Topics in Applied Math: Logic and Foundations of Mathematics

Part 6. Reverse Mathematics

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Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Reverse mathematics**
- Part 7. Second order arithmetic and non-standard methods

Part 6. Schedule

- Nov. 19, (1) Introduction and RCA_0
- Nov. 21, (2) Real numbers in RCA_0
- Nov. 26, (3) Completeness of the reals and ACA_0
- **Nov. 28, (4) Continuous functions and** WKL_0

§1. Recap

Reverse Mathematics: Which axioms are needed to prove a theorem?

The Reverse Mathematics Phenomenon

Many theorems of mathematics are either provable in RCA_0, *or logically equivalent (over* RCA_0*) to one of* WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Definition 1.2 The system of **recursive comprehension axioms** (RCA_0) consists of:

(0) Axioms and inference rules of first-order logic with axioms of equality for numbers.

(1) Basic arithmetic axioms: Same as $\mathsf{Q}_<$ (Chapter 4).

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and neither includes X as a free variable.

(3) Σ^0_1 induction: $\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

§2. Real numbers in RCA_0

- RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$. (**Lemma 1.3**)
- Primitive recursive functions (e.g., sequence numbers and Gödel numbers) are available in RCA_0.

So, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and their arithmetical operations are naturally defined in RCA_0.

A sequence of rational numbers $\{q_n\}$ is a **real number**, $\{q_n\} \in \mathbb{R}$, if it satisfies

$$\forall n \forall i (|q_n - q_{n+i}| \leq 2^{-n}).$$

The equality $=$, the inequality $<$ and the operations $+, \cdot$ on real numbers are defined as

$$\{p_n\} = \{q_n\} \leftrightarrow \forall n(|p_n - q_n| \leq 2^{-n+1}), \quad \{p_n\} < \{q_n\} \leftrightarrow \exists n(q_n - p_n > 2^{-n+1}),$$
$$\{p_n\} + \{q_n\} = \{p_{n+1} + q_{n+1}\}, \quad \{p_n\} \cdot \{q_n\} = \{p_{n+m} \cdot q_{n+m}\} \text{ for large enough } m.$$

Summary

It is provable in RCA_0 that $(\mathbb{R}, +, \cdot, 0, 1, <, =)$ is an **Archimedean ordered field**.

Sequences of real numbers

We define a **sequence of real numbers** as a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that:

for each n, letting $f_n(m) = f(n, m)$, $f_n : \mathbb{N} \to \mathbb{Q}$ is a real number.

The **limit** of $\{f_n\}$, denoted $\lim_{n\to\infty} f_n$, is defined as a unique real number a such that

$$\forall \varepsilon > 0 \exists n \forall i (|a - f_{n+i}| < \varepsilon).$$

Theorem 2.1 (Nested interval property of $\mathbb{R}$)

It is provable in RCA_0 that if $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers such that

$$a_n \le a_{n+1} \le b_{n+1} \le b_n (\text{for all } n), \text{ and } \lim_{n\to\infty} |a_n - b_n| = 0,$$

then there exists a real number c such that $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Theorem 2.2 (Uncountability of $\mathbb{R}$)

It is provable in RCA_0 that for any sequence of real numbers $\{a_n\}$, there exists a real number c such that $\forall n (a_n \neq c)$.

We will introduce continuous functions on $\mathbb{R}$ in RCA_0.

Definition 2.3

A set $\Phi \subseteq \mathbb{Q}^4$ that satisfies the following conditions is called the **code** for a **continuous function** $f : \mathrm{dom}\ f(\subseteq \mathbb{R}) \to \mathbb{R}$.

(1) $(p, q, r, s) \in \Phi \to p < q \land r \leq s$,

(2) $(p, q, r, s), (p', q', r', s') \in \Phi, p' < q \land p < q' \ \to \ r' \leq s \land r \leq s'$.

Intuitively, $(p, q, r, s) \in \Phi$ means $\forall x(p < x < q \to r \leq f(x) \leq s)$.
A real number x belongs to the **domain** of a continuous function f coded by Φ, if

$$\forall n \exists (p, q, r, s) \in \Phi(p < x < q \land s - r < 2^{-n}), \text{ denoted } x \in \mathrm{dom} f.$$

It is provable in RCA_0 that if $x \in \mathrm{dom} f$, there exists a unique real y such that $\forall (p, q, r, s) \in \Phi(p < x < q \to r \leq y \leq s)$. We denote this y as $f(x)$.

Theorem 2.4 (Intermediate Value Theorem)

The following is provable in RCA_0. Given a continuous function f such that its domain includes $[0, 1]$ and $f(0) < 0 < f(1)$, there exists an $x \in [0, 1]$ such that $f(x) = 0$.

§3. ACA_0 and WKL_0

Definition 3.1

The **system of arithmetical comprehension axioms** (ACA_0) is RCA_0 extended with (Π^1_0-CA) : $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is a Π^1_0 (arithmetical) formula with no free occurrence of X.

Lemma 3.2

ACA_0 is a conservative extension of Peano Arithmetic PA.

Lemma 3.3

In RCA_0, the following are equivalent:
(1) ACA_0, (2) (Σ^0_1-CA), (3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

Theorem 3.4

Each of the following propositions is equivalent to ACA_0 in RCA_0:
(1) Bolzano-Weierstrass Theorem, (2) Every Cauchy sequence converges,
(3) Monotone Convergence Theorem: Every bounded increasing sequence converges.

Defining WKL_0

Let Seq_2 denote the set of (finite) binary sequences. A subset T of Seq_2 is called a **tree**, if

$$\forall s, t \; (s \in \mathrm{Seq}_2 \wedge s \subseteq t \wedge t \in T \implies s \in T),$$

where $s \subseteq t$ means $\forall i < \mathrm{leng}(s)(s(i) = t(i))$.
An (infinite) **path** through T is a function $g : \mathbb{N} \to T$ such that $g(0) = \varnothing$, $g(n) \subseteq g(n+1)$ and $\neg \exists t \; (g(n) \subsetneq t \subsetneq g(n+1))$ for all $n \in \mathbb{N}$. A path g is often identified with its range.

Definition 3.5

Weak König's lemma asserts that every infinite subtree of Seq_2 has an infinite path. The system obtained from RCA_0 by adding this lemma as an axiom is called WKL_0.

Lemma 3.6

In RCA_0, WKL_0 is equivalent to the following statement:

$$(\Sigma^0_1\text{-SP}) : \forall n(\varphi(n) \to \psi(n)) \to \exists X \forall n \{(\varphi(n) \to n \in X) \wedge (n \in X \to \psi(n))\},$$

where $\varphi(n)$ is Σ^0_1 and $\psi(n)$ is Π^0_1, neither containing X as a free variable.

Corollary 3.7

WKL_0 is strictly stronger than RCA_0.

Proof Let Rec be the set of recursive subsets of ω. Then, (ω, Rec) is the minimal model of RCA_0, but it is not a model of WKL_0, since there are two disjoint Σ^0_1 sets A and B that are recursively inseparable. □

However, as shown later, WKL_0 and RCA_0 prove the same arithmetical formulas as $\mathrm{I}\Sigma_1$, though the first-order part of ACA_0 is just PA. Thus, these three systems have different strength.

Next, we show that the Heine-Borel theorem is equivalent to WKL_0 over RCA_0. First, we define some basic notions. An **open interval** with rational endpoints p, q $(p < q)$ is represented by the code of the pair (p, q). An **open set** of $\mathbb{R}$ is defined (encoded) as a set of codes of open intervals. We say that an open set U of $\mathbb{R}$ **covers** the closed unit interval $[0, 1]$, if for any real number $x \in [0, 1]$, there exists a code $(p, q) \in U$ such that $p < x < q$.

The **Heine-Borel (Covering) Theorem** asserts that if an open set U covers the closed interval $[0, 1]$, then there exists a finite subset U' of U that also covers $[0, 1]$.

Lemma 3.8

The Heine-Borel Theorem is provable in WKL_0.

Proof For each $s \in \mathrm{Seq}_2$, we associate an open interval (a_s, b_s) such that

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{s(i)}{2^{i+1}}, \quad b_s = a_s + \frac{1}{2^{\mathrm{leng}(s)}}.$$

We can easily see that if $s \subseteq t$, then $[a_t, b_t] \subseteq [a_s, b_s]$. For any infinite binary decimal $x = 0.s(0)s(1)s(2)\cdots$, we have $x \in [a_{s'}, b_{s'}]$ for all finite segments $s' \subset s$.

Now, take any open covering U of $[0, 1]$. For convenience, let's denote the open interval with code i as (p_i, q_i). Then, we define a tree $T \subseteq \mathrm{Seq}_2$ as follows:

$$s \in T \leftrightarrow \neg\exists i \leq \mathrm{leng}(s)(i \in U \wedge p_i < a_s < b_s < q_i)$$

$$\leftrightarrow \forall i \leq \mathrm{leng}(s)(i \in U \to [a_s, b_s] \not\subset (p_i, q_i)).$$

We first show that T has no infinite path.

By way of contradiction, we assume that there exists a path $f \subseteq T$, where

$$s \in T \leftrightarrow \forall i \leq \mathrm{leng}(s)(i \in U \to [a_s, b_s] \not\subset (p_i, q_i)).$$

By the nested interval property, there exists a (unique) real number x such that $a_s \leq x \leq b_s$ for all $s \in f$. Since the open set U covers $[0,1]$, there exists some $i \in U$ such that the real number x is contained in the open interval (p_i, q_i). Then, there exists an $s \in f$ with sufficient length such that $p_i < a_s \leq x \leq b_s < q_i$, which implies $s \notin T$, a contradiction.

Since T has no infinite path, by weak König's lemma, T is a finite set. This means that there exists a sufficiently large n such that all sequences belonging to T have a length shorter than n. Thus,

$$\forall s(\mathrm{leng}(s) = n \to \exists i \leq n(i \in U \land p_i < a_s < b_s < q_i)).$$

Then, since $\cup\{[a_s, b_s] : \mathrm{leng}(s) = n\} = [0,1]$, $\{i \in U : i \leq n\}$ also covers $[0,1]$. $\square$

Theorem 3.9

In RCA_0, the Heine-Borel Theorem is equivalent to WKL_0.

Proof We have already shown that the Heine-Borel Theorem holds in WKL_0. Now, we assume the Heine-Borel Theorem and derive the weak König's lemma.

First, let's discuss the idea behind the proof. The Heine-Borel Theorem implies the compactness of $[0, 1]$, which leads to the compactness of the ternary set (a closed set)

$$\left\{ \sum_{i=0}^{\infty} \frac{f(i)}{3^{i+1}} \mid f \in \{0, 2\}^{\mathbb{N}} \right\},$$

and hence also the compactness of the Cantor space $\{0, 1\}^{\mathbb{N}}$ since it is homeomorphic to the ternary set. Finally, the compactness of $\{0, 1\}^{\mathbb{N}}$ implies WKL_0.

For preparation, for each $s \in \mathrm{Seq}_2$, we associate the rational interval (a_s, b_s) defined as:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{2s(i)}{3^{i+1}}, \quad b_s = a_s + \frac{1}{3^{\mathrm{leng}(s)}}.$$

Then, the closed intervals $[a_{s^\frown 0}, b_{s^\frown 0}]$ and $[a_{s^\frown 1}, b_{s^\frown 1}]$ respectively become the left and right thirds of the interval $[a_s, b_s]$. Here, $s^\frown i$ denotes the sequence s followed by $i = 0, 1$. So, for any real number x not belonging to the ternary set, there exists exactly one open interval $(b_{s^\frown 0}, a_{s^\frown 1})$ containing it. Hence,

$$\bigcup\{(b_{s^\frown 0}, a_{s^\frown 1}) \mid s \in \mathrm{Seq}_2\}$$

is the complement of the ternary set.
Furthermore, for each $s \in \mathrm{Seq}_2$, define

$$a'_s = a_s - \frac{1}{3^{\mathrm{leng}(s)+1}}, \quad b'_s = b_s + \frac{1}{3^{\mathrm{leng}(s)+1}}.$$

Then, we have $[a_s, b_s] \subset (a'_s, b'_s)$ for each $s \in \mathrm{Seq}_2$, but (a'_s, b'_s) and (a'_t, b'_t) intersect only if either s or t is an initial segment of the other. For any real number x in the ternary set, there exists a unique $f \in \{0,1\}^{\mathbb{N}}$ such that $x \in (a'_s, b'_s)$ for any finite initial sequence $s \subset f$.

Now, to prove the weak König's lemma, take any (nonempty) tree $T \subseteq \mathrm{Seq}_2$ without infinite paths and show that T is finite.

Let B be the set of minimal binary sequences not in T, that is,

$$s \in B \Leftrightarrow s \notin T \wedge \forall t \subset s(t \neq s \to t \in T).$$

It's clear that any infinite path $f \subseteq \mathrm{Seq}_2$ shares exactly one element $s \in B$ and $s \subset f$.

Then, the following set U covers $[0,1]$:

$$U = \bigcup\{(a'_s, b'_s) : s \in B\} \cup \bigcup\{(b_{s^\frown 0}, a_{s^\frown 1}) : s \in \mathrm{Seq}_2\}.$$

So, by the Heine-Borel Theorem, U has a finite subcover U'.

Since for any $s \in B$, (a'_s, b'_s) does not intersect with any other $(a'_t, b'_t) \in U$ and is not a subset of $\bigcup\{(b_{s^\frown 0}, a_{s^\frown 1}) : s \in \mathrm{Seq}_2\}$, U' must contain $\{(a'_s, b'_s) : s \in B\}$. Therefore, B is finite.

Since T is obtained from the set of all initial segments of elements in B by removing the elements of B, it is also finite. □

The Heine-Borel property of $[0,1]$ allows us to derive various properties of continuous functions $f:[0,1]\to\mathbb{R}$.

Lemma 3.10

In WKL_0, a continuous function $f:[0,1]\to\mathbb{R}$ is uniformly continuous.

Proof Fix any $n\in\mathbb{N}$. We want to show the existence of $d>0$ such that

$$\forall x,y\in[0,1](\mid x-y\mid<d\to\mid f(x)-f(y)\mid<2^{-n}).$$

Let F be the code for the continuous function f, and denote the open interval with code i as (p_i,q_i). Then, define the open set U as follows:

$$i\in U\Leftrightarrow\exists j<i((p_i,q_i,p_j,q_j)\in F\wedge q_j-p_j<2^{-n-1}).$$

First, we show that U is a covering of $[0,1]$. For any real number $x\in[0,1]$, since $x\in\mathrm{dom}f$, there exists $(p_k,q_k,p_j,q_j)\in F$, such that

$$p_k<x<q_k\wedge q_j-p_j<2^{-n-1}.$$

Furthermore, there are infinitely many i such that $p_k\le p_i<x<q_i\le q_k$, so taking such an $i>j$, we have $i\in U$ with $p_i<x<q_i$. Therefore, U forms an open covering of $[0,1]$.

By the Heine-Borel Theorem, U has a finite subcover U'.

Let d be the minimum width $q_i - p_i$ among the intervals (p_i, q_i) in U'. We shall show that this d satisfies the uniform convergence condition.

Now, choose any real numbers $x, y \in [0, 1]$ such that $0 < y - x < d$. Then, there must exist intervals $(p_i, q_i), (p_{i'}, q_{i'})$ in U' such that $x \in (p_i, q_i)$, $y \in (p_{i'}, q_{i'})$ and that they have a common point z.

By way of contradiction, we assume that there are no such intervals. Then, we take an interval $(p_i, q_i) \ni x$ in U' with maximal q_i, and an interval $(p_{i'}, q_{i'}) \ni y$ in U' with minimal $p_{i'}$, and still have $q_i < p_{i'}$. Since U' is a covering, there exists $(p_k, q_k) \ni q_i$ in U'. By the maximality of q_i, $x \notin (p_k, q_k)$. Since $|q_k - p_k| \geq d > |x - y|$, we have $y \in (p_k, q_k)$, which contradicts with the minimality of $p_{i'}$.

By the definition of U, we have $\mid f(x) - f(z) \mid < 2^{-n-1}$ and $\mid f(y) - f(z) \mid < 2^{-n-1}$, thus $\mid f(x) - f(y) \mid < 2^{-n}$, which fulfills the lemma. □

Lemma 3.11

In WKL_0, a continuous function $f : [0,1] \to \mathbb{R}$ attains a maximum value.

Proof First, we show that the supremum M of the range of f exists.

As in the proof of the previous lemma, we define U by a Σ^0_0 formula:

$$i \in U \Leftrightarrow \exists j < i((p_i, q_i, p_j, q_j) \in F \wedge q_j - p_j < 2^{-n-1}).$$

We can finitely calculate whether or not a given finite set of open rational intervals covers $[0,1]$. Therefore, in WKL_0, we can construct a function extracting U' according to n.

For each $i \in U'$, select $j_i < i$ such that $(p_i, q_i, p_{j_i}, q_{j_i}) \in F \wedge q_{j_i} - p_{j_i} < 2^{-n-1}$, and let $M_n = \max\{q_{j_i} : i \in U'\}$. Thus, we can construct a sequence $\{M_n\}$, which becomes a real number, and it is clear that it is the supremum M of the range of f.

What remains is to show that the existence of a point $x = a$ such that $f(a) = M$. For the sake of the following argument, we redefine $M_n = \max\{p_{j_i} : i \in U'\}$. This ensures that for any n, $M_n \leq M = \{M_n\}$.

By way of contradiction, assume that $f(x) < M$ for all $x \in [0,1]$. Then, we define an open set V as follows:

$$i \in V \Leftrightarrow \exists j < i \ \exists n < i((p_i, q_i, p_j, q_j) \in F \wedge q_j < M_n).$$

To show that this set forms a covering of $[0,1]$, take any real number $x \in [0,1]$. Since $f(x) < M$, there exists n such that $f(x) < M_n \leq M$, and hence there exists $(p_k, q_k, p_j, q_j) \in F$ and n such that

$$p_k < x < q_k \ \wedge \ p_j \leq f(x) \leq q_j < M_n \leq M.$$

As there are infinitely many i such that $p_k \leq p_i < x < q_i \leq q_k$, taking $i > \max\{j, n\}$ ensures $i \in V$ with $p_i < x < q_i$. Therefore, V forms an open covering of $[0,1]$.

Again, by the Heine-Borel Theorem, V has a finite subcover V'. Let M' be the maximum of q_i for (p_i, q_i) in V'. Then, by the definition of values of a continuous function, obviously M' is an upper bound of the range. However, due to the finiteness of V', for some n, $M' < M_n \leq M$, which contradicts the fact that M is the supremum. □

Conversely, the properties described in the two lemmas above allow us to derive WKL_0. In sum, the following theorem holds:

Theorem 3.12

The following assertions are pairwise equivalent in RCA_0:

(1) WKL_0,

(2) A continuous function $f : [0, 1] \to \mathbb{R}$ is uniformly continuous,

(3) A continuous function $f : [0, 1] \to \mathbb{R}$ is bounded,

(4) A bounded continuous function $f : [0, 1] \to \mathbb{R}$ has a supremum,

(5) A continuous function $f : [0, 1] \to \mathbb{R}$ that has a supremum attains its maximum value.

Proof By Lemmas 3.10 and 3.11, we can deriving (2), (3), (4), and (5) from (1). Hence, it suffices to obtain counterexamples for (2), (3), (4) and (5) from the negation of (1). Now, assume the negation of (1). Then, there exists an infinite tree $T \subseteq \mathrm{Seq}_2$ without infinite paths.

As shown in the proof of Heine-Borel's theorem, for each $s \in \mathrm{Seq}_2$, define the two rational numbers a_s and b_s as follows:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{s(i)}{2^{i+1}},$$

$$b_s = a_s + \frac{1}{2^{\mathrm{leng}(s)}}.$$

Let B be the infinite set of all minimal binary sequences not in T,

$$s \in B \Leftrightarrow s \notin T \wedge \forall t \subset s(t \neq s \rightarrow t \in T)$$

and J be the set of closed intervals $[a_s, b_s]$ for all $s \in B$.

Each real number $x \in [0, 1]$ is either an interior point of exactly one interval in J or an endpoint of one or two intervals. Such an infinite set J is called a **singular closed cover**.

$\neg$ WKL$_0$ $\rightarrow$ $\neg$ (3) bounded, $\neg$ (5) a maximum value.

We will construct a counterexample for (3) using this singular closed cover J. This also serves as a counterexample for (2) since (2) implies (3). We define a continuous function f_s for each interval $[a_s, b_s]$ in J as follows:

$$f_s(x) = \begin{cases} \mathrm{leng}(s)\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ \mathrm{leng}(s)\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s. \end{cases}$$

That is, f_s takes 0 at the endpoints $x = a_s, b_s$, takes $\mathrm{leng}(s)$ at the midpoint $x = \frac{a_s+b_s}{2}$, and is linearly interpolated otherwise.

Let f be a function obtained by composing all such functions f_s. Then, it is clearly continuous but unbounded. (Eexercise: construct a continuous function code for f.)

A counterexample for (5) can be constructed in the similar way as follows:

$$f_s(x) = \begin{cases} (1-2^{-\mathrm{leng}(s)})\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ (1-2^{-\mathrm{leng}(s)})\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s. \end{cases}$$

Then, f clearly has 1 as its supremum, but it can not attain the maximum value 1 in $[0, 1]$.

$\neg\,\mathrm{WKL}_0 \to \neg$ (4) a supremum.

Theorem 3.4.

($\mathrm{RCA}_0 \vdash$) $\mathrm{ACA}_0 \Leftrightarrow$ (5) Every bounded increasing sequence of reals has a supremum.

Negating WKL_0, we have the negation of ACA_0, which implies the existence of a bounded increasing sequence of rational numbers $\{c_n\}$ that lacks a supremum.
Then, replace the maximum value of f_s with $c_{\mathrm{leng}(s)}$ and proceed similarly.

$$f_s(x) = \begin{cases} c_{\mathrm{leng}(s)} \frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ c_{\mathrm{leng}(s)} \frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s, \end{cases}$$

HW Problem # 5-3

Show that in the theorem 3.12 (4) and (5), "continuous function" can be replaced with "uniformly continuous function". Hint: It is beneficial to use a singular closed cover for the ternary set.

- The deadline for the fifth homework is changed to December 10th.
- The final report problems will be assigned on Dec. 10 and 12, and due Jan. 12.

Complete separable metric spaces and continuous functions

Let A be a non-empty subset of $\mathbb{N}$. Suppose $d : A \times A \to \mathbb{R}$ is a (pseudo) metric on A. A sequence $\{a_n\}$ from A satisfying $\forall n \forall i\ d(a_n, a_{n+i}) \leq 2^{-n}$ is called a point of $\hat{A}$, and we write $\{a_n\} \in \hat{A}$. $\hat{A}$ can be viewed as a **complete separable metric space**.

Example. If $A = \mathbb{Q}$ and $d(p, q) = |p - q|$, then $\hat{A}$ is nothing but $\mathbb{R}$.
Also, if $A = \mathbb{Q}^2$ and $d((p, q), (p', q')) = \sqrt{(p - p')^2 + (q - q')^2}$, then $\hat{A}$ is $\mathbb{R}^2$.

In a metric space $\hat{A}$, an **open ball** $B_r(a)$ centered at $a \in A$ with a rational radius $r > 0$ is coded by the pair $(a, r)(\in A \times \mathbb{Q}^+)$. An **open set** is a set of codes of open balls.

The code F of a **continuous function** f from a metric space $\hat{A}$ to a metric space $\hat{B}$ is a subset of $A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$, fulfilling conditions similar to those for a continuous function from $\mathbb{R}$ to $\mathbb{R}$, by which

$$(a, r, b, s) \in F \text{ means } x \in B_r(a) \to f(x) \in \overline{B_s(b)} \text{ (closed ball).}$$

Brouwer's Fixed-Point Theorem

Brouwer's Fixed-Point Theorem states that any continuous function $f : [0,1]^n \to [0,1]^n$ has a fixed point, i.e., a point x such that $f(x) = x$.

Theorem 3.13 (Shioji-T.)

Brouwer's Fixed-Point Theorem is equivalent to WKL_0 over RCA_0.

In WKL_0, Brouwer's Fixed-Point Theorem can be extended to the infinite-dimensional space $[0,1]^{\mathbb{N}} (\subseteq \mathbb{R}^{\mathbb{N}})$, which is known as the **Tychonoff-Schauder fixed-point theorem**.

By utilizing this fixed-point theorem, the **Cauchy-Peano theorem** for the existence of local solutions to ordinary differential equations can be derived within WKL_0, and the converse is also provable.

Various fixed-point theorems and their applications (e.g., the Hahn-Banach theorem) have been studied by N. Shioji and K. Tanaka [Fixed point theory in weak second-order arithmetic *Ann. Pure Appl. Logic*, 47, 167-188, 1990].

§4. König's Lemma and Ramsey's theorem

We begin with König's Lemma, not the "weak" version.

Let Seq denote the set of finite sequences from $\mathbb{N}$, that is, the set of functions with domain $\{i \in \mathbb{N} : i < n\}$ for some $n \in \mathbb{N}$.

A subset T of Seq, which is closed under initial segment, is called a **tree**.

A tree T is said to be **finitely branching**, if each node $s \in T$ has at most finitely many children, i.e.,

$$\forall s(s \in T \to \exists n \forall m(s^\frown m \in T \to m < n))$$

A subtree of T that never branches is called a **path** of T.

König's Lemma asserts that "every infinite, finitely branching tree has an infinite path." Weak König's Lemma is König's Lemma about special trees consisting of binary sequences. As we will see, König's Lemma is equivalent to ACA_0, and thus it is properly stronger than weak König's Lemma.

Theorem 4.1

Over RCA_0, the following are pairwise equivalent:

(1) ACA_0

(2) König's Lemma

(3) An infinite tree T, such that each node $s \in T$ has at most two children $s^\frown m \in T$ $(m \in \mathbb{N})$, has an infinite path.

Note: In the above (3), it is crucial that m such that $s^\frown m \in T$ is not bounded over T. If m were bounded, the assertion would be equivalent to weak König's Lemma.

Proof $(1) \Rightarrow (2)$. Given an infinite finitely branching tree T, let T' be the set of $s \in T$ that have an infinitely many descendants $t \supseteq s$ (by (Π^1_0-CA)).

Then, using primitive recursion, define a path g in T' as follows:

$g(0) =$ empty sequence,

$g(n+1) = g(n)^\frown m,$ where m is the smallest number such that $g(n)^\frown m \in T'$.

$(2) \Rightarrow (3)$ is trivial. To show $(3) \Rightarrow (1)$, assume (3) and show the existence of range of a given 1-1 function $f : \mathbb{N} \to \mathbb{N}$, which is equivalent to ACA_0, by Lemma 3.3.(3).

Define a tree T as follows: $s \in T \Leftrightarrow$

(a) $\forall m, n < \mathrm{leng}(s)(f(m) = n \leftrightarrow s(n) = m + 1)$,

(b) $\forall n < \mathrm{leng}(s)(s(n) > 0 \to f(s(n) - 1) = n)$.

Then, each node $t \in T$ has at most two children $t^\frown k \in T$. This is because letting $s = t^\frown k$, $n = \mathrm{leng}(s) - 1$ in (b), we have $k = 0$ or $k = f^{-1}(n) + 1$ if $f^{-1}(n)$ exists.

Next, show that T is infinite. For this, it suffices to show that for any $k \in \mathbb{N}$, there exists a sequence $s \in T$ with $\mathrm{leng}(s) = k$. First, by bounded $(\Sigma^0_1\text{-CA})$, $Y = \mathrm{ran} f \cap k$, that is, $\{n \in \mathrm{ran} f : n < k\}$ exists. Then, define a sequence s of length k as follows: for $n < k$,

$$s(n) = \begin{cases} 0 & \text{if } n \notin Y \\ m+1 & \text{if } n \in Y \wedge f(m) = n \end{cases}$$

Obviously, $s \in T$. So, T satisfies the conditions of (3).
Now, by (3), the tree T has an infinite path g. From the condition (a) of T,

$$\forall m, n(f(m) = n \leftrightarrow g(n) = m + 1).$$

Thus, setting $X = \{n : g(n) > 0\}$, we have $X = \mathrm{ran} f$. □

Ramsey's Theorem

Ramsey's Theorem was first invented by F. Ramsey in order to settle Hilbert's decision problem for first-order logic, though he only succeeded partially and we have no space to explain his original motivation and results.

For a set $X \subseteq \mathbb{N}$, we denote by $[X]^k$ the set of all sequences $(m_1, \ldots, m_k)$ of k elements from X such that $m_1 < \ldots < m_k$. Somewhat naïvely, (infinite) Ramsey's theorem RT^k_l states that for a coloring of $[\mathbb{N}]^k$ into l colors, there exists an infinite subset $X \subseteq \mathbb{N}$ such that $[X]^k$ is monochromatic[1]. More precisely, we state it as follows.

Definition 4.2 (Ramsey's Theorem)

Let $k, l > 0$ be natural numbers. Ramsey's Theorem RT^k_l is the following assertion:

$$\forall f : [\mathbb{N}]^k \to \{0, 1, \ldots, l-1\} \exists X \subseteq \mathbb{N} (X \text{ is infinite} \land f \text{ is constant on } [X]^k).$$

[1] Finite Ramsey's Theorem, denoted $m \to (n)^k_l$, is the statement that if $[\{0, \ldots, n-1\}]^k$ is painted in l colors, there exists a subset $X \subseteq \{0, \ldots, n-1\}$ of m elements such that $[X]^k$ is monochromatic. The finite version can be derived from the infinite version by the compactness argument.

For example, RT^2_l can be interpreted as follows: If all pairs $\{m, n\}$ of natural numbers are painted in l colors, then there always exists an infinite set X such that all pairs of elements from X are painted the same color. Such an X is called a **homogeneous** set.

If we consider the statement of painting any finite number of colors, we denote it as RT^k, i.e., $\mathsf{RT}^k \equiv \forall l \in \mathbb{N}(\mathsf{RT}^k_l)$.

Although RT^k_l for any standard natural number $l \geq 2$ can be deduced from RT^k_2 by meta-induction in RCA_0, the equivalence of RT^k to RT^k_2 may require Π^1_2-induction, since RT^k_l is a Π^1_2 formula.

First consider the strength of RT^1, which is also known as the (infinite version of) **pigeonhole principle** (PHP). For a standard natural number $l \geq 1$, RT^1_l obviously holds even in RCA_0. The question is how much restricted induction is needed to derive $\forall l\, \mathsf{RT}^1_l$.

Recall: the collection principle ($\mathsf{B}\varphi$) for $\varphi(x, y_1, \cdots, y_k)$ in $\mathcal{L}_{\mathrm{O}R}$ is as follows

$$\forall x < u \exists y_1 \cdots \exists y_k \varphi(x, y_1, \cdots, y_k) \to \exists v \forall x < u \exists y_1 < v \cdots \exists y_k < v \varphi(x, y_1, \cdots, y_k).$$

$\mathsf{B}\Pi^0_1$ denotes $\{(\mathsf{B}\varphi) \mid \varphi \in \Pi^0_1\}$. $\mathsf{B}\Pi^0_1$ is equivalent to $\mathsf{B}\Sigma^0_2$, and $\mathsf{I}\Sigma^0_1 \subsetneq \mathsf{B}\Sigma^0_2 \subsetneq \mathsf{I}\Sigma^0_2$. $\mathsf{B}\Pi^0_1$ is not provable in WKL_0, but obviously provable in ACA_0.

Theorem 4.3 (J. Hirst)

In RCA_0, RT^1 is equivalent to $\mathsf{B}\Pi^0_1$.

The above theorem indicates that the strength of RT^1 is intermediate between ACA_0 and RCA_0, and it is incomparable with WKL_0.

Theorem 4.4

In ACA_0, both RT^1 and $\forall k(\mathsf{RT}^k \to \mathsf{RT}^{k+1})$ are provable.

Since RT^k is a Π^1_2 statement, the theorem above does not allow us to derive $\forall k\, \mathsf{RT}^k$ within induction of ACA_0. Paris and Harrington formulated a proposition PH in first-order arithmetic to express something like $\forall k\, \mathsf{RT}^k$, and proved that PH is independent from PA.

Lemma 4.5

Within RCA_0, ACA_0 can be derived from RT^3_2.

Theorem 4.6

For any standard numbers $k \geq 3$, $l \geq 2$, RT^k_l, RT^k, and ACA_0 are equivalent over RCA_0.

Finally, concerning RT^2 and RT^2_2, it is known that both are between ACA_0 and RCA_0, and are incomparable with WKL_0. Within RCA_0, RT^2 implies $\mathsf{B}\Pi^0_2$, but RT^2_2 does not.

Thank you for your attention!