfrak{M}}(\varnothing)$ should be replaced with $\mathrm{Rec}^{\mathfrak{M}}(A)$. Note that the generic sequence is determined dependent on $\mathfrak{M}$.

In the following, we will proceed with the forcing argument like in axiomatic set theory. We here fix a countable model $\mathfrak{M} = (M, S)$ of WKL_0 as a ground model for our discussion.

Definition 3.10

Regarding $\{X_n \mid n \in M\}$ as set constants, consider a sentence φ in $\mathcal{L}_2(M \cup \{X_n\})$. We say that $p \in \mathbb{P}$ **forces** φ, denoted by $p \Vdash \varphi$, if for any generic sequence $\langle G_n \mid n \in M \rangle$ that meets p, $(M, \{G_n\}) \models \varphi$, where the set constants X_n in φ are interpreted as G_n.

Then, the following fundamental lemma links the forcing relation and satisfaction relation.

Lemma 3.11

For any generic sequence $\langle G_n \mid n \in M \rangle$ and any sentence φ in $\mathcal{L}_2(M \cup \{X_n \mid n \in M\})$,

$$(M, \{G_n \mid n \in M\}) \models \varphi \iff \text{there exists a } p \text{ that meets } \langle G_n \mid n \in M \rangle \text{ with } p \Vdash \varphi.$$

The proof of this is quite lengthy, so only an outline will be given here. Since the basic discussion follows that of set theory's forcing method, those who find the explanation too brief may refer to literature such as [K. Kunen, Set Theory, North-Holland, 1980].

We first introduce another forcing relation $p \Vdash^* \varphi$, a formal alternative to $p \Vdash \varphi$. This is defined inductively based on the complexity of a formula φ as follows:

$$p \Vdash^* t \in X_m \Leftrightarrow \text{for any sequence } \langle X_n \rangle \text{ meeting } p, (M, \{X_n\}) \models t \in X_m$$

$$p \Vdash^* \neg\varphi \Leftrightarrow \neg\exists q \leq p \; q \Vdash^* \varphi$$

$$p \Vdash^* \varphi \land \psi \Leftrightarrow p \Vdash^* \varphi \text{ and } p \Vdash^* \psi$$

$$p \Vdash^* \forall x \varphi(x) \Leftrightarrow p \Vdash^* \varphi(m) \text{ for all } m \in M$$

$$p \Vdash^* \forall X \varphi(X) \Leftrightarrow p \Vdash^* \varphi(X_m) \text{ for all } m \in M$$

Other operations are defined by De Morgan's laws. Note that the right-hand side for the atomic formula $t \in X_m$ is Σ^0_1 within the model (M, S) of WKL_0, independent of the choice of the second-order part S. In general, $p \Vdash^* \varphi$ is an arithmetical relation.
Furthermore, the following claims can also be demonstrated.

Claim 1 If $p \Vdash^* \varphi$ and $q \leq p$, then $q \Vdash^* \varphi$.

Claim 2 For any generic sequence $\langle G_n \mid n \in M \rangle$,

$$(M, \{G_n \mid n \in M\}) \models \varphi \;\Leftrightarrow\; \text{there exists a } p \text{ that meets } \langle G_n \mid n \in M \rangle \text{ with } p \Vdash^* \varphi.$$

Claim 3 $p \Vdash \varphi$ is equivalent to $p \Vdash^* \neg\neg\varphi$.

Proof of Lemma 3.11 Obviously, $(M, \{G_n \mid n \in M\}) \models \varphi \ \Leftrightarrow$ $(M, \{G_n \mid n \in M\}) \models \neg\neg\varphi$.
By Claim 2, the latter is equivalent to the existence of a p that $\langle G_n \mid n \in M\rangle$ meets and that satisfies $p \Vdash^* \neg\neg\varphi$. From Claim 3, $p \Vdash^* \neg\neg\varphi$ is equivalent to $p \Vdash \varphi$, thereby proving the lemma. □

The following corollary is easily proved:

Corollary 3.12

If $p \nVdash \varphi$, then there exists some $q \leq p$ such that $q \Vdash \neg\varphi$.

Proof If $p \nVdash \varphi$, then there exists a generic sequence $\langle G_n \mid n \in M\rangle$ that meets p and satisfies $(M, \{G_n \mid n \in M\}) \models \neg\varphi$. By Lemma 3.11, there exists a q that $\langle G_n \mid n \in M\rangle$ meets and that satisfies $q \Vdash \neg\varphi$. Therefore, $p \cup q \leq p$, and $p \cup q \Vdash \neg\varphi$. □

Problem: Show that $p \Vdash \neg\varphi$ is equivalent to $\neg\exists q \leq p\ (q \Vdash \varphi)$.

Next, I would like to discuss the symmetry in models constructed by forcing methods.

Given a bijection $\pi : M \to M$ $(\in S)$, we define the following notation, for $p \in \mathbb{P}$ and a sentence φ in $\mathcal{L}_2(M \cup \{X_n \mid n \in M\})$,

$$\pi(p) = \{(e, \langle \pi(n_1), \cdots, \pi(n_k)\rangle) \mid (e, \langle n_1, \cdots, n_k\rangle) \in p\},$$
$$\pi(\varphi) = (\text{the formula obtained by replacing each } X_n \text{ in } \varphi \text{ with } X_{\pi(n)}).$$

It is easy to see that the following holds:

Lemma 3.13

(1) For any $p \in \mathbb{P}$ and sentence φ, $\quad p \Vdash \varphi \Leftrightarrow \pi(p) \Vdash \pi(\varphi)$.

(2) For any $p, q \in \mathbb{P}$, if $\mathrm{supp}(p) \cap \mathrm{supp}(q) = \varnothing$, then $p \cup q \in \mathbb{P}$. Here, $\mathrm{supp}(p) := \bigcup\{\{n_1, \cdots, n_k\} \mid (e, \langle n_1, \cdots, n_k\rangle) \in p\}$.

(3) (Symmetry) If $\langle G_n \mid n \in M\rangle$ is a generic sequence, then $\langle G_{\pi(n)} \mid n \in M\rangle$ is also a generic sequence.

With these preparations, the following lemma can be demonstrated:

Lemma 3.14

Any $p, q \in \mathbb{P}$ force the same sentences in $\mathcal{L}_2(M)$. That is, for any sentence φ in $\mathcal{L}_2(M)$, either $\forall p \in \mathbb{P}\,(p \Vdash \varphi)$ or $\forall p \in \mathbb{P}\,(p \nVdash \varphi)$ holds.

Proof By way of contradiction, take $p, q \in \mathbb{P}$ and φ in $\mathcal{L}_2(M)$ such that $p \Vdash \varphi$ and $q \nVdash \varphi$. From Corollary 4.12, there exists $q' \leq q$ such that $q' \Vdash \neg\varphi$. Choose a bijection $\pi : M \to M$ such that $\mathrm{supp}(\pi(p)) \cap \mathrm{supp}(q') = \varnothing$. Since φ is a sentence in $\mathcal{L}_2(M)$, $\pi(\varphi) = \varphi$. Therefore, $\pi(p) \Vdash \varphi$. Consequently, $\pi(p) \cup q' \Vdash \varphi$, which contradicts $q' \Vdash \neg\varphi$. □

From Lemma 3.11 and Lemma 3.14, the following corollary can be derived:

Corollary 3.15

For any sentence φ in $\mathcal{L}_2(M)$,

either $(M, \{G_n \mid n \in M\}) \models \varphi$ for any generic sequence $\langle G_n \mid n \in M \rangle$,
or $(M, \{G_n \mid n \in M\}) \not\models \varphi$ for any generic sequence $\langle G_n \mid n \in M \rangle$.

From Corollary 3.15 and several other lemmas, Lemma 3.2 can be proved.

Proof of Lemma 3.2 Consider a generic sequence $\langle G_n \mid n \in M \rangle$. From Lemma 3.9, there exists another generic sequence $\langle G'_n \mid n \in M \rangle$ that meets p such that $\{G_n \mid n \in M\} \cap \{G'_n \mid n \in M\} = \mathrm{Rec}^{\mathfrak{M}}(A)$. Let us define $S_1 := \{G_n \mid n \in M\}$ and $S_2 := \{G'_n \mid n \in M\}$. Then, by Lemma 3.7, each (M, S_i) forms a model of WKL_0. Furthermore, according to Corollary 3.15, (M, S_1) and (M, S_2) satisfy the same sentences in $\mathcal{L}_2(M \cup \{A\})$.

□

Application

Fundamental Theorem of Algebra (FTA)

Any monic complex polynomial has a unique factorization into linear terms,

$$\mathsf{RCA}_0 \vdash \forall p(x) \in \mathbb{C}[x]\ \exists! \overrightarrow{\alpha} \in \mathbb{C}^{<\mathbb{N}}\ p(x) = \prod_i (x - \alpha_i).$$

- FTA is proved by combining two non-standard proof methods.
  1. A self-embedding theorem (T. 1997).

$$V = (M, S) \qquad\qquad {}^*V = ({}^*M, {}^*S)$$

$$f : \mathbb{Q}[x] \to (\mathbb{C} \cap \mathbb{Q}^2)^{<\mathbb{N}} \quad \Longrightarrow \quad {}^*f : \{p_i\}_{i<a} \to ({}^*\mathbb{C} \cap {}^*\mathbb{Q}^2)^{<b}$$

$\|$

$\{p_i\}_{i \in M}$ with infinite repetition and $f(p_i)$ is a list of rational approx. of the roots of p_i with error $< 2^{-i}$

$(a, b \in {}^* M - M, f = {}^* f \cap M)$

$\Downarrow$

$${}^*f(p_{i_j}) \lceil M \text{ is the list of roots of } p \quad \Longleftarrow \quad p(x) = \{p_{i_n}\}_{n \in M} = p_{i_j} \lceil M (\exists j \notin M)$$

  2. STY theorem: $\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{RCA}_0 \vdash \sigma$ for $\sigma \equiv \forall X \exists! Y \varphi(X, Y)$ with $\varphi \in \Sigma^1_0$.

Outer model method for ACA_0

- Suppose $(M,S) \models \mathsf{ACA}_0$, countable, $M \neq \omega$.
  Then $\exists {}^*M \supsetneq_e M$ $\exists {}^*S$ such that $({}^*M, {}^*S) \models \mathsf{ACA}_0$, $S =^* S\lceil M$ and
  $\exists * : S \to^* S$ $\forall \varphi(x,X) \in \Sigma^1_1 \cup \Pi^1_1$

$$(M,S) \models \varphi(m,A) \leftrightarrow ({}^*M, {}^*S) \models \varphi(m, {}^*A).$$

- This easily follows from

Theorem(Gainfman)

Every model M of PA has a conservative extension K, i.e., (the sets definable in K) $K\lceil M$ = the sets definable in M.

Applications

$\mathsf{ACA}_0 \vdash$ Any Cauchy sequence converges.

Proof.

$$V = (M, S) \qquad {}^*V = ({}^*M, {}^*S)$$

$$\{a_i\}_{i\in M} \text{ a Cauchy sequence} \quad \Longrightarrow \quad {}^*(\{a_i\}_{i\in M}) = \{({}^*a)_i\}_{i\in {}^*M}$$

Pick $j \in {}^*M - M$.

$$\forall n \exists m \forall k > m \mid a_k - b \mid < 2^{-n} \quad \Longleftarrow \quad \begin{array}{c} \forall n \in M \exists m \in M \forall k > m \\ \mid ({}^*a)_k - ({}^*a)_j \mid < 2^{-n} \\ {}^*a \approx ({}^*a)_j \end{array}$$

Final report problems

Choose one or more problems below. Prepare a self-contained report of 4 to 10 pages on them and submit it to me at tanaka.math@tohoku.ac.jp by Jan. 9, 2026.

1. How far can the STY theorem be extended? Can you prove that it holds for the sentences obtained from $\forall X \exists ! Y \varphi(X, Y)$ (where $\varphi(X, Y)$ is arithmetic) by applying $\vee, \wedge, \forall x, \exists y$ and $\forall X$ (cf. Shore, JSL 2023). Can you extend it further?
2. Which axioms are needed to prove the STY or Harrington's conservation results?
3. For which class of formulas is Δ^1_1-CA$_0$ conservative over ACA$_0$?
4. Can you weaken the assumption of the self-embedding theorem of WKL$_0$ from IΣ^0_1 to IΣ^0_0+ exp.
5. Discuss other conservation results that you are interested in.

Thank you for your attention!

Please check our WeChat sometimes.

The solution set of the homework problems will be uploaded there next week.

I will give an online course in the next semester.

Appendix

Claim 2 For any generic sequence $\langle G_n \mid n \in M \rangle$,

$$(M, \{G_n \mid n \in M\}) \models \varphi \;\Leftrightarrow\; \text{there exists a } p \text{ that meets } \langle G_n \mid n \in M \rangle \text{ with } p \Vdash^* \varphi.$$

Proof By induction on the complexity of φ. The essential step is negation.

($\Leftarrow$) Suppose $\langle G_n \mid n \in M \rangle$ meets p, and $p \Vdash^* \neg\varphi$. For contradiction, assume $(M, \{G_n \mid n \in M\}) \models \varphi$. By the inductive hypothesis, there exists a q meeting $\langle G_n \mid n \in M \rangle$ such that $q \Vdash^* \varphi$. Since $\langle G_n \mid n \in M \rangle$ meets $p \cup q$, and $p \cup q \in \mathbb{P}$, and since $p \cup q \leq q$, $p \cup q \Vdash^* \varphi$ holds, contradicting the definition of $p \Vdash^* \neg\varphi$.

($\Rightarrow$) Suppose $(M, \{G_n \mid n \in M\}) \models \neg\varphi$. Let $D = \{p \mid p \Vdash^* \varphi \text{ or } p \Vdash^* \neg\varphi\}$. From the definition of $p \Vdash^* \neg\varphi$, it is easy to see that D is dense. Therefore, there exists a $p \in D$ meeting $\langle G_n \mid n \in M \rangle$. If $p \Vdash^* \varphi$, then by the inductive hypothesis, $(M, \{G_n \mid n \in M\}) \models \varphi$, which contradicts the premise.
If not $p \Vdash^* \varphi$, then by the definition of D, $p \Vdash^* \neg\varphi$. □

Claim 3 $p \Vdash \varphi$ is equivalent to $p \Vdash^* \neg\neg\varphi$.

Proof

($\Leftarrow$) Assume $p \Vdash^* \neg\neg\varphi$. For contradiction, also assume that $p \Vdash \varphi$ is false. Then, there exists a generic sequence $\langle G_n \mid n \in M\rangle$ that meets p with $(M, \{G_n \mid n \in M\}) \models \neg\varphi$.
From Claim 2, there exists a q that meets $\langle G_n \mid n \in M\rangle$ with $q \Vdash^* \neg\varphi$.
Hence, $p \cup q \Vdash^* \neg\varphi$, which contradicts the assumption of $p \Vdash^* \neg\neg\varphi$.

($\Rightarrow$) To show this, assume $p \Vdash \varphi$. For any $q \leq p$, by Lemma 3.5, there exists a generic sequence $\langle G_n \mid n \in M\rangle$ that meets q. Since this sequence also meets p, $(M, \{G_n \mid n \in M\}) \models \varphi$. From Claim 2, there exists an r that meets $\langle G_n \mid n \in M\rangle$ such that $r \Vdash^* \varphi$. Since $\langle G_n \mid n \in M\rangle$ also meets $q \cup r$, and $q \cup r \Vdash^* \varphi$, it follows that $\forall q \leq p \exists r \leq q\ r \Vdash^* \varphi$ holds. Finally, $p \Vdash^* \neg\neg\varphi$ is equivalent to $\forall q \leq p \exists r \leq q\ r \Vdash^* \varphi$. □