Topics in Applied Math: Logic and Foundations of Mathematics

Part 6. Reverse Mathematics

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November 26, 2025

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Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
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Part 6. Schedule

- Nov. 19, (1) Introduction and RCA_0
- Nov. 21, (2) Real numbers in RCA_0
- Nov. 26, (3) Completeness of the reals and ACA_0
- Nov. 28, (4) Continuous functions and WKL_0

§1. Recap

Reverse Mathematics: Which axioms are needed to prove a theorem?

The Reverse Mathematics Phenomenon

Many theorems of mathematics are either provable in RCA_0, *or logically equivalent* (*over* RCA_0) *to one of* WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Definition 1.2 The system of **recursive comprehension axioms** (RCA_0) consists of:

(0) Axioms and inference rules of first-order logic with axioms of equality for numbers.

(1) Basic arithmetic axioms: Same as $\mathsf{Q}_<$ (Chapter 4).

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ^0_1, $\psi(n)$ is Π^0_1, and neither includes X as a free variable.

(3) Σ^0_1 induction: $\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

Recap

- RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$. (**Lemma 1.3**)

The **pair** of natural numbers (m, n) is coded by a natural number $\frac{(m+n)(m+n+1)}{2} + m$.
The **product** $X \times Y$ is the set of pairs (codes) of one from X and the other from Y.
A **function** $f : X \to Y$ is a subset $F \subseteq X \times Y$ such that $\forall x \in X \exists! y \in Y (x, y) \in F$.
A function f whose domain is $X = \{i : i < n\}$ is called a **finite sequence** with **length** n.

In RCA_0, the following holds:

- (1) Π^0_1 induction. (2) The class of Σ^0_1 formulas is closed under bounded quantification.
- The set of total functions is closed under primitive recursion.
- The set of (partial) functions is closed under minimization μ. Moreover, if $\forall \vec{x} \exists y f(\vec{x}, y) = 0$ is provable, then $\mu y(f(\vec{x}, y) = 0)$ exists as a total function.
- For any Σ^0_1 formula $\varphi(x)$, there exists a finite set X s.t. $\forall x(x \in X \leftrightarrow \varphi(x))$, or there exists a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ s.t. $\forall y(\exists x f(x) = y \leftrightarrow \varphi(y))$.
- (Bounded Σ^0_1-CA) : $\forall x \exists X \forall y(y \in X \leftrightarrow (y < x \land \varphi(y)))$, where $\varphi(y)$ is Σ^0_1.

§2. Real numbers in RCA_0

First, we denote the set of all natural numbers $\{n : n = n\}$ by $\mathbb{N}$. Arithmetical operations on $\mathbb{N}$ such as $+$ and $\cdot$ are simply taken as the corresponding operations in RCA_0.

We will use ω to denote the totality of standard natural numbers.

An equivalence relation $=_{\mathbb{Z}}$ on $\mathbb{N} \times \mathbb{N}$ is defined by $(k, l) =_{\mathbb{Z}} (m, n) \leftrightarrow k + n = l + m$. Then, a pair with the smallest code in each equivalence class of $=_{\mathbb{Z}}$ is called an **integer**. The set of all integers is denoted by $\mathbb{Z}$. The operations on $\mathbb{Z}$ are defined as

$$(k, l) + (m, n) =_{\mathbb{Z}} (k + m, l + n), \quad (k, l) \cdot (m, n) =_{\mathbb{Z}} (km + ln, kn + lm), \text{ etc.}$$

Next, $=_{\mathbb{Q}}$ on $\mathbb{Z} \times (\mathbb{Z} - \{(0, 0)\})$ is defined by $(k, l) =_{\mathbb{Q}} (m, n) \leftrightarrow kn = lm$. Then, a pair with the smallest code in each equivalence class of $=_{\mathbb{Q}}$ is called a **rational number**, and denote the set of all such numbers by $\mathbb{Q}$. The operations on $\mathbb{Q}$ are defined as

$$(k, l) + (m, n) =_{\mathbb{Q}} (kn + lm, ln), \quad (k, l) \cdot (m, n) =_{\mathbb{Q}} (km, ln), \text{ etc.}$$

Real numbers $\mathbb{R}$

A sequence of rational numbers $\{q_n\}$ is called a **real number**, $\{q_n\} \in \mathbb{R}$, if it satisfies

$$\forall n \forall i (|q_n - q_{n+i}| \leq 2^{-n}).$$

The equality $=$ and inequality $<$ on real numbers are defined as

$$\{p_n\} = \{q_n\} \leftrightarrow \forall n (|p_n - q_n| \leq 2^{-n+1}),$$
$$\{p_n\} < \{q_n\} \leftrightarrow \exists n (q_n - p_n > 2^{-n+1}).$$

It is easy to see that for any two reals $\{p_n\}$ and $\{q_n\}$, exactly one of the following holds:

$$\{p_n\} = \{q_n\}, \{p_n\} < \{q_n\} \text{ or } \{q_n\} < \{p_n\}.$$

The **sum** of reals $\{p_n\}$ and $\{q_n\}$ is defined as $\{p_n\} + \{q_n\} = \{p_{n+1} + q_{n+1}\}$.
The **product** of reals $\{p_n\}$ and $\{q_n\}$ is defined as $\{p_n\} \cdot \{q_n\} = \{p_{n+m} \cdot q_{n+m}\}$, where m is the smallest natural number such that $\max(|p_0|, |q_0|) + 1 \leq 2^{m-1}$.

Summary

It is provable in RCA_0 that $(\mathbb{R}, +, \cdot, 0, 1, <, =)$ is an **Archimedean ordered field**.

Sequences of real numbers

We define a **sequence of real numbers** as a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that:

for each n, letting $f_n(m) = f(n, m)$, $f_n : \mathbb{N} \to \mathbb{Q}$ is a real number.

The **limit** of $\{f_n\}$, denoted $\lim\limits_{n\to\infty} f_n$, is defined as a unique real number a such that

$$\forall \varepsilon > 0 \exists n \forall i (|a - f_{n+i}| < \varepsilon).$$

Theorem 2.1 (Nested interval property of $\mathbb{R}$)

It is provable in RCA_0 that if $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers such that

$$a_n \le a_{n+1} \le b_{n+1} \le b_n (\text{for all } n), \text{ and } \lim_{n\to\infty} |a_n - b_n| = 0,$$

then there exists a real number c such that $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Theorem 2.2 (Uncountability of $\mathbb{R}$)

It is provable in RCA_0 that for any sequence of real numbers $\{a_n\}$, there exists a real number c such that $\forall n(a_n \neq c)$.

We will introduce continuous functions on $\mathbb{R}$ in RCA_0.

Definition 2.3

A set $\Phi \subseteq \mathbb{Q}^4$ that satisfies the following conditions is called the **code** for a **continuous function** $f : \mathrm{dom}\ f(\subseteq \mathbb{R}) \to \mathbb{R}$.

(1) $(p, q, r, s) \in \Phi \to p < q \wedge r \leq s$,

(2) $(p, q, r, s), (p', q', r', s') \in \Phi, p' < q \wedge p < q' \ \to \ r' \leq s \wedge r \leq s'$.

In (2), $p' < q \wedge p < q'$ is $(p, q) \cap (p', q') \neq \varnothing$, and $r' \leq s \wedge r \leq s'$ is $[r, s] \cap [r', s'] \neq \varnothing$. Intuitively, $(p, q, r, s) \in \Phi$ means $\forall x(p < x < q \to r \leq f(x) \leq s)$.
A real number x belongs to the **domain** of a continuous function f coded by Φ, if

$$\forall n \exists (p, q, r, s) \in \Phi(p < x < q \wedge s - r < 2^{-n}), \text{ denoted } x \in \mathrm{dom} f.$$

It is provable in RCA_0 that if $x \in \mathrm{dom} f$, there exists a unique real y such that $\forall (p, q, r, s) \in \Phi(p < x < q \to r \leq y \leq s)$. We denote this y as $f(x)$.

Theorem 2.4 (Intermediate Value Theorem)

The following is provable in RCA_0. Given a continuous function f such that its domain includes $[0, 1]$ and $f(0) < 0 < f(1)$, there exists an $x \in [0, 1]$ such that $f(x) = 0$.

§3. ACA_0

Definition 3.1

The **system of arithmetical comprehension axioms** (ACA_0) is RCA_0 extended with
(Π^1_0-CA) : $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is a Π^1_0 (arithmetical) formula with no free occurrence of X.

Lemma 3.2

ACA_0 is a conservative extension of Peano Arithmetic PA.

Lemma 3.3

In RCA_0, the following are equivalent:
(1) ACA_0, (2) (Σ^0_1-CA), (3) The range of any 1-1 function $f : \mathbb{N} \to \mathbb{N}$ exists.

Theorem 3.4

Each of the following propositions is equivalent to ACA_0 in RCA_0:
(1) Bolzano-Weierstrass Theorem, (2) Every Cauchy sequence converges,
(3) Monotone Convergence Theorem: Every bounded increasing sequence converges.

Trees and paths

Despite the necessity of ACA_0 for the completeness of real numbers, many important theorems in analysis can be proved in a weaker system WKL_0, and moreover they often turn out to be equivalent to it.
First, we introduce necessary concepts to define the system WKL_0.

(A code s ($\in \mathbb{N}$) for) a function $f : \{i \in \mathbb{N} : i < n\} \to \mathbb{N}$ (where $n \in \mathbb{N}$) is called a **finite sequence** of length $n = \mathrm{leng}(s)$. Particularly, finite sequences that only take values 0 or 1 are called **binary sequences**, and the set of binary sequences is denoted by Seq_2.

A subset T of Seq_2 is called a **tree**, if it is closed under **initial segments**, i.e.,

$$\forall s, t\ (s \in \mathrm{Seq}_2 \land s \subseteq t \land t \in T \implies s \in T),$$

where $s \subseteq t$ means $\forall i < \mathrm{leng}(s)(s(i) = t(i))$.

A subset of a tree T that is itself a tree and has no branches (i.e., for any two elements, one is an initial segment of the other) is called a **branch** or **path** of T.
An (infinite) **path** through T is also expressed as a function $g : \mathbb{N} \to \mathrm{Seq}_2$ such that $g(0) = \varnothing$, $g(n) \subseteq g(n+1) \in T$ and $\neg\exists t\ (g(n) \subsetneqq t \subsetneqq g(n+1))$ for all $n \in \mathbb{N}$.

Defining WKL_0

Definition 3.5

Weak König's lemma asserts that every infinite subtree of Seq_2 has an infinite path. The system obtained from RCA_0 by adding this as an axiom is called WKL_0.

To show the existence of a specific path through a given infinite tree such as the left-most path, we need ACA_0, which is properly stronger than WKL_0, as we will see later. We first show that WKL_0 is strictly stronger than RCA_0.

Lemma 3.6

In RCA_0, WKL_0 is equivalent to the following statement:

$$(\Sigma^0_1\text{-SP}) : \forall n(\varphi(n) \to \psi(n)) \to \exists X \forall n\{(\varphi(n) \to n \in X) \land (n \in X \to \psi(n))\},$$

where $\varphi(n)$ is Σ^0_1 and $\psi(n)$ is Π^0_1, neither containing X as a free variable.

SP stands for the **Separation Principle**. Note that in the above, replacing the premise $\forall n(\varphi(n) \to \psi(n))$ with $\forall n(\psi(n) \leftrightarrow \psi(n))$ leads to the Δ^0_1 comprehension axiom. That is, WKL_0 is obtained from RCA_0 by replacing $(\Delta^0_1\text{-CA})$ with $(\Sigma^0_1\text{-SP})$.

Proof of Lemma 3.6 First, we show that the Σ^0_1 Separation Principle holds in WKL_0. Given two Σ^0_1 formulas $\varphi_0(n)$ and $\varphi_1(n)$ with

$$\forall n(\varphi_0(n) \to \neg\varphi_1(n)), \text{ or equivalently } \forall n \neg(\varphi_0(n) \land \varphi_1(n)).$$

Assume $\varphi_i(n) \equiv \exists m \theta_i(m, n)$, where $\theta_i(m, n) \in \Sigma^0_0$. Then, define a set $T \subseteq \mathrm{Seq}_2$ as follows

$$t \in T \Leftrightarrow \forall m, n < \mathrm{leng}(t)[(\theta_0(m, n) \to t(n) = 0) \land (\theta_1(m, n) \to t(n) = 1)].$$

It is easy to see that T forms an infinite tree, and so by weak König's lemma, there exists an infinite path f. Finally, setting $X = \{n : f(n) = 0\}$, we clearly have

$$\forall n\{(\varphi_0(n) \to n \in X) \land (n \in X \to \neg\varphi_1(n))\}.$$

Conversely, we prove weak König's lemma from the Σ^0_1 Separation Principle. Fix any infinite tree $T \subseteq \mathrm{Seq}_2$. For $i = 0, 1$, let $\varphi_i(s)$ denote the Σ^0_1 formula expressing

"$s \in \mathrm{Seq}_2$ and the set $\{t \in T : s^\frown i \subseteq t\}$ is finite".

Here, $s^\frown i$ denotes the binary sequence s followed by i, that is, $s \cup \{(\mathrm{leng}(s), i)\}$. "Tree T' is finite" can be expressed as $T' \cap \{0, 1\}^n \neq \varnothing$ for some sufficiently large n, which can be written as a Σ^0_1 formula.

Now, let $\varphi_i(s) \equiv \exists m \theta_i(m, s)$ with $\theta_i(m, s) \in \Sigma^0_0$, and modify them as

$$\varphi'_0(s) \equiv \exists m(\theta_0(m, s) \land \forall k < m \neg\theta_1(k, s)),$$

$$\varphi'_1(s) \equiv \exists m(\theta_1(m, s) \land \forall k \leq m \neg\theta_0(k, s)).$$

Since $\forall n \neg(\varphi'_0(n) \land \varphi'_1(n))$, or $\forall n(\varphi'_0(n) \to \neg\varphi'_1(n))$, the Σ^0_1 Separation Principle ensures the existence of X such that $\forall n\{(\varphi'_0(n) \to n \in X) \land (n \in X \to \neg\varphi'_1(n))\}$.

Using X, we recursively define an increasing sequence of binary sequences $s_0 \subset s_1 \subset \cdots$ as follows: Let s_0 be the empty sequence. If $s_n \in X$, then

$$s_{n+1} = \begin{cases} s_n{}^\frown\{1\} & \text{if } s_n \in X, \\ s_n{}^\frown\{0\} & \text{o.w.} \end{cases}$$

We show by induction that for all n, $T_n = \{t \in T : s_n \subseteq t\}$ is infinite. So in particular, it's easy to verify that $s_n \in T$. Thus, $f = \{s_n\}$ forms an infinite path through T. $\square$

With the above lemma, the following is straightforward.

Corollary 3.7

WKL_0 is strictly stronger than RCA_0.

Proof Let Rec be the set of recursive subsets of ω. Then, (ω, Rec) is the minimal model of RCA_0, but it is not a model of WKL_0, since as shown below, there are two disjoint Σ^0_1 sets A and B that are recursively inseparable.

Let A and B be defined as follows:

$A = \{\ulcorner\sigma\urcorner \mid \mathsf{R} \vdash \sigma\}$ (the set of Gödel numbers of the theorems of R)

$B = \{\ulcorner\sigma\urcorner \mid \mathsf{R} \vdash \neg\sigma\}$ (the set of Gödel numbers of the negations of the theorems of R)

Then, by way of contradiction, we assume the existence of a recursive set C such that $A \subset C \subset B^c$.

Since C is a recursive, by theorem in Ch.4, there exists Σ^0_1 formula $\varphi(x)$ such that

$$n \in C \to \mathsf{R} \vdash \varphi(\bar{n})$$
$$n \notin C \to \mathsf{R} \vdash \neg\varphi(\bar{n})$$

By Diagonalization Lemma in Ch.4, there exists a fixed point σ for $\neg\varphi(x)$ such that

$$\ulcorner\sigma\urcorner \notin C \to \mathsf{R} \vdash \neg\varphi(\overline{\ulcorner\sigma\urcorner}) \to \mathsf{R} \vdash \sigma \to \ulcorner\sigma\urcorner \in A.$$

$$\ulcorner\sigma\urcorner \in C \to \mathsf{R} \vdash \varphi(\overline{\ulcorner\sigma\urcorner}) \to \mathsf{R} \vdash \neg\sigma \to \ulcorner\sigma\urcorner \in B,$$

Then, C does not separate A and B.

Therefore, (Σ^0_1-SP) does not hold in (ω, Rec).

□

Hine-Borel theorem in WKL_0

An **open interval** with rational endpoints p, q $(p < q)$ is represented by the code of the pair (p, q).

An **open set** of $\mathbb{R}$ is defined (encoded) as a set of codes of open intervals.

Now, we say that an open set U of $\mathbb{R}$ **covers** the closed unit interval $[0, 1]$, if for any real number $x \in [0, 1]$, there exists a code $(p, q) \in U$ such that $p < x < q$.

The **Heine-Borel (Covering) Theorem** asserts that if an open set U covers the closed interval $[0, 1]$, then there exists a finite subset U' of U that also covers $[0, 1]$.

We show that this theorem is equivalent to WKL_0 over RCA_0. First, we show ...

Lemma 3.8

The Heine-Borel Theorem is provable in WKL_0.

Proof For each $s \in \mathrm{Seq}_2$, we associate an open interval (a_s, b_s) such that

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{s(i)}{2^{i+1}}, \quad b_s = a_s + \frac{1}{2^{\mathrm{leng}(s)}}.$$

We can easily see that if $s \subseteq t$, then $(a_t, b_t) \subseteq (a_s, b_s)$.

Now, take any open covering U of $[0,1]$. Let's denote the open interval with code i as (p_i, q_i). Then, define a tree $T \subseteq \mathrm{Seq}_2$ as follows:

$$s \in T \leftrightarrow \neg \exists i \leq \mathrm{leng}(s)(i \in U \land p_i < a_s < b_s < q_i).$$

We show that T has no infinite path.

By way of contradiction, we assume that there exists a path $f \subseteq T$, where

$$s \in T \leftrightarrow \forall i \leq \text{leng}(s)(i \in U \to [a_s, b_s] \not\subset (p_i, q_i)).$$

By the nested interval property, there exists a (unique) real number x such that $a_s \leq x \leq b_s$ for all $s \in f$. Since the open set U covers $[0, 1]$, there exists some $i \in U$ such that the real number x is contained in the open interval (p_i, q_i). Then, there exists an $s \in f$ with sufficient length such that $p_i < a_s \leq x \leq b_s < q_i$, which implies $s \notin T$, a contradiction.

If T has no infinite path, then by weak König's lemma, T is a finite set. This means that there exists a sufficiently large n such that all sequences belonging to T have a length shorter than n. Thus,

$$\forall s(\text{leng}(s) = n \to \exists i \leq n(i \in U \land p_i < a_s < b_s < q_i)).$$

Therefore, $\{i \in U : i \leq n\}$ forms a finite covering of $[0, 1]$. □

Theorem 3.9

In RCA_0, the Heine-Borel Theorem is equivalent to WKL_0.

Proof We have already shown that the Heine-Borel Theorem holds in WKL_0. Now, we assume the Heine-Borel Theorem and derive the weak König's lemma.

First, let's discuss the idea behind the proof. The Heine-Borel Theorem implies the compactness of $[0,1]$, which leads to the compactness of the ternary set (a closed set)

$$\left\{ \sum_{i=0}^{\infty} \frac{f(i)}{3^{i+1}} \mid f \in \{0,2\}^{\mathbb{N}} \right\},$$

and hence also the compactness of the Cantor space $\{0,1\}^{\mathbb{N}}$ since it is homeomorphic to the ternary set. Finally, the compactness of $\{0,1\}^{\mathbb{N}}$ implies WKL_0.

For preparation, for each $s \in \mathrm{Seq}_2$, we associate the rational open interval (a_s, b_s) defined as follows:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{2s(i)}{3^{i+1}},$$

$$b_s = a_s + \frac{1}{3^{\mathrm{leng}(s)}}.$$

Let $s^\frown i$ simply denote the binary sequence s followed by $i = 0, 1$, i.e., $s \cup \{(\mathrm{leng}(s), i)\}$.

Then, the closed intervals $[a_{s^\frown 0}, b_{s^\frown 0}]$ and $[a_{s^\frown 1}, b_{s^\frown 1}]$ respectively become the left and right thirds of the closed interval $[a_s, b_s]$.

Thus, for any real number x not belonging to the ternary set, there exists exactly one open interval $(b_{s^\frown 0}, a_{s^\frown 1})$ containing it. Especially,

$$\bigcup \{(b_{s^\frown 0}, a_{s^\frown 1}) \mid s \in \mathrm{Seq}_2\}$$

is the complement of the ternary set.

Furthermore, for each $s \in \mathrm{Seq}_2$, define

$$a'_s = a_s - \frac{1}{3^{\mathrm{leng}(s)+1}},$$

$$b'_s = b_s + \frac{1}{3^{\mathrm{leng}(s)+1}}.$$

Then, for any real number x in the ternary set, there exists a unique $f \in \{0,1\}^{\mathbb{N}}$ such that: for any finite initial sequence $s \subset f, x \in (a'_s, b'_s)$. Note that two open intervals (a'_s, b'_s) and (a'_t, b'_t) intersect only if either s or t is an initial segment of the other.

Now, let's consider any (nonempty) tree $T \subseteq \mathrm{Seq}_2$ without infinite paths and show that T is finite.

Let B be the set of minimal binary sequences not in T, that is,

$$s \in B \Leftrightarrow s \notin T \wedge \forall t \subset s(t \neq s \rightarrow t \in T).$$

It's clear that any infinite path $f \subseteq \mathrm{Seq}_2$ shares exactly one element $s \in B$ and $s \subset f$.

Thus, if we set

$$U = \bigcup\{(a'_s, b'_s) : s \in B\} \cup \bigcup\{(b_{s^\frown 0}, a_{s^\frown 1}) : s \in \mathrm{Seq}_2\},$$

then, it forms an open cover of $[0,1]$.

By the Heine-Borel Theorem, there exists a finite subcover U'.

Since for any $s \in B$, (a'_s, b'_s) does not intersect with any other $(a'_t, b'_t) \in U$ and is not a subset of $\bigcup\{(b_{s^\frown 0}, a_{s^\frown 1}) : s \in \mathrm{Seq}_2\}$, U' must contain $\{(a'_s, b'_s) : s \in B\}$. Therefore, B is finite.

Since T is obtained from the set of all initial segments of elements in B by removing the elements of B, it is also finite. □

The Heine-Borel property of $[0,1]$ allows us to derive various properties of continuous functions $f:[0,1]\to\mathbb{R}$.

Lemma 3.10

In WKL_0, a continuous function $f:[0,1]\to\mathbb{R}$ is uniformly continuous.

Proof Fix any $n\in\mathbb{N}$. We want to show the existence of $d>0$ such that

$$\forall x,y\in[0,1](\mid x-y\mid<d\to\mid f(x)-f(y)\mid<2^{-n}).$$

Let F be the code for the continuous function f, and denote the open interval with code i as (p_i,q_i). Then, define the open set U as follows:

$$i\in U\Leftrightarrow\exists j<i((p_i,q_i,p_j,q_j)\in F\wedge q_j-p_j<2^{-n-1}).$$

First, we show that U is a covering of $[0,1]$. For any real number $x\in[0,1]$, since $x\in\mathrm{dom}f$, there exists $(p_k,q_k,p_j,q_j)\in F$, such that

$$p_k<x<q_k\wedge q_j-p_j<2^{-n-1}.$$

Furthermore, there are infinitely many i such that $p_k\le p_i<x<q_i\le q_k$, so taking such an $i>j$, we have $i\in U$ with $p_i<x<q_i$. Therefore, U forms an open covering of $[0,1]$.

By the Heine-Borel Theorem, U has a finite subcover U'.

Let d be the minimum width $q_i - p_i$ among the intervals (p_i, q_i) in U'. We shall show that this d satisfies the uniform convergence condition.

Now, choose any real numbers $x, y \in [0, 1]$ such that $| x - y |< d$. Then, there must exist intervals $(p_i, q_i), (p_{i'}, q_{i'})$ in U' such that $x \in (p_i, q_i)$, $y \in (p_{i'}, q_{i'})$ and they have a common point z.

Otherwise, take an interval $(p_i, q_i) \ni x$ in U' with maximum q_i, and an interval $(p_{i'}, q_{i'}) \ni y$ in U' with minimum $p_{i'}$. If there is no common point, $q_i < p_{i'}$. Since U' is a covering, there exists $q_i \in (p_k, q_k)$ in U'. By the maximality of q_i, $x \notin (p_k, q_k)$. From $|q_k - p_k| \geq d > |x - y|$, we have $y \in (p_k, q_k)$, which contradicts with the minimality of $p_{i'}$.

By the definition of U, we have $| f(x) - f(z) |< 2^{-n-1}$ and $| f(y) - f(z) |< 2^{-n-1}$, thus $| f(x) - f(y) |< 2^{-n}$, which fulfills the lemma. □

Lemma 3.11

In WKL_0, a continuous function $f : [0,1] \to \mathbb{R}$ attains a maximum value.

Proof First, we show that the supremum M of the range of f exists.

As in the proof of the previous lemma, we define U by a Σ^0_0 formula:

$$i \in U \Leftrightarrow \exists j < i((p_i, q_i, p_j, q_j) \in F \wedge q_j - p_j < 2^{-n-1}).$$

We can finitely calculate whether or not a given finite set of open rational intervals covers $[0,1]$. Therefore, by arranging all finite subsets of U and checking sequentially whether they cover $[0,1]$, we eventually obtain a finite subcover U'. That is, in WKL_0, we can construct a function extracting U' according to n.

For each $i \in U'$, select $j_i < i$ such that $(p_i, q_i, p_{j_i}, q_{j_i}) \in F \wedge q_{j_i} - p_{j_i} < 2^{-n-1}$, and let $M_n = \max\{q_{j_i} : i \in U'\}$. Then, $\{M_n\}$ itself is a real number, and it is clear that it is the supremum M of the range of f.

What remains is to show that the existence of a point $x = a$ such that $f(a) = M$. For the sake of the following argument, we redefine $M_n = \max\{p_{j_i} : i \in U'\}$. This ensures that for any n, $M_n \leq M = \{M_n\}$.

By way of contradiction, assume that $f(x) < M$ for all $x \in [0, 1]$. Then, we define an open set V as follows:

$$i \in V \Leftrightarrow \exists j < i \; \exists n < i((p_i, q_i, p_j, q_j) \in F \wedge q_j < M_n).$$

To show that this set forms a covering of $[0, 1]$, take any real number $x \in [0, 1]$. Since $f(x) < M$, there exists n such that $f(x) < M_n \leq M$, and hence there exists $(p_k, q_k, p_j, q_j) \in F$ and n such that

$$p_k < x < q_k \;\wedge\; p_j \leq f(x) \leq q_j < M_n \leq M.$$

As there are infinitely many i such that $p_k \leq p_i < x < q_i \leq q_k$, taking $i > j, n$ ensures $i \in V$ with $p_i < x < q_i$. Therefore, V forms an open covering of $[0, 1]$.

Again, by the Heine-Borel Theorem, V has a finite subcover V'. Let M' be the maximum of q_i for (p_i, q_i) in V'. Then, by the definition of values of a continuous function, obviously M' is an upper bound of the range. However, due to the finiteness of V', for some n, $M' < M_n \leq M$, which contradicts the fact that M is the supremum. □

Conversely, the properties described in the two lemmas above allow us to derive WKL_0. In sum, the following theorem holds:

Theorem 3.12

The following assertions are pairwise equivalent in RCA_0:

(1) WKL_0,

(2) A continuous function $f : [0,1] \to \mathbb{R}$ is uniformly continuous,

(3) A continuous function $f : [0,1] \to \mathbb{R}$ is bounded,

(4) A bounded continuous function $f : [0,1] \to \mathbb{R}$ has a supremum,

(5) A continuous function $f : [0,1] \to \mathbb{R}$ that has a supremum attains its maximum value.

Proof By Lemmas 3.10 and 3.11, we can deriving (2), (3), (4), and (5) from (1). Hence, it suffices to obtain counterexamples for (2), (3), (4) and (5) from the negation of (1). Now, assume the negation of (1). Then, there exists an infinite tree $T \subseteq \mathrm{Seq}_2$ without infinite paths.

As shown in the proof of Heine-Borel's theorem, for each $s \in \mathrm{Seq}_2$, define the two rational numbers a_s and b_s as follows:

$$a_s = \sum_{i<\mathrm{leng}(s)} \frac{s(i)}{2^{i+1}},$$

$$b_s = a_s + \frac{1}{2^{\mathrm{leng}(s)}}.$$

Let B be the infinite set of all minimal binary sequences not in T,

$$s \in B \Leftrightarrow s \notin T \wedge \forall t \subset s(t \neq s \to t \in T)$$

and J be the set of closed intervals $[a_s, b_s]$ for all $s \in B$.

Each real number $x \in [0, 1]$ is either an interior point of exactly one interval in J or an endpoint of one or two intervals. Such an infinite set J is called a **singular closed cover**.

$\neg\,\mathsf{WKL}_0 \to \neg$ (3) bounded.

We will construct a counterexample for (3) using this singular closed cover J. This also serves as a counterexample for (2) since (2) implies (3). We define a continuous function f_s for each interval $[a_s, b_s]$ in J as follows:

$$f_s(x) = \begin{cases} \mathrm{leng}(s)\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \leq x \leq \frac{a_s+b_s}{2}, \\ \mathrm{leng}(s)\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \leq x \leq b_s. \end{cases}$$

That is, f_s takes 0 at the endpoints $x = a_s, b_s$, takes $\mathrm{leng}(s)$ at the midpoint $x = \frac{a_s+b_s}{2}$, and is linearly interpolated otherwise.

Let f be a function obtained by composing all such functions f_s. Then, it is clearly continuous but unbounded. (It is left as an exercise for the reader to construct a continuous function code for f.)

$\neg\,\mathsf{WKL}_0 \to \neg$ (5) a maximum value.

A counterexample for (5) can be constructed in the way similar to that for (3) in the previous slide. We just replace the maximum value of f_s from $\mathrm{leng}(s)$ to $1 - 2^{-\mathrm{leng}(s)}$ as follows:

$$f_s(x) = \begin{cases} (1 - 2^{-\mathrm{leng}(s)})\frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \leq x \leq \frac{a_s+b_s}{2}, \\ (1 - 2^{-\mathrm{leng}(s)})\frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \leq x \leq b_s. \end{cases}$$

Then, a composed function f clearly has 1 as its supremum, but it can not attain the maximum value 1 in $[0,1]$.

$\neg\,\mathsf{WKL}_0 \to \neg$ (4) a supremum.

Recall:

Theorem 3.4.

($\mathsf{RCA}_0 \vdash$) $\mathsf{ACA}_0 \Leftrightarrow$ (5) Every bounded increasing sequence of reals has a supremum.

Negating WKL_0, we have the negation of ACA_0, which implies the existence of a bounded increasing sequence of rational numbers $\{c_n\}$ that lacks a supremum.

Then, replace the maximum value of f_s with $c_{\mathrm{leng}(s)}$ and proceed similarly.

$$f_s(x) = \begin{cases} c_{\mathrm{leng}(s)} \frac{2(x-a_s)}{a_s+b_s} & \text{if } a_s \le x \le \frac{a_s+b_s}{2}, \\ c_{\mathrm{leng}(s)} \frac{2(b_s-x)}{a_s+b_s} & \text{if } \frac{a_s+b_s}{2} \le x \le b_s, \end{cases}$$

HW Problem # 5-3

Show that in the theorem 3.12 (4) and (5), "continuous function" can be replaced with "uniformly continuous function". Hint: It is beneficial to use a singular closed cover for the ternary set.

Thank you for your attention!