Topics in Applied Math:

Logic and Foundations of Mathematics

Part 7. Models of second order arithmetic

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Logic and Foundations

- **Part 1. Equational theory**
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Part 7. Schedule

- Dec. 3, (1) Introduction and RCA_0
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Part 7. §1. A self-embedding theorem of WKL_0

In this section, we introduce a self-embedding theorem of WKL_0, by which we can devise methods of nonstandard analysis in WKL_0.

Gödel stated in 1973 that "nonstandard analysis is the future of analysis." However, Henson and Keisler have shown in 1986 that nonstandard arguments in n-th order arithmetic require $(n+1)$-th order arithmetic. Therefore, conducting complete nonstandard analysis for second-order arithmetic Z_2 is impossible within the framework of second-order arithmetic alone. Nevertheless, as demonstrated in my paper[1], certain amount of nonstandard analysis can still be developed within WKL_0.

The main tool of our nonstandard method is a self-embedding theorem of WKL_0 (Theorem 1.9), which extends Friedman's self-embedding theorem (§5.3) to WKL_0. In this lecture, we primarily discuss the proof of this theorem.

[1] K. Tanaka, The self-embedding theorem of WKL_0 and a non-standard method, *Annals of Pure and Applied Logic* 84 (1997), pp.41–49. See also A. Enayat, A new proof of Tanaka's theorem, *CSLI Lecture Notes* 211 (2013), pp.93-102. S. Bahrami, Tanaka's theorem revisited, *Archive for Math. Logic* 59 (2020), 865-877.

Theorem 1.1 (Self-Embedding Theorem)

Let $\mathfrak{M} = (M, S)$ be a countable model of WKL_0 with $M \neq \omega$. Then, there exists a proper initial segment I of M such that $\mathfrak{M}\lceil I = (I, S\lceil I)$ is isomorphic to $\mathfrak{M}$. Here, $S\lceil I = \{X \cap I \mid X \in S\}$.

Before proving this theorem, we need some preparations. We first prove the following lemma, which will be frequently used later.

Lemma 1.2 (Compactness in WKL_0)

(1) For any Π^0_1 formula $\varphi(X)$, there exists a Π^0_1 formula $\hat{\varphi}$ such that WKL_0 proves:

$$\hat{\varphi} \leftrightarrow \exists X\, \varphi(X).$$

(2) For any Π^0_1 formula $\varphi(k, X)$, WKL_0 proves:

$$\forall n\, \exists X\, \forall k < n\, \varphi(k, X) \rightarrow \exists X\, \forall k\, \varphi(k, X).$$

From now on, we adopt the notation $[T]$ for the set of all infinite paths through a tree T.

Proof. (1) We identify a set X with its characteristic function, which is also represented as an infinite binary sequence. Then, a Π^0_1 formula $\varphi(X)$ can be expressed as $\forall x\ \theta(X{\restriction}x)$, where θ is Σ^0_0 and $X{\restriction}x$ is a code for a finite binary sequence. We set $T = \{t \mid \forall s \subseteq t\ \theta(s)\}$. Then T is a tree, and $X \in [T]$ iff $\varphi(X)$ holds. Thus, $\exists X\, \varphi(X)$ is equivalent to $[T] \neq \varnothing$, which is expressed as a Π^0_1 formula "T is infinite ($\forall n \exists t \in \{0,1\}^n t \in T$)".

(2) Express a Π^0_1 formula $\varphi(k, X)$ as $\forall x\ \theta(k, X{\restriction}x)$ (where θ is Σ^0_0), and define a tree $T = \{t \mid \forall k \leq \mathsf{leng}(t) \forall x \leq \mathsf{leng}(t)\ \theta(k, t\lceil x)\}$. Here, $\mathsf{leng}(t)$ denotes the length of the finite binary sequence t. If $\forall n\, \exists X\, \forall k < n\, \varphi(k, X)$ holds, then $\forall n\, \exists X\, \forall k < n\, \forall x < n\ \theta(k, X\lceil x)$, so $t = X\lceil n \in T$ for all n, thus T is infinite. Hence, in WKL_0, there exists an infinite path $X \in [T]$ satisfying $\forall k\, \varphi(k, X)$. □

Here is another demonstration for (2). If we express $\varphi(k, X)$ as $X \in [T_k]$, then $\exists X\, \forall k < n\, \varphi(k, X)$ can be expressed as $\bigcap_{k<n}[T_k] \neq \varnothing$. Since this is true for any n, we have $\bigcap_{k<\infty}[T_k] \neq \varnothing$ by the compactness of the Cantor space since $[T_k]$'s are closed sets.

Both (1) and (2) are referred to as "compactness (of binary trees) in WKL_0".

We define G-Σ^0_1 **formulas** or simply G **formulas** by generalizing Σ^0_1 formulas as follows. The G formulas are obtained from Σ^0_1 formulas by using $\wedge, \vee$, bounded universal quantifier $\forall x < y$ and unbounded existential quantifier $\exists x$, and set quantifiers $\forall X, \exists X$.

In WKL_0, we can prove that a G formula is equivalent to a Σ^0_1 formula.

(Proof)

- The closure condition under $\forall x < y$ is nothing but the collection principle $\mathsf{B}\Sigma^0_1$ derivable from Σ^0_1 induction.
- The closure condition under $\forall X$ can be obtained from Lemma 1.2(1) by taking the negation on both sides.
- The closure condition under $\exists X$ can be demonstrated by noting that $\exists X \exists x\ \theta(x, X\lceil x)$ (where θ is Σ^0_0) can be rewritten as $\exists t \exists x\ \theta(x, t)$.
- The other closure conditions are almost obvious.

Now, we redefine the G-formulas explicitly in RCA_0 in the next slide.

Definition 1.3 (G-formulas)

A sequence $G_0 \subset G_1 \subset G_2 \subset \cdots$ of sets of $\mathcal{L}^2_{\mathrm{OR}}$-formulas is defined inductively modulo 4 as follows: for each $e \in \mathbb{N}$,

$$
\begin{aligned}
G_0 &= \{\text{finite disjunctions } (\vee) \text{ of atomic formulas or their negations}\},\\
G_{4e+1} &= \{\exists x\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e} \text{ formulas}\} \cup G_{4e},\\
G_{4e+2} &= \{\forall x < y\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+1} \text{ formulas}\} \cup G_{4e+1},\\
G_{4e+3} &= \{\exists X\, \phi \mid \phi \text{ is a finite conjunction } (\wedge) \text{ of } G_{4e+2} \text{ formulas}\} \cup G_{4e+2},\\
G_{4e+4} &= \{\forall X\, \phi \mid \phi \text{ is a finite disjunction } (\vee) \text{ of } G_{4e+3} \text{ formulas}\} \cup G_{4e+3}.
\end{aligned}
$$

Finally, we set $\mathbf{G} = \bigcup_{e\in\mathbb{N}} G_e$. The formulas in G are called **G formulas**.

By Lemma 5.3.3, there is no formula that defines the truth values of all formulas. But, Lemma 5.3.4 shows that if we restrict the formulas to a class like Σ_n, then there exists a formula Sat_{Σ_n} to define the truth values of formulas in the class. This is also the case for Σ^0_n in second order arithmetic. In the following, we will define Sat for G formulas.

From now on, a structure $\mathfrak{M} = (M, S)$ is denoted by V. Then, for each $p \in M$, set $M_p = \{a \in M \mid \mathfrak{M} \models a < p\}$, $S_p = \{X \cap M_p \mid X \in S\}$ and denote $V_p = (M_p, S_p)$.

Since M_p may not be closed under operations such as addition, V_p may not be a substructure of V. However, just by restricting the ranges of variables to these sets, the **satisfaction predicate** $\mathrm{Sat}^p(z, \xi)$ for V_p can be naturally defined within $V = (M, S)$. Here, z represents the code of a formula φ, and ξ is a finite function that assigns elements of $M_p \cup S_p$ to free variables appearing in φ. Thus, supposing that a formula $\varphi(\vec{x}, \vec{X})$ has no free variables other than $\vec{x}, \vec{X}$, and $\xi(\vec{x}) = \vec{a}, \xi(\vec{X}) = \vec{U}$, we have in V,

$$\mathrm{Sat}^{\mathbf{p}}(\ulcorner\varphi\urcorner, \xi) \equiv \varphi(\vec{a}, \vec{U})^{V_p}, \text{ roughly } V_p \models \varphi(\vec{a}, \vec{U}).$$

Here, in $\varphi(\vec{a}, \vec{U})^{V_p}$, quantification over numbers is bounded by p, and quantification over sets is also considered as ranging binary sequences of length p, which can be coded by numbers $< 2^p$. Thus, $\mathrm{Sat}^p(z, \xi)$ can be defined as a Δ^0_1 formula in V (cf. Lemma 5.3.4).

We also remark that a variable z in $\mathrm{Sat}^p(z, \xi)$ can potentially express a non-standard number. In V, it can be easily verified that Sat^p satisfies Tarski's truth definition clauses for all standard formulas (cf. Theorem IV.2.26 in [P. Hájek and P. Pudlák, *Metamathematics of First-oder Arithmetic*, Springer, 1993.]).

Next, we define the **satisfaction relation for G formulas** as follows:

Definition 1.4

For each $z \in G$, define the satisfaction relation $\mathrm{Sat}(z, \xi)$ as follows:

$$\mathrm{Sat}(z, \xi) \leftrightarrow \exists p \, \mathrm{Sat}^p(z, \xi \upharpoonright V_p).$$

Here, $\xi \upharpoonright V_p$ is the assignment obtained by restricting the values of ξ to V_p.

For simplicity, we abbreviate $\mathrm{Sat}^p(z, \xi \upharpoonright V_p)$ as $\mathrm{Sat}^p(z, \xi)$. It is provable in RCA_0 that for the code z of a Σ^0_1 formula, if $\mathrm{Sat}^p(z, \xi)$ holds, then $\mathrm{Sat}^{p'}(z, \xi)$ also holds for any $p' \geq p$.

Moreover, we will show in WKL_0 that it is also the case for the codes z of G.

In the following, we identify a formula with its code.

Lemma 1.5

In a model V of WKL_0, $\mathrm{Sat}(z,\xi)$ satisfies Tarski's truth definition clauses for G formulas.

Proof. We prove the statement by induction on the complexity of the formula z.
If z is an atomic formula or its negation, $\mathrm{Sat}(z,\xi) \Leftrightarrow \exists p\, \mathrm{Sat}^p(z,\xi) \Leftrightarrow \exists p\, z(\xi)^{V_p} \Leftrightarrow z(\xi)$.
If $z = \bigvee_{i<n} z_i$ (where each z_i is a G formula),

$$\mathrm{Sat}\left(\bigvee_{i<n} z_i, \xi\right) \Leftrightarrow \exists p\, \mathrm{Sat}^p\left(\bigvee_{i<n} z_i, \xi\right) \Leftrightarrow \exists p \bigvee_{i<n} \mathrm{Sat}^p\left(z_i, \xi\right)$$
$$\Leftrightarrow \bigvee_{i<n} \exists p\, \mathrm{Sat}^p\left(z_i, \xi\right) \Leftrightarrow \bigvee_{i<n} \mathrm{Sat}\left(z_i, \xi\right).$$

If z is $\exists x\, z'$ or $\exists X\, z'$ (where z' is a G formula), the proofs are analogous.

When $z = \bigwedge_{i<n} z_i$ (where each z_i is a G formula),

$$\begin{aligned}\mathrm{Sat}\left(\bigwedge_{i<n} z_i, \xi\right) &\Leftrightarrow \exists p\, \mathrm{Sat}^p\left(\bigwedge_{i<n} z_i, \xi\right) \Leftrightarrow \exists p \bigwedge_{i<n} \mathrm{Sat}^p(z_i, \xi)\\ &\Leftrightarrow \bigwedge_{i<n} \exists p\, \mathrm{Sat}^p(z_i, \xi) \qquad (\Leftarrow \text{ by } \Sigma^0_1 \text{ collection principle})\\ &\Leftrightarrow \bigwedge_{i<n} \mathrm{Sat}(z_i, \xi).\end{aligned}$$

If z is $\forall x < y\, z'$ (where z' is a G formula), the proof is analogous.
If $z = \forall X\, z'$ (where z' is a G formula),

$$\begin{aligned}\mathrm{Sat}(\forall X\, z', \xi) &\Leftrightarrow \exists p\, \mathrm{Sat}^p(\forall X\, z', \xi) \Leftrightarrow \exists p\, \forall U\, \mathrm{Sat}^p(z', \xi \cup \{(X, U)\})\\ &\Leftrightarrow \forall U\, \exists p\, \mathrm{Sat}^p(z', \xi \cup \{(X, U)\}) \qquad (\Leftarrow \text{ by compactness (Lemma 1.2(2))})\\ &\Leftrightarrow \forall U\, \mathrm{Sat}(z', \xi \cup \{(X, U)\}),\end{aligned}$$

where $\xi \cup \{(X, U)\}$ is an extension of ξ with X assigned to U. □

Lemma 1.6

In a model $V = (M, S)$ of WKL_0, we fix any $e \in M$ and an M-finite assignment map ξ. Then, there exists a $p \in M$ such that for all G_e formulas z whose free variables all belong to the domain of ξ, then $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^p(z, \xi)$ holds.

Proof. Since the domain of the assignment map ξ is M-finite, the set of G_e formulas whose free variables are in the domain of ξ is essentially M-finite (disregarding repetitions of the same formulas within a disjunction or conjunction). This fact can be demonstrated by Σ^0_1 induction on e.

Therefore, for M-finitely many G_e formulas z, if $\mathrm{Sat}(z, \xi)$ holds, let p_z be p such that $\mathrm{Sat}^p(z, \xi)$, or otherwise let $p_z = 0$. Then, if we put $q = \max\{p_z\}$, [2] then we have $\mathrm{Sat}(z, \xi) \Leftrightarrow \mathrm{Sat}^q(z, \xi)$. □

[2]Strictly speaking, strong Σ^0_1 collection principle ($\mathrm{S}\Sigma_1$) is used here. (cf. HW # 5-1.)

Definition 1.7 (Reflection)

In a model V of WKL_0, for any e, p, and for two assignment maps ξ, ξ' with the same domain, the relation $\mathrm{Ref}_e^p(\xi, \xi')$ is defined as follows:

$$\mathrm{Sat}(z, \xi) \Rightarrow \mathrm{Sat}^p(z, \xi'), \text{for each } G_e \text{ formula } z \text{ with free variables in the domain of } \xi.$$

Lemma 1.8

In a model V of WKL_0, supposing $\mathrm{Ref}_e^p(\xi, \xi')$ with M-finite ξ, ξ', the following holds:

(1) If $e = 4d + 1$, $\forall a \, \exists a' < p \, \mathrm{Ref}_{e-1}^p(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$, where y is a variable not in the domain of ξ.

(2) If $e = 4d + 2$, for each numerical variable x belonging to ξ, $\forall a' < \xi'(x) \, \exists a < \xi(x) \, \mathrm{Ref}_{e-1}^p(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$, with y not in ξ.

(3) If $e = 4d + 3$, $\forall U \, \exists U' \, \mathrm{Ref}_{e-1}^p(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$, where Y is a variable not belonging to the domain of ξ.

(4) If $e = 4d + 4$, $\forall U' \, \exists U \, \mathrm{Ref}_{e-1}^p(\xi \cup \{(Y, U)\}, \xi' \cup \{(Y, U')\})$, with Y not in ξ.

Proof Let $V = (M, S)$ be a model of WKL_0, and let ξ, ξ' be M-finite assignments with the same domain such that $\mathrm{Ref}^p_e(\xi, \xi')$ is satisfied.

(1) For $e = 4d + 1$. Show $\forall a\, \exists a' < p\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$.
Fix any $a \in M$. Let Z be the set of all codes of G_{e-1} formulas z satisfying $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ and in a non-redundant form (i.e., no same formula is repeated in disjunctions or conjunctions), whose free variables are either y or belong to the domain of ξ. According to the argument in the proof of Lemma 1.6, this set Z is M-finite within V. Thus, by (bounded Σ^0_1-CA) (Lemma 7.1.8), Z exists.

Now, consider a G_e-formula $z' = \exists y \bigwedge_{z \in Z} z$. Since $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ for each $z \in Z$, it follows from Lemma 1.5 that $\mathrm{Sat}(\bigwedge_{z \in Z} z, \xi \cup \{(y, a)\})$ and so $\mathrm{Sat}(z', \xi)$.

Therefore, by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds. Thus, there exists $a' < p$ such that $\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$ holds for each $z \in Z$, fulfilling the requirement.

(2) For $e = 4d + 2$. Show $\forall a' < \xi'(x)\, \exists a < \xi(x)\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(y, a)\}, \xi' \cup \{(y, a')\})$.
Fix any $a' < \xi'(x)$. To prove by contradiction, assume that for any $a < \xi(x)$ there exists a G_{e-1} formula z such that $\mathrm{Sat}(z, \xi \cup \{(y, a)\})$ and $\neg\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$.
Let Z be the set of all $z \in G_{e-1}$ satisfying $\neg\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$ and in a non-redundant form, whose free variables are either y or belong to the domain of ξ.

(2) (continued) Like in case (1), Z exists by (bounded Σ^0_1-CA). Consider a G_e formula $z' = \forall y < x \bigvee_{z \in Z} z$. By the other assumption, for each $a < \xi(x)$, there exists $z \in Z$ such that $\mathrm{Sat}(z, \xi \cup \{(y,a)\})$, so $\mathrm{Sat}(z', \xi)$ holds.

Therefore, by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds. Thus for each $a' < \xi'(x)$, there exists $z \in Z$ such that $\mathrm{Sat}^p(z, \xi' \cup \{(y, a')\})$, which contradicts the definition of Z.

(3) For $e = 4d+3$. $\forall U\, \exists U'\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y,U)\}, \xi' \cup \{(Y,U')\})$ can be shown like (1).

(4) For $e = 4d+4$. Show $\forall U'\, \exists U\, \mathrm{Ref}^p_{e-1}(\xi \cup \{(Y,U)\}, \xi' \cup \{(Y,U')\})$.
Fix any U'. Let Z be the set of $z \in G_{e-1}$ satisfying $\neg\mathrm{Sat}^p(z, \xi' \cup \{(Y,U')\})$ and in a non-redundant form, whose free variables are either y or belong to the domain of ξ.
Consider a G_e formula $z' = \forall Y \bigvee_{z \in Z} z$. By contradiction, assume for each U, there exists $z \in Z$ such that $\mathrm{Sat}(z, \xi \cup \{(Y,U)\})$. Thus, $\mathrm{Sat}(z', \xi)$ holds, and by the hypothesis, $\mathrm{Sat}^p(z', \xi')$ holds, which contradicts the definition of Z.

Thus, the proof is complete. □

Theorem 1.9 (Self-Embedding Theorem)

Let $\mathfrak{M} = (M, S)$ be a countable model of WKL_0 with $M \neq \omega$. Then, there exists a proper initial segment I of M such that $\mathfrak{M}\lceil I = (I, S\lceil I)$ is isomorphic to $\mathfrak{M}$.

Proof Let $V = (M, S)$ be a countable nonstandard model of WKL_0, and fix $q \in M$. Since V_q is M-finite within V, we can also make an M-finite mapping ξ_0 that assigns each number and set in V_q to distinct variables.

Now, take any nonstandard number $e \in M$. By Lemma 1.6, for any G_e-formula z whose free variables belong to the domain of ξ_0, there exists p such that $\mathrm{Sat}(z, \xi_0) \Leftrightarrow \mathrm{Sat}^p(z, \xi_0)$ holds.

In the following, by repeatedly using Lemma 1.8 (the back-and-forth method), we construct two ω-sequences of assignment mappings $\xi_0 \subseteq \xi_1 \subseteq \cdots \subseteq \xi_k \subseteq \dots$ and $\xi'_0\,(=\xi_0) \subseteq \xi'_1 \subseteq \cdots \subseteq \xi'_k \subseteq \dots$ $(k \in \omega)$, where $\mathrm{Ref}^p_{e-k}(\xi_k, \xi'_k)$ holds for all $k \in \omega$, and $\bigcup_k \mathrm{range}(\xi_k) = V$ and $\bigcup_k \mathrm{range}(\xi'_k)$ forms the desired initial segment of the model V.

To begin with, we enumerate the elements of V as $M = \{a_i \mid i \in \omega\}$, $S = \{U_i \mid i \in \omega\}$. We inductively construct ξ_k, ξ'_k with the same domain $(k \in \omega)$ by cases:

(i) For $e - k = 4d + 1$. Let a be the element a_i in $M - \mathrm{range}(\xi_k)$ with the smallest index i, and let $a' < p$ be obtained by Lemma 1.8(1). Then, let y be a new numerical variable not in the domain of ξ_k, and set $\xi_{k+1} = \xi_k \cup \{(y, a)\}$, $\xi'_{k+1} = \xi_k \cup \{(y, a')\}$.

(ii) For $e-k=4d+2$. Let $\xi'_k(x_0)$ be the largest in the order in M among all $\xi'_k(x)$'s. Then, let a' be the element a_i in $M-\mathrm{range}(\xi'_k)$ and satisfying $a_i<\xi'_k(x_0)$ with the smallest index i, and let $a<\xi(x_0)$ be obtained by Lemma 1.8(2). Then, let y be a new numerical variable, and set $\xi_{k+1}=\xi_k\cup\{(y,a)\}$, $\xi'_{k+1}=\xi_k\cup\{(y,a')\}$.

(iii) For $e-k=4d+3$. Let U be $U_i\in S$ with the smallest index i, that is different from any set in $\mathrm{range}(\xi_k)$ with regards to the numbers in $\mathrm{range}(\xi_k)$. Also, let U' be obtained by Lemma 1.8(3). Then, let Y be a new set variable, and set $\xi_{k+1}=\xi_k\cup\{(Y,U)\}$, $\xi'_{k+1}=\xi_k\cup\{(Y,U')\}$.

(iv) For $e-k=4d+4$. Let U' be $U_i\in S$, with the smallest index i, that is different from any set in $\mathrm{range}(\xi'_k)$ with regards to the numbers in $\mathrm{range}(\xi'_k)$. Also, let U be obtained by Lemma 1.8(4). Then, let Y be a new set variable, and set $\xi_{k+1}=\xi_k\cup\{(Y,U)\}$, $\xi'_{k+1}=\xi'_k\cup\{(Y,U')\}$

From the above construction, it is easy to see that $\mathrm{Ref}^p_{e-k}(\xi_k,\xi'_k)$ holds for each $k\in\omega$.

From (i) and (iii), it is obvious that $\bigcup_k \mathrm{range}(\xi_k)=(M,S)$. Also, from (ii), we can easily see that the set I consisting of a belonging to $\bigcup_k \mathrm{range}(\xi'_k)$ forms an initial segment of M. Then, from (iv) it follows that $\bigcup_k \mathrm{range}(\xi'_k)=(I,S\lceil I)$.

Next, we prove by induction that both ξ_k, ξ'_k are injective for all $k \in \omega$. It is clear from the definition that $\xi_0 = \xi'_0$ is injective.

In (i), we first extend the injective mapping ξ_k to an injective ξ_{k+1}, and then extend the injective ξ'_k to a mapping ξ'_{k+1} that satisfies $\mathrm{Ref}^p_{e-k-1}(\xi_{k+1}, \xi'_{k+1})$. The injectivity of ξ_{k+1} is clear from the construction. Since the injectivity is expressed by a G_2 formula, ξ'_{k+1} is also injective.

Similarly for (ii), (iii) and (iv).

Thus, $\bigcup_k \xi_k$ and $\bigcup_k \xi'_k$ are also injective.
Let $f = (\bigcup_k \xi'_k) \circ (\bigcup_k \xi_k)^{-1}$, which becomes a bijection from V to $V\lceil I$. It is evident that f acts as the identity map on V_q.
Furthermore, since $\mathrm{Ref}^p_0(\xi_k, \xi'_k)$ holds for each $k \in \omega$, it is clear that f is an isomorphism.
Thus, the proof of the theorem is complete. □

Applications of the Self-Embedding Theorem

Let's briefly describe how our theorem can be applied to nonstandard analysis.

- According to Gödel's completeness theorem and compactness theorem,

$$\mathsf{WKL}_0 \vdash \varphi \Leftrightarrow \text{for any non-}\omega \text{ model } \mathfrak{M} \text{ of } \mathsf{WKL}_0, \mathfrak{M} \models \varphi.$$

- Since any infinite structure has an elementarily equivalent countable structure by the Löwenheim-Skolem Theorem,

$$\mathsf{WKL}_0 \vdash \varphi \Leftrightarrow \text{for any countable non-}\omega \text{ model } \mathfrak{M} \text{ of } \mathsf{WKL}_0, \mathfrak{M} \models \varphi.$$

- Choose a countable non-ω model $\mathfrak{M} = (M, S)$ of WKL_0. Theorem 1.9 states that $\mathfrak{M}$ has an initial segment isomorphic to itself. But by swapping their roles of $\mathfrak{M}$ and an isomorphic initial segment, $\mathfrak{M}$ is seen to have an isomorphic end-extension ${}^*\mathfrak{M} = ({}^*M, {}^*S)$, which allows us to carry out some nonstandard analysis arguments.
- For example, in $\mathfrak{M} = (M, S)$, a real number a is indeed a set in S. Thus, a is an initial segment ${}^*a\lceil M$ of some set ${}^*a \in {}^*S$. Since ${}^*a$ may be taken bounded in ${}^*\mathfrak{M}$, it can be coded by an element of ${}^*M$. Therefore, a real number in $\mathfrak{M}$ can be treated like a rational number in ${}^*\mathfrak{M}$.

Application (The Maximum Principle)

$\mathsf{WKL}_0 \vdash$ Any continuous function $f : [0,1] \to [0,1]$ has a maximum value.

Proof.

$$\mathfrak{M} = (M, S) \qquad\qquad {}^*\mathfrak{M} = ({}^*M, {}^*S)$$

$$\begin{array}{ccc} f : \underset{\displaystyle \{q_i\}_{i \in M}}{\underset{\|}{[0,1] \cap \mathbb{Q}}} \to \underset{\displaystyle 2^M}{\underset{\|}{[0,1]}} & \Longrightarrow & \begin{array}{c} {}^*f : \{q_i\}_{i<a} \to 2^b \\ (a, b \in {}^*M - M, f = {}^*f \cap M) \end{array} \\ & & \Downarrow \\ {}^*m \cap M \text{ is } \sup f & \Longleftarrow & {}^*m = \max\{{}^*f(q_i)\}_{i<a} \end{array}$$

Other Applications

$\mathsf{WKL}_0 \vdash$ The Cauchy-Peano Theorem (Tanaka, 1997)

$\mathsf{WKL}_0 \vdash$ The existence of Haar measure for a compact group (Tanaka-Yamazaki, 2000)

$\mathsf{WKL}_0 \vdash$ The Jordan curve theorem (Sakamoto-Yokoyama, 2007)

HW # 5-4

$\mathsf{WKL}_0 \vdash \sigma \ \Rightarrow ? \ \mathsf{RCA}_0 \vdash \sigma$ for $\sigma \equiv \forall X \exists ! Y \varphi(X, Y)$.

- The above conservation holds for any arithmetical formula $\varphi(Y)$.
- (1) Show that the conservation does not hold for some Σ^1_1 formula $\varphi(Y)$.
- (2) Show that the conservation does not hold for some Π^1_1 formula $\varphi(Y)$.

§2. Forcing and Harrington's Theorem

In this section, we introduce Harrington's theorem that "WKL_0 is a Π^1_1 conservative extension of RCA_0." The forcing argument of adding infinite paths of an infinite tree as generic paths to a ground model was invented by Jockusch and Soare (Π^0_1 classes and degrees of theories, *Trans. of the A. M. S.* 173 (1972), pp.35–56). Subsequently, Harrington cleverly applied it to non-ω models in second-order arithmetic.

The basic idea of forcing is to generate something that does not exist in the world without causing confusion. First, a set of conditions $\mathbb{P}$ for what to generate is given, and a partial order is defined on $\mathbb{P}$. Ways to interpret these conditions varies depending on applications, and we first proceed without giving particular meanings.

Fix an arbitrary partially ordered set $(\mathbb{P}, <)$, and let $p, q, r, \ldots$ denote elements of $\mathbb{P}$.
A set $G \subseteq \mathbb{P}$ is called an **open set**, if it satisfies the following condition

$$\forall p, q\ (q < p \wedge p \in G \to q \in G).$$

Thus, $(\mathbb{P}, <)$ becomes a topological space. Now, let

$$[p] = \{q \in \mathbb{P} \mid q \leq p\}.$$

Any open set G coincides with $\bigcup_{p \in G}[p]$, and so $\{[p] \mid p \in \mathbb{P}\}$ forms a basis for the topology.

Any set $D \subseteq \mathbb{P}$ is called a **dense** set, if it has a non-empty intersection with every non-empty open set. The condition for D to be dense is equivalent to

$$\forall p \in \mathbb{P}\ \ [p] \cap D \neq \emptyset, \text{ in other words, } \forall p \in \mathbb{P}\ \ \exists d \in D\ \ d \leq p.$$

Definition 2.1

A set $F \subseteq \mathbb{P}$ is called a **filter**, if it satisfies the following conditions:

1) $p \in F \wedge p < q \to q \in F$,
2) $\forall p, q \in F\ \ [p] \cap [q] \cap F \neq \emptyset$.

Definition 2.2

Given a family of sets $\mathcal{D}$, a filter G is called a **$\mathcal{D}$-generic filter** if it intersects every dense set $D \subseteq \mathbb{P}$ belonging to $\mathcal{D}$.

Lemma 2.3

If $\mathcal{D}$ contains at most countably many dense subsets of $\mathbb{P}$, then for any $p \in \mathbb{P}$, there exists a $\mathcal{D}$-generic filter G that contains p.

Proof Enumerate the dense subsets of $\mathbb{P}$ contained in $\mathcal{D}$ as $D_0, D_1, \cdots, D_i, \cdots (i \in \omega)$. For a given $p \in \mathbb{P}$, construct a decreasing sequence $p_0 \geq p_1 \geq \cdots$ from $\mathbb{P}$ as follows: $p_0 = p$, and $p_n \in [p_{n-1}] \cap D_{n-1}$ for each $n > 0$. Then, we set $G = \{q \mid \exists i\ p_i \leq q\}$. Thus, it is obvious that $p \in G$ and G is a $\mathcal{D}$-generic filter. □

Now, we will introduce the forcing conditions used in Harrington's proof.
Let $\mathfrak{M} = (M, S)$ be a countable model of RCA_0. Here, M is the first-order part (the domain corresponding to the natural numbers), and S is the second-order part consisting of subsets of M, that is, $S \subseteq \mathcal{P}(M)$. Then, set

$$\mathbb{P} = \{T \in S \mid \mathfrak{M} \models \text{“}T(\subseteq \mathrm{Seq}_2) \text{ is an infinite binary tree”}\},$$

and define a partial order on $\mathbb{P}$ by

$$T_1 \leq T_2 \Leftrightarrow T_1 \subseteq T_2.$$

For each $T \in \mathbb{P}$, we want to generate an infinite path and put it into S. But if we bring in an arbitrary path of T from outside, it might break the condition of $\mathfrak{M}$, e.g., induction axiom. Instead, we approximate an infinite path by $T' \leq T$, and for this purpose, the concept of density is important, namely

$$D \subseteq \mathbb{P} \text{ is dense } \Leftrightarrow \forall T \in \mathbb{P}\ \exists T' \in D\ T' \leq T.$$

$E \subseteq \mathbb{P}$ is said to be **definable in $\mathfrak{M}$** if there exists a formula $\varphi(X)$ (with parameters from $M \cup S$) such that $E = \{T \in \mathbb{P} \mid \mathfrak{M} \models \varphi(T)\}$. The totality of such sets is denoted by $\mathrm{Def}(\mathfrak{M})$. Since we only consider a countable model $\mathfrak{M} = (M, S)$ in a countable language, $\mathrm{Def}(\mathfrak{M})$ is a countable set. By Lemma 2.3, any $T \in \mathbb{P}$ is contained in some $\mathrm{Def}(\mathfrak{M})$-generic filter. Such a filter is simply referred to as an $\mathfrak{M}$-generic filter.

Lemma 2.4

If $F \subseteq \mathbb{P}$ is an $\mathfrak{M}$-generic filter, then there exists a unique infinite path $G = \cap F = \cap_{T \in F} T$ common to all $T \in F$. That is, F is contained in the principal filter generated by G.

Proof For each $k \in M$, let $E_k = \{T \in \mathbb{P} \mid \exists! s \in \{0,1\}^k \; s \in T\}$ be dense and definable in $\mathfrak{M}$. If F is an $\mathfrak{M}$-generic filter, then for each k, there exists some $s_k \in \{0,1\}^k$ such that there is $T_k \in F$ with $T_k \cap \{0,1\}^k = \{s_k\}$. Moreover, if $k < k'$, then s_k is an initial segment of $s_{k'}$, and $s_{k'} \in T_k$. If not, $[T_k] \cap [T_{k'}] = \emptyset$[3], which would contradict the filter condition of F. Thus, let $G = \bigcup_{k \in M} s_k$; then $G = \bigcap_k T_k$ as well. Finally, to show $G = \cap F$, if $G \not\subseteq T \in F$, then there exists some k such that $s_k \notin T$, and $[T] \cap [T_k] = \varnothing$, which contradicts the filter condition of F. □

[3]Here, $[T]$ denotes $\{T' \in \mathbb{P} \mid T' \subset T\}$. In the latter half of part 8, the same notation $[T]$ represents the set of infinite paths of T. Since both are conventional, we would use both as they are.

Definition 2.5

$G(\subseteq M)$ is called an **$\mathfrak{M}$-generic path**, if for every dense set $D \in \mathrm{Def}(\mathfrak{M})$, there exists a tree $T \in D$ such that G is an infinite path through T.

Lemma 2.6

Every $T \in \mathbb{P}$ has an $\mathfrak{M}$-generic path G.

Proof By Lemma 2.3, every T is contained in some $\mathfrak{M}$-generic filter F. Then, by Lemma 2.4, there is a common infinite path G in the trees of F. It is clear from the definition that this G is an $\mathfrak{M}$-generic path. □

From now on, an $\mathfrak{M}$-generic path will simply be referred to as a generic path.

Lemma 2.7

If G is a generic path, then $(M, S \cup \{G\}) \models \Sigma^0_1$-induction.

Thank you for your attention!