# *Topics in Applied Math:*
# Logic and Foundations of Mathematics

## Part 7. Models of second order arithmetic

**Kazuyuki Tanaka**

BIMSA

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清华大学 求真书院
Qiuzhen College, Tsinghua University

## Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First-order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Reverse mathematics**
- **Part 7. Second-order arithmetic and non-standard methods**

## Part 7. Schedule

- Dec. 3, (1) A self-embedding theorem I
- **Dec. 5, (2) A self-embedding thm II and Harrington's conservation thm**
- Dec. 10, (3) STY theorem I
- Dec. 12, (4) STY theorem II

# Part 7. §1. A self-embedding theorem of $\mathsf{WKL}_0$

## Theorem 1.1 (Self-Embedding Theorem)

Let $\mathfrak{M} = (M, S)$ be a countable model of $\mathsf{WKL}_0$ with $M \neq \omega$. Then, there exists a proper initial segment $I$ of $M$ such that $\mathfrak{M}\lceil I = (I, S\lceil I)$ is isomorphic to $\mathfrak{M}$. Here, $S\lceil I = \{X \cap I \mid X \in S\}$.

We first prove the following lemma, which will be frequently used later.

## Lemma 1.2 (Compactness in $\mathsf{WKL}_0$)

(1) For any $\Pi^0_1$ formula $\varphi(X)$, there exists a $\Pi^0_1$ formula $\hat{\varphi}$ such that $\mathsf{WKL}_0$ proves:

$$\hat{\varphi} \leftrightarrow \exists X\, \varphi(X).$$

(2) For any $\Pi^0_1$ formula $\varphi(k, X)$, $\mathsf{WKL}_0$ proves:

$$\forall n\, \exists X\, \forall k < n\, \varphi(k, X) \rightarrow \exists X\, \forall k\, \varphi(k, X).$$

We define $G$-$\Sigma^0_1$ **formulas** or simply $G$ **formulas** by generalizing $\Sigma^0_1$ formulas as follows. The $G$ formulas are obtained from $\Sigma^0_1$ formulas by using $\wedge, \vee$, bounded universal quantifier $\forall x < y$ and unbounded existential quantifier $\exists x$, and set quantifiers $\forall X, \exists X$.
In $\mathsf{WKL}_0$, we can prove that a $G$ formula is equivalent to a $\Sigma^0_1$ formula.

### Definition 1.3 ($G$ formulas in $\mathsf{RCA}_0$)

A sequence $G_0 \subset G_1 \subset G_2 \subset \cdots$ of sets of $\mathcal{L}^2_{\mathrm{OR}}$-formulas is defined inductively modulo 4 as follows: for each $e \in \mathbb{N}$,

$$\begin{aligned}
G_0 &= \{\text{finite disjunctions } (\vee) \text{ of atomic formulas or their negations}\},\\
G_{4e+1} &= \{\exists x\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e} \text{ formulas}\} \cup G_{4e},\\
G_{4e+2} &= \{\forall x < y\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+1} \text{ formulas}\} \cup G_{4e+1},\\
G_{4e+3} &= \{\exists X\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e+2} \text{ formulas}\} \cup G_{4e+2},\\
G_{4e+4} &= \{\forall X\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+3} \text{ formulas}\} \cup G_{4e+3}.
\end{aligned}$$

Finally, we set $\mathbf{G} = \bigcup_{e\in\mathbb{N}} G_e$. The formulas in G are called $G$ **formulas**.

In the following, we will define Sat for $G$ formulas.

From now on, a structure $\mathfrak{M} = (M, S)$ is denoted by $V$. Then, for each $p \in M$, let $M_p = \{a \in M \mid \mathfrak{M} \models a < p\}$, $S_p = \{X \cap M_p \mid X \in S\}$ and $V_p = (M_p, S_p)$.

For any formula $\varphi$ in $\mathcal{L}^2_{\mathrm{OR}}$, let $\varphi^{V_p}$ be a formula obtained by restricting the ranges of variables to $V_p = (M_p, S_p)$. More precisely, in $\varphi^{V_p}$, quantification over numbers is bounded by $p$, and quantification over sets is also considered as ranging binary sequences of length $p$, which can be coded by numbers $< 2^p$. So, $\varphi^{V_p}$ can be regarded as a $\Delta^0_1$ formula in $V$. Thus, by using $\mathsf{Sat}_{\Sigma^0_1}$, we define the **satisfaction predicate** $\mathsf{Sat}^p(z, \xi)$ as follows:

$$\mathsf{Sat}^{\mathbf{p}}(\ulcorner \varphi \urcorner, \xi) \equiv \mathsf{Sat}_{\Sigma^0_1}(\ulcorner \varphi^{V_p} \urcorner, \xi \restriction V_p), \quad \text{i.e., } \varphi(\xi)^{V_p}.$$

Here, $\xi$ is a finite function that assigns elements of $M_p \cup S_p$ to free variables in $\varphi$, and $\xi \restriction V_p$ is the assignment obtained from $\xi$ by restricting its values to $V_p$.

We also remark that a variable $z$ in $\mathsf{Sat}^p(z, \xi)$ can potentially express a non-standard number. In $V$, we can verify that $\mathsf{Sat}^p$ satisfies Tarski's truth definition clauses (cf. Theorem IV.2.26 in [P. Hájek and P. Pudlák, *Metamathematics of First-oder Arithmetic*, Springer, 1993.]).

Next, we define the **satisfaction relation for $G$ formulas** as follows:

## Definition 1.4

For each $z \in G$, define the satisfaction relation $\mathrm{Sat}(z, \xi)$ as follows:

$$\mathrm{Sat}(z, \xi) \leftrightarrow \exists p\, \mathrm{Sat}^p(z, \xi \restriction V_p).$$

For simplicity, we abbreviate $\mathrm{Sat}^p(z, \xi \restriction V_p)$ as $\mathrm{Sat}^p(z, \xi)$.
In the following, we identify a formula with its code.

## Lemma 1.5

In a model $V$ of WKL$_0$, $\mathrm{Sat}(z, \xi)$ satisfies Tarski's truth definition clauses for $G$ formulas.

**Proof idea.** In fact, if $z$ is $\Sigma^0_1$, $\mathrm{Sat}(z, \xi) \Leftrightarrow \exists p\, \mathrm{Sat}^p(z, \xi) \Leftrightarrow \exists p\, z(\xi)^{V_p} \Leftrightarrow z(\xi)$.
The critical case is $z = \forall X\, z'$ (where $z'$ is a $G$ formula).

$$\begin{aligned}\mathrm{Sat}\left(\forall X\, z', \xi\right) &\Leftrightarrow \exists p\, \mathrm{Sat}^p\left(\forall X\, z', \xi\right) \Leftrightarrow \exists p\, \forall U\, \mathrm{Sat}^p\left(z', \xi \cup \{(X, U)\}\right) \\ &\Leftrightarrow \forall U\, \exists p\, \mathrm{Sat}^p\left(z', \xi \cup \{(X, U)\}\right) \qquad (\Leftarrow \text{ by compactness (Lemma 1.2(2))}) \\ &\Leftrightarrow \forall U\, \mathrm{Sat}\left(z', \xi \cup \{(X, U)\}\right),\end{aligned}$$

where $\xi \cup \{(X, U)\}$ is an extension of $\xi$ with $X$ assigned to $U$.

## Lemma 1.6

In a model $V = (M, S)$ of WKL$_0$, we fix any $e \in M$ and an $M$-finite assignment map $\xi$. Then, there exists a $p \in M$ such that for all $G_e$ formulas $z$ whose free variables all belong to the domain of $\xi$, then $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^p(z, \xi)$ holds.

**Proof.** Since the domain of the assignment map $\xi$ is $M$-finite, the set of $G_e$ formulas whose free variables are in the domain of $\xi$ is essentially $M$-finite (disregarding repetitions of the same formulas within a disjunction or conjunction). This fact can be demonstrated by $\Sigma^0_1$ induction on $e$.

Therefore, for $M$-finitely many $G_e$ formulas $z$, if $\mathrm{Sat}(z, \xi)$ holds, let $p_z$ be $p$ such that $\mathrm{Sat}^p(z, \xi)$, or otherwise let $p_z = 0$ . Then, if we put $q = \max\{p_z\}$, [1] then we have $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^q(z, \xi)$. □

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[1] Strictly speaking, strong $\Sigma^0_1$ collection principle ($\mathrm{S}\Sigma_1$) is used here. (HW # 5-1.)

## Definition 1.7 (Reflection)

In a model $V$ of $\mathsf{WKL}_0$, for any $e, p$, and for two assignment maps $\xi, \xi'$ with the same domain, the relation $\mathrm{Ref}^p_e(\xi, \xi')$ is defined as follows:

$$\mathrm{Sat}(z, \xi) \Rightarrow \mathrm{Sat}^p(z, \xi'), \text{for each } G_e \text{ formula } z \text{ with free variables in the domain of } \xi.$$

## Lemma 1.8

In a model $V$ of $\mathsf{WKL}_0$, supposing $\mathrm{Ref}^p_e(\xi, \xi')$ with $M$-finite $\xi, \xi'$, the following holds:

(1) If $e = 4d + 1$, $\forall a \, \exists a' < p \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$, where $y$ is a variable not in the domain of $\xi$.

(2) If $e = 4d + 2$, for each numerical variable $x$ belonging to $\xi$, $\forall a' < \xi'(x) \, \exists a < \xi(x) \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$, with $y$ not in $\xi$.

(3) If $e = 4d + 3$, $\forall U \, \exists U' \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$, where $Y$ is a variable not belonging to the domain of $\xi$.

(4) If $e = 4d + 4$, $\forall U' \, \exists U \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$, with $Y$ not in $\xi$.

**Proof** Let $V = (M, S)$ be a model of $\mathsf{WKL}_0$, and let $\xi, \xi'$ be $M$-finite assignments with the same domain such that $\mathrm{Ref}^p_e(\xi, \xi')$ is satisfied.

(1) For $e = 4d + 1$. Show $\forall a \, \exists a' < p \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$. Fix any $a \in M$. Let $Z$ be the set of all codes of $G_{e-1}$ formulas $z$ satisfying $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ and in a non-redundant form (i.e., no same formula is repeated in disjunctions or conjunctions), whose free variables are either $y$ or belong to the domain of $\xi$. According to the argument in the proof of Lemma 1.6, this set $Z$ is $M$-finite within $V$. Thus, by (bounded $\Sigma^0_1$-CA) (Lemma 7.1.8), $Z$ exists.

Now, consider a $G_e$-formula $z' = \exists y \bigwedge_{z \in Z} z$. Since $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ for each $z \in Z$, it follows from Lemma 1.5 that $\mathrm{Sat}(\bigwedge_{z \in Z} z, \xi \cup \{(y, a)\})$ and so $\mathrm{Sat}(z', \xi)$.

Therefore, by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds. Thus, there exists $a' < p$ such that $\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$ holds for each $z \in Z$, fulfilling the requirement.

(2) For $e = 4d + 2$. Show $\forall a' < \xi'(x) \, \exists a < \xi(x) \, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$. Fix any $a' < \xi'(x)$. To prove by contradiction, assume that for any $a < \xi(x)$ there exists a $G_{e-1}$ formula $z$ such that $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ and $\neg\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$. Let $Z$ be the set of all $z \in G_{e-1}$ satisfying $\neg\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$ and in a non-redundant form, whose free variables are either $y$ or belong to the domain of $\xi$.

(2) (continued) Like in case (1), $Z$ exists by (bounded $\Sigma^0_1$-CA). Consider a $G_e$ formula $z' = \forall y < x \bigvee_{z \in Z} z$. By the other assumption, for each $a < \xi(x)$, there exists $z \in Z$ such that $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$, so $\mathrm{Sat}(z', \xi)$ holds.

Therefore, by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds. Thus for each $a' < \xi'(x)$, there exists $z \in Z$ such that $\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$, which contradicts the definition of $Z$.

(3) For $e = 4d + 3$. $\forall U\, \exists U'\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$ can be shown like (1).

(4) For $e = 4d + 4$. Show $\forall U'\, \exists U\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$.
Fix any $U'$. Let $Z$ be the set of $z \in G_{e-1}$ satisfying $\neg \mathrm{Sat}^p(z, \xi' \cup \{(Y, U')\})$ and in a non-redundant form, whose free variables are either $y$ or belong to the domain of $\xi$. Consider a $G_e$ formula $z' = \forall Y \bigvee_{z \in Z} z$. By contradiction, assume for each $U$, there exists $z \in Z$ such that $\mathrm{Sat}(z, \xi \cup \{(Y, U)\})$. Thus, $\mathrm{Sat}(z', \xi)$ holds, and by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds, which contradicts the definition of $Z$.

Thus, the proof is complete. □

## Theorem 1.1 (Self-Embedding Theorem)

Let $\mathfrak{M} = (M, S)$ be a countable model of $\mathsf{WKL}_0$ with $M \neq \omega$. Then, there exists a proper initial segment $I$ of $M$ such that $\mathfrak{M}\lceil I = (I, S\lceil I)$ is isomorphic to $\mathfrak{M}$.

**Proof** Let $V = (M, S)$ be a countable nonstandard model of $\mathsf{WKL}_0$, and fix $q \in M$. Since $V_q$ is $M$-finite within $V$, we can also make an $M$-finite mapping $\xi_0$ that assigns each number and set in $V_q$ to distinct variables.

Now, take any nonstandard number $e \in M$. By Lemma 1.6, for any $G_e$-formula $z$ whose free variables belong to the domain of $\xi_0$, there exists $p$ such that $\mathrm{Sat}(z, \xi_0) \Leftrightarrow \mathrm{Sat}^p(z, \xi_0)$ holds.

In the following, by repeatedly using Lemma 1.8 (the back-and-forth method), we construct two $\omega$-sequences of assignment mappings $\xi_0 \subseteq \xi_1 \subseteq \cdots \subseteq \xi_k \subseteq \ldots$ and $\xi'_0\,(=\xi_0) \subseteq \xi'_1 \subseteq \cdots \subseteq \xi'_k \subseteq \ldots$ $(k \in \omega)$, where $\mathrm{Ref}^p_{e-k}(\xi_k, \xi'_k)$ holds for all $k \in \omega$, and $\bigcup_k \mathrm{range}(\xi_k) = V$ and $\bigcup_k \mathrm{range}(\xi'_k)$ forms the desired initial segment of the model $V$.

To begin with, we enumerate the elements of $V$ as $M = \{a_i \mid i \in \omega\}$, $S = \{U_i \mid i \in \omega\}$. We inductively construct $\xi_k, \xi'_k$ with the same domain $(k \in \omega)$ by cases:

(i) For $e - k = 4d + 1$. Let $a$ be the element $a_i$ in $M - \mathrm{range}(\xi_k)$ with the smallest index $i$, and let $a' < p$ be obtained by Lemma 1.8(1). Then, let $y$ be a new numerical variable not in the domain of $\xi_k$, and set $\xi_{k+1} = \xi_k \cup \{(y, a)\}$, $\xi'_{k+1} = \xi_k \cup \{(y, a')\}$.

(ii) For $e - k = 4d + 2$. Let $\xi'_k(x_0)$ be the largest in the order in $M$ among all $\xi'_k(x)$'s. Then, let $a'$ be the element $a_i$ in $M - \mathrm{range}(\xi'_k)$ and satisfying $a_i < \xi'_k(x_0)$ with the smallest index $i$, and let $a < \xi(x_0)$ be obtained by Lemma 1.8(2). Then, let $y$ be a new numerical variable, and set $\xi_{k+1} = \xi_k \cup \{(y, a)\}$, $\xi'_{k+1} = \xi_k \cup \{(y, a')\}$.

(iii) For $e - k = 4d + 3$. Let $U$ be $U_i \in S$ with the smallest index $i$, that is different from any set in $\mathrm{range}(\xi_k)$ with regards to the numbers in $\mathrm{range}(\xi_k)$. Also, let $U'$ be obtained by Lemma 1.8(3). Then, let $Y$ be a new set variable, and set $\xi_{k+1} = \xi_k \cup \{(Y, U)\}$, $\xi'_{k+1} = \xi_k \cup \{(Y, U')\}$.

(iv) For $e - k = 4d + 4$. Let $U'$ be $U_i \in S$, with the smallest index $i$, that is different from any set in $\mathrm{range}(\xi'_k)$ with regards to the numbers in $\mathrm{range}(\xi'_k)$. Also, let $U$ be obtained by Lemma 1.8(4). Then, let $Y$ be a new set variable, and set $\xi_{k+1} = \xi_k \cup \{(Y, U)\}$, $\xi'_{k+1} = \xi'_k \cup \{(Y, U')\}$

From the above construction, it is easy to see that $\mathrm{Ref}^p_{e-k}(\xi_k, \xi'_k)$ holds for each $k \in \omega$.

From (i) and (iii), it is obvious that $\bigcup_k \mathrm{range}(\xi_k) = (M, S)$. Also, from (ii), we can easily see that the set $I$ consisting of $a$ belonging to $\bigcup_k \mathrm{range}(\xi'_k)$ forms an initial segment of $M$. Then, from (iv) it follows that $\bigcup_k \mathrm{range}(\xi'_k) = (I, S\lceil I)$.

Next, we prove by induction that both $\xi_k, \xi'_k$ are injective for all $k \in \omega$. It is clear from the definition that $\xi_0 = \xi'_0$ is injective.

In (i), we first extend the injective mapping $\xi_k$ to an injective $\xi_{k+1}$, and then extend the injective $\xi'_k$ to a mapping $\xi'_{k+1}$ that satisfies $\mathrm{Ref}^p_{e-k-1}(\xi_{k+1}, \xi'_{k+1})$. The injectivity of $\xi_{k+1}$ is clear from the construction. Since the injectivity is expressed by a $G_2$ formula, $\xi'_{k+1}$ is also injective.

Similarly for (ii), (iii) and (iv).

Thus, $\bigcup_k \xi_k$ and $\bigcup_k \xi'_k$ are also injective.
Let $f = (\bigcup_k \xi'_k) \circ (\bigcup_k \xi_k)^{-1}$, which becomes a bijection from $V$ to $V\lceil I$. It is evident that $f$ acts as the identity map on $V_q$.
Furthermore, since $\mathrm{Ref}^p_0(\xi_k, \xi'_k)$ holds for each $k \in \omega$, it is clear that $f$ is an isomorphism.
Thus, the proof of the theorem is complete. □

Let's briefly describe how the Self-Embedding Theorem 1.1 can be applied to nonstandard analysis.

- According to Gödel's completeness theorem and compactness theorem,

$$\mathsf{WKL}_0 \vdash \varphi \Leftrightarrow \text{for any non-}\omega\text{ model } \mathfrak{M} \text{ of } \mathsf{WKL}_0, \mathfrak{M} \models \varphi.$$

- Since any infinite structure has an elementarily equivalent countable structure by the Löwenheim-Skolem Theorem,

$$\mathsf{WKL}_0 \vdash \varphi \Leftrightarrow \text{for any countable non-}\omega\text{ model } \mathfrak{M} \text{ of } \mathsf{WKL}_0, \mathfrak{M} \models \varphi.$$

- Choose a countable non-$\omega$ model $\mathfrak{M} = (M, S)$ of $\mathsf{WKL}_0$. Theorem 1.1 states that $\mathfrak{M}$ has an initial segment isomorphic to itself. But by swapping their roles of $\mathfrak{M}$ and an isomorphic initial segment, $\mathfrak{M}$ is seen to have an isomorphic end-extension ${}^*\mathfrak{M} = ({}^*M, {}^*S)$, which allows us to carry out some nonstandard analysis arguments.
- For example, in $\mathfrak{M} = (M, S)$, a real number $a$ is indeed a set in $S$. Thus, $a$ is an initial segment ${}^*a \lceil M$ of some set ${}^*a \in {}^*S$. Since ${}^*a$ may be taken bounded in ${}^*\mathfrak{M}$, it can be coded by an element of ${}^*M$. Therefore, a real number in $\mathfrak{M}$ can be treated like a rational number in ${}^*\mathfrak{M}$.

# Application (The Maximum Principle)

WKL$_0$ $\vdash$ Any continuous function $f : [0,1] \to [0,1]$ has a maximum value.

**Proof.**

$$\mathfrak{M} = (M, S) \qquad\qquad {}^*\mathfrak{M} = ({}^*M, {}^*S)$$

$$\begin{tikzcd}
f : \underset{\{q_i\}_{i\in M}}{\underset{\parallel}{[0,1]\cap\mathbb{Q}}} \to \underset{2^M}{\underset{\parallel}{[0,1]}} \arrow[r, Rightarrow] & \begin{array}{c} {}^*f : \{q_i\}_{i<a} \to 2^b \\ (a, b \in {}^* M - M, f = {}^* f \cap M) \end{array} \arrow[d, Rightarrow] \\
{}^*m \cap M \text{ is } \sup f & {}^*m = \max\{{}^*f(q_i)\}_{i<a} \arrow[l, Rightarrow]
\end{tikzcd}$$

# Other Applications

$\mathsf{WKL}_0 \vdash$ The Cauchy-Peano Theorem (Tanaka, 1997)

$\mathsf{WKL}_0 \vdash$ The existence of Haar measure for a compact group (Tanaka-Yamazaki, 2000)

$\mathsf{WKL}_0 \vdash$ The Jordan curve theorem (Sakamoto-Yokoyama, 2007)

**HW # 5-4**

$\mathsf{WKL}_0 \vdash \sigma \;\; \Rightarrow ? \;\; \mathsf{RCA}_0 \vdash \sigma \quad$ for $\sigma \equiv \forall X \exists ! Y \varphi(X, Y)$.

- The above conservation holds for any arithmetical formula $\varphi(Y)$.
- (1) Show that the conservation does not hold for some $\Sigma^1_1$ formula $\varphi(Y)$.
- (2) Show that the conservation does not hold for some $\Pi^1_1$ formula $\varphi(Y)$.

# §2. Forcing and Harrington's Theorem

In this section, we introduce Harrington's theorem that "$\mathrm{WKL}_0$ is a $\Pi^1_1$ conservative extension of $\mathrm{RCA}_0$." The forcing argument of adding infinite paths of an infinite tree as generic paths to a ground model was invented by Jockusch and Soare ($\Pi^0_1$ classes and degrees of theories, *Trans. of the A. M. S.* 173 (1972), pp.35–56). Subsequently, Harrington cleverly applied it to non-$\omega$ models in second-order arithmetic.

The basic idea of forcing is to generate something that does not exist in the world without causing confusion. First, a set of conditions $\mathbb{P}$ for what to generate is given, and a partial order is defined on $\mathbb{P}$. Ways to interpret these conditions varies depending on applications, and we first proceed without giving particular meanings.

Fix an arbitrary partially ordered set $(\mathbb{P}, <)$, and let $p, q, r, \ldots$ denote elements of $\mathbb{P}$. A set $G \subseteq \mathbb{P}$ is called an **open set**, if it satisfies the following condition

$$\forall p, q \ (q < p \land p \in G \to q \in G).$$

Thus, $(\mathbb{P}, <)$ becomes a topological space. Now, let

$$[p] = \{q \in \mathbb{P} \mid q \leq p\}.$$

Any open set $G$ coincides with $\bigcup_{p \in G} [p]$, and so $\{[p] \mid p \in \mathbb{P}\}$ forms a basis for the topology.

Any set $D \subseteq \mathbb{P}$ is called a **dense** set, if it has a non-empty intersection with every non-empty open set. The condition for $D$ to be dense is equivalent to

$$\forall p \in \mathbb{P} \ \ [p] \cap D \neq \emptyset, \text{ in other words, } \forall p \in \mathbb{P} \ \ \exists d \in D \ \ d \leq p.$$

### Definition 2.1

A set $F \subseteq \mathbb{P}$ is called a **filter**, if it satisfies the following conditions:

1) $p \in F \land p < q \to q \in F$,
2) $\forall p, q \in F \ \ [p] \cap [q] \cap F \neq \emptyset$.

## Definition 2.2

Given a family of sets $\mathcal{D}$, a filter $G$ is called a **$\mathcal{D}$-generic filter** if it intersects every dense set $D \subseteq \mathbb{P}$ belonging to $\mathcal{D}$.

## Lemma 2.3

If $\mathcal{D}$ contains at most countably many dense subsets of $\mathbb{P}$, then for any $p \in \mathbb{P}$, there exists a $\mathcal{D}$-generic filter $G$ that contains $p$.

**Proof** Enumerate the dense subsets of $\mathbb{P}$ contained in $\mathcal{D}$ as $D_0, D_1, \cdots, D_i, \cdots (i \in \omega)$. For a given $p \in \mathbb{P}$, construct a decreasing sequence $p_0 \geq p_1 \geq \cdots$ from $\mathbb{P}$ as follows: $p_0 = p$, and $p_n \in [p_{n-1}] \cap D_{n-1}$ for each $n > 0$. Then, we set $G = \{q \mid \exists i \; p_i \leq q\}$. Thus, it is obvious that $p \in G$ and $G$ is a $\mathcal{D}$-generic filter. □

Now, we will introduce the forcing conditions used in Harrington's proof.
Let $\mathfrak{M} = (M, S)$ be a countable model of $\mathsf{RCA}_0$. Here, $M$ is the first-order part (the domain corresponding to the natural numbers), and $S$ is the second-order part consisting of subsets of $M$, that is, $S \subseteq \mathcal{P}(M)$. Then, set

$$\mathbb{P} = \{T \in S \mid \mathfrak{M} \models \text{“}T (\subseteq \mathrm{Seq}_2) \text{ is an infinite binary tree”}\},$$

and define a partial order on $\mathbb{P}$ by

$$T_1 \leq T_2 \Leftrightarrow T_1 \subseteq T_2.$$

For each $T \in \mathbb{P}$, we want to generate an infinite path and put it into $S$. But if we bring in an arbitrary path of $T$ from outside, it might break the condition of $\mathfrak{M}$, e.g., induction axiom. Instead, we approximate an infinite path by $T' \leq T$, and for this purpose, the concept of density is important, namely

$$D \subseteq \mathbb{P} \text{ is dense } \Leftrightarrow \forall T \in \mathbb{P}\ \exists T' \in D\ T' \leq T.$$

$E \subseteq \mathbb{P}$ is said to be **definable in $\mathfrak{M}$** if there exists a formula $\varphi(X)$ (with parameters from $M \cup S$) such that $E = \{T \in \mathbb{P} \mid \mathfrak{M} \models \varphi(T)\}$. The totality of such sets is denoted by $\mathrm{Def}(\mathfrak{M})$. Since we only consider a countable model $\mathfrak{M} = (M, S)$ in a countable language, $\mathrm{Def}(\mathfrak{M})$ is a countable set. By Lemma 2.3, any $T \in \mathbb{P}$ is contained in some $\mathrm{Def}(\mathfrak{M})$-generic filter. Such a filter is simply referred to as an $\mathfrak{M}$-generic filter.

### Lemma 2.4

If $F \subseteq \mathbb{P}$ is an $\mathfrak{M}$-generic filter, then there exists a unique infinite path $G = \cap F = \cap_{T \in F} T$ common to all $T \in F$. That is, $F$ is contained in the principal filter generated by $G$.

**Proof** For each $k \in M$, let $E_k = \{T \in \mathbb{P} \mid \exists! s \in \{0,1\}^k \ s \in T\}$ be dense and definable in $\mathfrak{M}$. If $F$ is an $\mathfrak{M}$-generic filter, then for each $k$, there exists some $s_k \in \{0,1\}^k$ such that there is $T_k \in F$ with $T_k \cap \{0,1\}^k = \{s_k\}$. Moreover, if $k < k'$, then $s_k$ is an initial segment of $s_{k'}$, and $s_{k'} \in T_k$. If not, $[T_k] \cap [T_{k'}] = \emptyset$[2], which would contradict the filter condition of $F$. Thus, let $G = \bigcup_{k \in M} s_k$; then $G = \bigcap_k T_k$ as well. Finally, to show $G = \cap F$, if $G \not\subseteq T \in F$, then there exists some $k$ such that $s_k \notin T$, and $[T] \cap [T_k] = \varnothing$, which contradicts the filter condition of $F$. □

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[2] Here, $[T]$ denotes $\{T' \in \mathbb{P} \mid T' \subset T\}$. In the latter half of part 8, the same notation $[T]$ represents the set of infinite paths of $T$. Since both are conventional, we would use both as they are.

## Definition 2.5

$G (\subseteq M)$ is called an **$\mathfrak{M}$-generic path**, if for every dense set $D \in \mathrm{Def}(\mathfrak{M})$, there exists a tree $T \in D$ such that $G$ is an infinite path through $T$.

## Lemma 2.6

Every $T \in \mathbb{P}$ has an $\mathfrak{M}$-generic path $G$.

**Proof** By Lemma 2.3, every $T$ is contained in some $\mathfrak{M}$-generic filter $F$. Then, by Lemma 2.4, there is a common infinite path $G$ in the trees of $F$. It is clear from the definition that this $G$ is an $\mathfrak{M}$-generic path. □

From now on, an $\mathfrak{M}$-generic path will simply be referred to as a generic path.

## Lemma 2.7

If $G$ is a generic path, then $(M, S \cup \{G\}) \models \Sigma^0_1$-induction.

**Proof** Let $\varphi(i, X)$ be any $\Sigma^0_1$ formula, and choose any $b \in M$, and we will show that $A = \{a \leq_M b \mid \varphi(a, G)\} \in S$ [3]. If $A \in S$, induction on $\varphi(n, G)$ can be shown as follows.

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[3] See Lemma 1.8 of part 7 for $\mathsf{RCA}_0 \vdash$ (bounded $\Sigma^0_1$-CA). We show (bounded $\Sigma^0_1$-CA) $\rightarrow$ $\Sigma^0_1$ induction.

Suppose $A \in S$. Then, $B = \{a \mid a \in A \vee a >_M b\} \in S$ since $\mathfrak{M} \models (\Delta^0_1\text{-CA})$. Now, assume $\varphi(0, G)$ and $\forall n(\varphi(n, G) \to \varphi(n+1, G))$. Then, we have $0 \in B$ and $\forall m(m \in B \to m+1 \in B)$. Since $\mathfrak{M} \models \Sigma^0_1$-induction, by induction on $B$, we have $B = M$. Therefore, $b \in A$, that is, $\varphi(b, G)$. Since $b \in M$ is arbitrary, we get $\forall n \varphi(n, G)$.

Now we show $A \in S$. Let $\varphi(i, X) \equiv \exists j \theta(i, X\lceil j)$ (where $\theta \in \Sigma^0_0$)[4], and set

$$\begin{aligned} D_b = \{T \in \mathbb{P} \mid \mathfrak{M} \models \forall a \leq b \ (1)\ \forall t \in T \neg\theta(a, t) \vee \\ (2)\ \exists k \forall t \in T \cap \{0,1\}^k \exists s \subseteq t \theta(a, s)\}. \end{aligned}$$

Of course, $D_b$ is definable in $\mathfrak{M}$. Here, note that if $T \in D_b$ and $T' \subseteq T$, then $T' \in D_b$. And as shown below, $D_b$ is dense, so there exists a tree $T_0$ in $D_b$ that has $G$ as an infinite path. Fix such a $T_0$. For simplicity, we write $(1)_{T_0}$ for above condition (1) with $T = T_0$, and $(2)_{T_0}$ for condition (2) with $T = T_0$.

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[4] $X \upharpoonright j$ represents the code of the initial segment $(f(0), \cdots, f(j-1))$ of the characteristic function $f$ of $X$. The truth value of the $\Sigma^0_0$ formula $\theta(X)$ depends only on a finite part of $X$, so for sufficiently large $j$, $X$ can be replaced by $X \upharpoonright j$. See [Simpson, Theorem II.2.7] for details.

Then, for each $a \leq_M b$,

$$\mathfrak{M} \models (1)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \neg\varphi(a, G),$$
$$\mathfrak{M} \models (2)_{T_0} \Rightarrow (M, S \cup \{G\}) \models \varphi(a, G).$$

Since $\mathfrak{M} \models (1)_{T_0} \vee (2)_{T_0}$, we have

$$\mathfrak{M} \models (2)_{T_0} \Leftrightarrow (M, S \cup \{G\}) \models \varphi(a, G)$$

Since (2) is a $\Sigma^0_1$ formula, and $\mathfrak{M} \models (\text{bounded}\Sigma^0_1\text{-CA})$ (Lemma 1.8, Chapter 7), $A = \{a \leq_M b \mid \mathfrak{M} \models (2)_{T_0}\} \in S$.

Finally, we show that $D_b$ is dense. Choose any $\tilde{T} \in \mathbb{P}$. For each $\sigma \in \{0,1\}^{\leq b}$, define a tree $T_\sigma$ inductively as follows:

$T_\varnothing = \tilde{T}$,
$T_{\sigma^\frown 0} = \{t \in T_\sigma \mid \forall s \subseteq t\ \neg\theta(a, s)\}$, where $a = \mathrm{leng}(\sigma)$,
$T_{\sigma^\frown 1} = T_\sigma$.

Here, $\varnothing$ is the empty sequence, and $\sigma^\frown i$ denotes the sequence $\sigma$ followed by $i(= 0, 1)$.

Next, let $S_b = \{\sigma \in \{0,1\}^{b+1} \mid T_\sigma$ is an infinite tree$\}$. Then, since "$T_\sigma$ is an infinite tree" is expressed by a $\Pi^0_1$ formula $\forall n \exists \tau \in \{0,1\}^n\ \tau \in T_\sigma$, by (bounded $\Sigma^0_1$-CA), we have $S_b \in S$. Also, since $\overbrace{<1,1,\cdots,1>}^{b+1} \in S_b$, we get $S_b \neq \varnothing$.
Thus, let $\sigma_b$ be the lexicographically first element in $S_b$.

Take any $a \leq_M b$. $\sigma_b(a) = 0$, then $(\sigma_b\lceil a)^\cap 0 \subset \sigma_b$, so

$$T_{\sigma_b} \subseteq T_{(\sigma_b\lceil a)\cap 0} \subseteq \{t \mid \neg\theta(a,t)\},$$

from which we have $(1)_{T_{\sigma_b}}$.

If $\sigma_b(a) = 1$, then $T_{(\sigma_b\lceil a)\cap 0}$ is finite, and thus $(2)_{T_{\sigma_b\lceil a}}$ and also $(2)_{T_{\sigma_b}}$ holds.

From all the above, $T_{\sigma_b} \in D_b$, which proves that $D_b$ is dense. □

Fix a generic path $G$ for $T \in \mathbb{P}$, and let

$$S^T = \{X \subseteq M \mid X \text{ is definable in } (M, S \cup \{G\}) \text{ by a } \Delta^0_1 \text{ formula}\}.$$

## Lemma 2.8

$(M, S^T) \models \mathsf{RCA}_0 + T$ has an infinite path.

**Proof** For a $\Sigma^0_1$ formula $\varphi$ with parameters from $S^T$, there exists an equivalent $\Sigma^0_1$ formula $\psi$ with parameters only from $S \cup \{G\}$, which is obtained from the former by replacing a parameter $X$ of $S^T$ with a $\Delta^0_1$ formula defining it. Recall that the same argument was used to show that $\mathsf{RCA}_0$ is a conservative extension of $\mathsf{I}\Sigma_1$ (in part 5, Lemma 1.3). Then, by Lemma 2.7, $(M, S^T) \models \mathsf{RCA}_0$. Also, in $(M, S^T)$, $T$ has an infinite path $G$. □

Notice that if $(M, S)$ is countable, then $S^T$ is also countable. In the following lemma, this process is repeated to construct a model $(M, S_\infty)$ of $\mathsf{WKL}_0$, which is also countable.

## Lemma 2.9

For any countable model $(M, S)$ of $\mathsf{RCA}_0$, there exists a countable set $S_\infty$ such that $S \subseteq S_\infty \subseteq \mathcal{P}(M)$ and $(M, S_\infty) \models \mathsf{WKL}_0$.

**Proof** Construct $S_0 \subseteq S_1 \subseteq \cdots$ as follows: $S_0 = S$, and

$$S_{(n,m)+1} = S^T_{(n,m)}, \text{ where } T \text{ is the } m\text{-th infinite tree in } S_n (\subseteq S_{(n,m)}).$$

Here, $(n,m) = \frac{(n+m)(n+m+1)}{2} + n$, and so $(n,m) \geq n$. Finally, let $S_\infty = \bigcup_{i \in \omega} S_i$. It is clear from the definition that this is the desired set. □

## Theorem 2.10 (Harrington)

For any $\Pi^1_1$ sentence $\sigma$, $\mathsf{WKL}_0 \vdash \sigma \Rightarrow \mathsf{RCA}_0 \vdash \sigma$.

**Proof** Suppose $\sigma$ is a $\Pi^1_1$ sentence that is not provable in $\mathsf{RCA}_0$. By Gödel's completeness theorem, there exists a countable model $(M, S) \models \mathsf{RCA}_0 + \neg\sigma$. Now, $\neg\sigma$ can be expressed as $\exists X \varphi(X)$ with $\varphi \in \Pi^1_0$. Then there exists $A \in S$ such that $(M, S) \models \mathsf{RCA}_0 + \varphi(A)$. By constructing $S_\infty$ by Lemma 2.9, we have $(M, S_\infty) \models \mathsf{WKL}_0 + \varphi(A)$. Note that since $\varphi(X)$ is arithmetical, the truth value of $\varphi(A)$ depends only on $M$ and $A$. Therefore, $(M, S_\infty) \models \mathsf{WKL}_0 + \neg\sigma$, which implies $\mathsf{WKL}_0 \nvdash \sigma$. □

Thank you for your attention!