

Topics in Applied Math: Logic and Foundations of Mathematics

Part 6. Reverse Mathematics

Kazuyuki Tanaka

BIMSA

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清华大学求真书院
Qiu Zhen College, Tsinghua University

Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Reverse mathematics**
- Part 7. Second order arithmetic and non-standard methods

Part 6. Schedule

- Nov. 19, (1) Introduction and RCA_0
- **Nov. 21, (2) Real numbers in RCA_0**
- Nov. 26, (3) Completeness of the reals and ACA_0
- Nov. 28, (4) Continuous functions and WKL_0

Reverse Mathematics: Which axioms are needed to prove a theorem?

The Reverse Mathematics Phenomenon

Many theorems of mathematics are either provable in RCA_0 , or logically equivalent (over RCA_0) to one of WKL_0 , ACA_0 , ATR_0 , $\Pi_1^1\text{-CA}_0$.

Definition 1.2 The system of **recursive comprehension axioms** (RCA_0) consists of:

- (0) Axioms and inference rules of first-order logic with axioms of equality for numbers.
- (1) Basic arithmetic axioms: Same as $\text{Q}_<$ (Chapter 4).
- (2) Δ_1^0 comprehension axiom ($\Delta_1^0\text{-CA}_0$): $\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))$, where $\varphi(n)$ is Σ_1^0 , $\psi(n)$ is Π_1^0 , and neither includes X as a free variable.
- (3) Σ_1^0 induction: $\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1)) \rightarrow \forall n\varphi(n)$, for any Σ_1^0 formula $\varphi(n)$.

- RCA_0 is a conservative extension of first-order arithmetic $\text{I}\Sigma_1$. (**Lemma 1.3**)

The **pair** of natural numbers (m, n) is coded by a natural number $\frac{(m+n)(m+n+1)}{2} + m$.

The **product** $X \times Y$ is the set of pairs (codes) of one from X and the other from Y .

A **function** $f : X \rightarrow Y$ is a subset $F \subseteq X \times Y$ such that $\forall x \in X \exists! y \in Y (x, y) \in F$.

A function f whose domain is $X = \{i : i < n\}$ is called a **finite sequence** with **length** n .

In RCA_0 , the following holds:

- (1) Π_1^0 induction. (2) The class of Σ_1^0 formulas is closed under bounded quantification. (**Lemma 1.4**)
- The set of total functions is closed under primitive recursion. (**Lemma 1.5**)
- The set of (partial) functions is closed under minimization μ . (**Lemma 1.6**)
Moreover, if $f : \mathbb{N}^{n+1} \rightarrow \mathbb{N}$ is total and $\forall \vec{x} \exists y f(\vec{x}, y) = 0$ is provable, then $\mu y (f(\vec{x}, y) = 0)$ exists as a total function.

Lemma 1.7

In RCA_0 , for any Σ_1^0 formula $\varphi(x)$, there exists a finite set X such that $\forall x(x \in X \leftrightarrow \varphi(x))$, or there exists a one-to-one function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall y(\exists x f(x) = y \leftrightarrow \varphi(y))$.

Proof

- Let $\varphi(x)$ be a Σ_1^0 formula. So, there exists a Σ_0^0 formula $\theta(x, y)$ such that $\varphi(x) \leftrightarrow \exists y \theta(x, y)$. By $(\Sigma_0^0\text{-CA})$,

$$Y = \{(x, y) : \theta(x, y) \wedge \forall y' < y \neg \theta(x, y')\}$$

exists. Note that $\exists y \theta(x, y) \leftrightarrow \exists y(x, y) \in Y \leftrightarrow \exists! y(x, y) \in Y$ for all x .

- If Y is bounded, there exist u, v such that $\varphi(x) \leftrightarrow (x < u \wedge \exists y < v \theta(x, y))$. Then, by $(\Sigma_0^0\text{-CA})$, there exists a finite set X such that $\forall x(x \in X \leftrightarrow \varphi(x))$.
- Next, suppose that Y is unbounded. By Lemmas 1.5 and 1.6, we can define a function which enumerates the elements of Y , and a function which extracts the first component x from (x, y) . Combining them, we can create a one-to-one function f such that $\forall y(\exists x f(x) = y \leftrightarrow \varphi(y))$. This proves the lemma. \square

Lemma 1.8

In RCA_0 , the following form of set existence axiom is provable:

$$(\text{Bounded } \Sigma_1^0\text{-CA}) : \forall x \exists X \forall y (y \in X \leftrightarrow (y < x \wedge \varphi(y))),$$

where $\varphi(y)$ is a Σ_1^0 formula, not containing X as a free variable.

Proof By way of contradiction, assume for some x , if there exist no finite set X such that

$$\forall y (y \in X \leftrightarrow (y < x \wedge \varphi(y))).$$

Then by Lemma 1.7, there must exist a one-to-one function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall y (\exists z f(z) = y \leftrightarrow (y < x \wedge \varphi(y))),$$

and so \mathbb{N} is finite, which is absurd. □

HW # 5-1 (Strong Σ_1^0 Collection Axiom)

Prove in RCA_0 : for a Σ_1^0 formula $\varphi(i, j)$ (not containing n as a free variable),

$$(\text{S}\Sigma_1^0) : \forall m \exists n \forall i < m (\exists j \varphi(i, j) \rightarrow \exists j < n \varphi(i, j)).$$

§2. Real numbers in RCA_0

In this lecture, we explore how real number theory is developed in RCA_0 .

First, we denote the set of all natural numbers $\{n : n = n\}$ by \mathbb{N} .

This is formally defined within the system, and the interpretation of \mathbb{N} relies on a model of RCA_0 . Arithmetical operations on \mathbb{N} such as $+$ and \cdot are simply taken as the corresponding operations in RCA_0 . So, $(\mathbb{N}, +, \cdot, 0, 1, <)$ is nothing but the first order part M of a model (M, S) of RCA_0 .

We will use ω to denote the totality of standard natural numbers.

In the following, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are all formally introduced, and so they may not coincide with their standard counterparts in the real world.

First, the equivalence relation $=_{\mathbb{Z}}$ on $\mathbb{N} \times \mathbb{N}$ is defined by

$$(k, l) =_{\mathbb{Z}} (m, n) \leftrightarrow k + n = l + m.$$

Here, the pair (k, l) intuitively represents the integer $k - l$.

Then, we select a pair with the smallest code from each equivalence class of $=_{\mathbb{Z}}$ to be the representative, called an **integer**. We denote the set of all such representatives by \mathbb{Z} .

The operations on \mathbb{Z} are defined as

$$(k, l) + (m, n) =_{\mathbb{Z}} (k + m, l + n),$$

$$(k, l) \cdot (m, n) =_{\mathbb{Z}} (km + ln, kn + lm), \text{ etc.}$$

Note that even if (k, l) and (m, n) belong to \mathbb{Z} , $(k + m, l + n)$ may not belong to \mathbb{Z} .

Strictly, $+$ on \mathbb{Z} is induced by taking the representatives of equivalent classes as follows:

$$[(k, l)] + [(m, n)] = [(k + m, l + n)].$$

Then, it can be verified that \mathbb{Z} satisfies the basic properties of an integer ring.

Next, the equivalence relation $=_{\mathbb{Q}}$ on $\mathbb{Z} \times (\mathbb{Z} - \{(0,0)\})$ is defined by

$$(k, l) =_{\mathbb{Q}} (m, n) \leftrightarrow kn = lm.$$

Here, the pair (k, l) intuitively means the rational number $\frac{k}{l}$.

Then, we select a pair with the smallest code from each equivalence class of $=_{\mathbb{Q}}$ as a **rational number**, and denote the set of all such representatives by \mathbb{Q} .

The operations on \mathbb{Q} are defined as

$$(k, l) + (m, n) =_{\mathbb{Q}} (kn + lm, ln),$$

$$(k, l) \cdot (m, n) =_{\mathbb{Q}} (km, ln), \text{ etc.}$$

Thus, \mathbb{Q} satisfies the basic properties of the field of rational numbers.

We will next define a real number as an infinite sequence of rational numbers, that is, a function $\subset \mathbb{N} \times \mathbb{Q} \subset \mathbb{N} \times \mathbb{Z}^2 \subset \mathbb{N} \times \mathbb{N}^4 = \mathbb{N}^5$, coded as a subset of \mathbb{N} after all.

A sequence of rational numbers $\{q_n\}$ is called a **real number**, $\{q_n\} \in \mathbb{R}$, if it satisfies

$$\forall n \forall i (|q_n - q_{n+i}| \leq 2^{-n}).$$

Note that \mathbb{Q} is a set in second-order arithmetic, whereas \mathbb{R} is a predicate about sets. The equality $=$ and inequality $<$ on real numbers are defined as

$$\{p_n\} = \{q_n\} \leftrightarrow \forall n (|p_n - q_n| \leq 2^{-n+1}),$$

$$\{p_n\} < \{q_n\} \leftrightarrow \exists n (q_n - p_n > 2^{-n+1}).$$

It is easy to see that for any two reals $\{p_n\}$ and $\{q_n\}$, exactly one of the following holds:

$$\{p_n\} = \{q_n\}, \{p_n\} < \{q_n\} \text{ or } \{q_n\} < \{p_n\}.$$

The **sum** of two real numbers $\{p_n\}$ and $\{q_n\}$ is defined as follows:

$$\{p_n\} + \{q_n\} = \{p_{n+1} + q_{n+1}\}.$$

Here, $\{p_{n+1} + q_{n+1}\}$ is a real, since

$$\begin{aligned} |(p_{n+1} + q_{n+1}) - (p_{n+1+i} + q_{n+1+i})| &\leq |p_{n+1} - p_{n+1+i}| + |q_{n+1} - q_{n+1+i}| \\ &\leq 2^{-n-1} + 2^{-n-1} \leq 2^{-n} \end{aligned}$$

Next, the **product** of two real numbers $\{p_n\}$ and $\{q_n\}$ is defined as follows:

$$\{p_n\} \cdot \{q_n\} = \{p_{n+m} \cdot q_{n+m}\},$$

where m is the smallest natural number such that $\max(|p_0|, |q_0|) + 1 \leq 2^{m-1}$. Then,

$$\begin{aligned} & |p_{n+m} \cdot q_{n+m} - p_{n+m+i} \cdot q_{n+m+i}| \\ & \leq |q_{n+m}| \cdot |p_{n+m} - p_{n+m+i}| + |p_{n+m+i}| \cdot |q_{n+m} - q_{n+m+i}| \\ & \leq (|q_0| + 1) \cdot 2^{-n-m} + (|p_0| + 1) \cdot 2^{-n-m} \\ & \leq 2 \cdot (\max(|p_0|, |q_0|) + 1) \cdot 2^{-n-m} \\ & \leq 2 \cdot 2^{m-1} \cdot 2^{-n-m} = 2^{-n}. \end{aligned}$$

Thus, $\{p_n\} \cdot \{q_n\}$ is also a real number.

Problem 2

In RCA_0 , show the existence of a real $y = 1/x$ for any real $x \neq 0$.

Summary

It is provable in RCA_0 that $(\mathbb{R}, +, \cdot, 0, 1, <, =)$ becomes an **Archimedean ordered field**.

Remark 1. The above relation $=$ is not the equality but an equivalent relation. In ordinary mathematics, the definition of real numbers \mathbb{R} is finished by dividing by the equivalence relation $=$. But in RCA_0 , we cannot construct equivalence classes of sets, or choose representatives for them.

Remark 2. There is another way to define reals from rationals by so-called “Dedekind cuts”. In this definition, a real r is identified with the unique set $\{q \in \mathbb{Q} : q < r\}$. Arithmetical operations on such reals can be defined easily. However, it is difficult to handle infinite sequences of such reals. In fact, it is not provable in RCA_0 that the element-wise sum of two sequence of such reals exists.

Sequences of real numbers

We define a **sequence of real numbers** as a function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}$ such that:

for each n , letting $f_n(m) = f(n, m)$, $f_n : \mathbb{N} \rightarrow \mathbb{Q}$ is a real number.

The sequence of real numbers is denoted by $\{f_n\}$, and for each n , f_n represented as $\{f_{nm}\}$. Furthermore, the **limit** of a sequence of real numbers $\{f_n\}$, denoted $\lim_{n \rightarrow \infty} f_n$, is

defined as the unique real number a such that

$$\forall \varepsilon > 0 \exists n \forall i (|a - f_{n+i}| < \varepsilon).$$

Under this definition, the next theorem can be proved in RCA_0 .

Theorem 2.1 (Nested interval property of \mathbb{R})

The following is provable in RCA_0 . Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \text{ (for all } n), \text{ and } \lim_{n \rightarrow \infty} |a_n - b_n| = 0.$$

Then, there exists a real number c such that $c = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.

Proof. We reason within RCA_0 .

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \text{ (for all } n), \text{ and } \lim_{n \rightarrow \infty} |a_n - b_n| = 0.$$

Now, suppose $a_n = \{p_{nm}\}$ and $b_n = \{q_{nm}\}$. We set $p'_{nm} = p_{n(m+1)} - 2^{-m-1}$ and $q'_{nm} = q_{n(m+1)} + 2^{-m-1}$. Then $\{p'_{nm}\}$ and $\{q'_{nm}\}$ are also sequences of real numbers. In addition, for any n , $a_n = \{p'_{nm}\}$ and $b_n = \{q'_{nm}\}$, and for any m ,

$$p'_{nm} \leq a_n \text{ and } b_n \leq q'_{nm}$$

Since $\lim_{n \rightarrow \infty} |a_n - b_n| = 0$, we have

$$\forall \varepsilon > 0 \exists n \forall m \geq n |a_m - b_m| < \varepsilon.$$

A simple calculation shows that

$$\forall \varepsilon > 0 \exists n \forall m \geq n |p'_{mm} - q'_{mm}| < \varepsilon$$

also holds.

We define an increasing sequence $\{p''_k\}$ by primitive recursion as follows: Let $p''_0 = p'_{00}$. For $k > 0$, set $p''_k = p'_{nn}$ with the smallest $n \geq k$ such that $|p'_{nn} - q'_{nn}| < 2^{-k}$ and $p''_{k-1} \leq p'_{nn}$.

Since $p''_k = p'_{nn} \leq p''_{k+i} \leq q'_{nn}$, it is clear that $\{p''_k\}$ becomes a real number.

Moreover, since $\forall k \exists n \forall m \geq n |a_m - p''_k| < 2^{-k}$, we have $\{p''_k\} = \lim_{n \rightarrow \infty} a_n$.

Similarly, we have $\{p''_k\} = \lim_{n \rightarrow \infty} b_n$. So we are done. □

Although the nested completeness property of \mathbb{R} is provable in RCA_0 , the sequential compactness or completeness of \mathbb{R} is not, which will be discussed in the next lecture. As an application of the above theorem, we prove that \mathbb{R} is uncountable.

Theorem 2.2 (Uncountability of \mathbb{R})

It is provable in RCA_0 that for any sequence of real numbers $\{a_n\}$, there exists a real number c such that $\forall n (a_n \neq c)$.

Proof. We reason within RCA_0 .

First, let $a_n = \{p_{nm}\}$. Using primitive recursion, we define a sequence of shrinking closed intervals with rational endpoints $\{[q_n, r_n]\}$ as follows:

$$[q_0, r_0] = [0, 1],$$

$$[q_{n+1}, r_{n+1}] = \begin{cases} [\frac{q_n+3r_n}{4}, r_n] & \text{if } p_{n,2n+3} \leq \frac{q_n+r_n}{2}, \\ [q_n, \frac{3q_n+r_n}{4}] & \text{otherwise.} \end{cases}$$

The two sequences $\{q_n\}$ and $\{r_n\}$ clearly satisfy the conditions of Theorem 2.1, and thus there exists a real number c such that $c = \lim_{n \rightarrow \infty} q_n = \lim_{n \rightarrow \infty} r_n$.

Now, take any n . If $p_{n,2n+3} \leq \frac{q_n+r_n}{2}$, then

$$a_n \leq p_{n,2n+3} + 2^{-2n-3} \leq \frac{q_n+r_n}{2} + 2^{-2n-3} < \frac{q_n+r_n}{2} + \frac{2^{-2n}}{4} = \frac{q_n+r_n}{2} + \frac{r_n-q_n}{4} = q_{n+1} \leq c.$$

Otherwise, $a_n \geq \frac{q_n+r_n}{2} - 2^{-2n-3} > r_{n+1} \geq c$.

In either case, $a_n \neq c$ is shown. □

We will introduce continuous functions on \mathbb{R} in RCA_0 .

Definition 2.3

A set $\Phi \subseteq \mathbb{Q}^4$ that satisfies the following conditions is called the **code** for a **continuous function** $f : \text{dom } f (\subseteq \mathbb{R}) \rightarrow \mathbb{R}$.

$$(1) (p, q, r, s) \in \Phi \rightarrow p < q \wedge r \leq s,$$

$$(2) (p, q, r, s), (p', q', r', s') \in \Phi, p' < q \wedge p < q' \rightarrow r' \leq s \wedge r \leq s'.$$

In (2), $p' < q \wedge p < q'$ is $(p, q) \cap (p', q') \neq \emptyset$, and $r' \leq s \wedge r \leq s'$ is $[r, s] \cap [r', s'] \neq \emptyset$. Intuitively, $(p, q, r, s) \in \Phi$ means $\forall x (p < x < q \rightarrow r \leq f(x) \leq s)$.

A real number x belongs to the **domain** of a continuous function f coded by Φ , if

$$\forall n \exists (p, q, r, s) \in \Phi (p < x < q \wedge s - r < 2^{-n}), \text{ denoted } x \in \text{dom } f.$$

It is provable in RCA_0 that if $x \in \text{dom } f$, there exists a unique real y such that $\forall (p, q, r, s) \in \Phi (p < x < q \rightarrow r \leq y \leq s)$. (Exercise: Use the nested interval property of \mathbb{R} to prove this.) We denote this y as $f(x)$.

Problem 3. HW # 5-2

Show in RCA_0 that $y = 1/x$ is a continuous function on $\mathbb{R} - \{0\}$. Cf. Problem 2.

Theorem 2.4 (Intermediate Value Theorem)

The following is provable in RCA_0 . Given a continuous function f such that its domain includes $[0, 1]$ and $f(0) < 0 < f(1)$, there exists an $x \in [0, 1]$ such that $f(x) = 0$.

Proof. We reason in RCA_0 . We may assume that $f(q) \neq 0$ for all rational numbers $q \in [0, 1]$. Otherwise, the theorem already holds. For any rational number $q \in [0, 1]$, if $\{p_n\}$ represents the real number $f(q)$, then for sufficiently large n , either $p_n < -2^{-n}$ or $2^{-n} < p_n$ holds, which allows us to determine whether $f(q) < 0$ or $f(q) > 0$. Therefore, we can recursively define a sequence of shrinking closed intervals with rational endpoints $\{[p_n, q_n]\}$ as follows:

$$[p_0, q_0] = [0, 1],$$

$$[p_{n+1}, q_{n+1}] = \begin{cases} [\frac{p_n+q_n}{2}, q_n] & \text{if } f(\frac{p_n+q_n}{2}) < 0, \\ [p_n, \frac{p_n+q_n}{2}] & \text{if } f(\frac{p_n+q_n}{2}) > 0. \end{cases}$$

Then, by Theorem 2.1, there exists a real number $x \in [0, 1]$ s.t. $x = \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} q_n$.

Finally, it is clear that $f(x) = 0$. □

One conclusion that can be drawn from the Intermediate Value Theorem is that \mathbb{R} forms a real closed ordered field.

However, this does not immediately imply that all theorems of the theory of real closed ordered fields hold in \mathbb{R} . Since \mathbb{R} is not a set but a formula in the sense of second-order arithmetic, treatment of quantifiers in \mathbb{R} requires formal methods like quantifier elimination.

As a corollary of the Intermediate Value Theorem, it is provable in RCA_0 that any continuous function $f : [0, 1] \rightarrow [0, 1]$ has a fixed point. (Apply the Intermediate Value Theorem to $x - f(x)$.)

However, the empirical fact that this cannot be simply extended to dimensions two or higher is supported by a result in the next section, which claims that Brouwer's Fixed Point Theorem is equivalent to WKL_0 .

Problem 4

Consider the following. Given an infinite sequence of continuous functions $\{f_n\}$ such that for each n , $\text{dom}f_n$ includes $[0, 1]$, and $f_n(0) < 0 < f_n(1)$. Can you show the existence of a sequence $\{x_n\} \subset [0, 1]$ such that $f_n(x_n) = 0$ in RCA_0 ? (Hint: Use two Σ_1^0 sets that cannot be recursively separated in the minimal model of RCA_0 , (ω, Rec) , to construct a counterexample. Refer to Lemma 3.6 and Corollary 3.7 in the next section.)

This section investigates properties of real numbers that cannot be proved within the system RCA_0 . In the previous section, we proved nested interval property of \mathbb{R} in RCA_0 , but important topological properties such as sequential compactness cannot be proved in RCA_0 , requiring the strictly stronger system ACA_0 .

Moreover, it is interesting that many famous theorems in mathematics are equivalent to the system WKL_0 , which lies between RCA_0 and ACA_0 . This fact has significant implications related to Hilbert's program, which we will revisit at the end of this section.

Definition 3.1

The **system of arithmetical comprehension axioms** (ACA_0) is RCA_0 extended with the following axiom:

$$(\Pi_0^1\text{-CA}) : \exists X \forall n (n \in X \leftrightarrow \varphi(n)).$$

Here, $\varphi(n)$ is an arithmetical formula (Π_0^1 formula), which does not have X as a free variable.

Lemma 3.2

ACA_0 is a conservative extension of Peano Arithmetic PA. So, ACA_0 is a proper extension of RCA_0 .

Proof To say that all theorems of PA can be proved in ACA_0 , it suffices to show that induction for any arithmetic formula can be proved in ACA_0 .

Let $\varphi(n)$ be any arithmetic formula, and assume

$$\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1)).$$

By the arithmetical comprehension axiom, there exists X such that

$$\forall n(n \in X \leftrightarrow \varphi(n)).$$

For this X , it follows that

$$0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X),$$

and applying Σ_1^0 induction yields $\forall n(n \in X)$, hence $\forall n\varphi(n)$.

For the converse direction, to prove that a proposition of first-order arithmetic provable in ACA_0 is a theorem of PA, we can use the same method that was used to show RCA_0 is a conservative extension of $I\Sigma_1$.

Lemma 3.3

In RCA_0 , the following are equivalent:

- (1) ACA_0 ,
- (2) $(\Sigma_1^0\text{-CA})$,
- (3) The range of any 1-1 function $f : \mathbb{N} \rightarrow \mathbb{N}$ exists.

Proof (1) \Rightarrow (2) is obvious. (2) \Rightarrow (3) is also clear since the range of f can be represented as Σ_1^0 , ($n \in \text{ran } f \leftrightarrow \exists m f(m) = n$). For (3) \Rightarrow (2), it immediately follows from Lemma 1.7 and Lemma 1.8 of the section 1.

To show (2) \Rightarrow (1), we prove $(\Sigma_1^0\text{-CA}) \rightarrow (\Sigma_k^0\text{-CA})$ by meta-induction on k . For $k \leq 1$, it is clear. Let $\varphi(n)$ be any Σ_{k+1}^0 formula. Then, we can write $\varphi(n)$ as

$$\varphi(n) \leftrightarrow \exists m \neg \theta(m, n), \text{ where } \theta(m, n) \text{ is } \Sigma_k^0$$

By $(\Sigma_k^0\text{-CA})$, the set $Y = \{(m, n) : \theta(m, n)\}$ exists. Then by $(\Sigma_1^0\text{-CA})$, the set

$$X = \{n : \exists m \neg ((m, n) \in Y)\}$$

also exists. Thus,

$$n \in X \leftrightarrow \varphi(n).$$

Therefore, $(\Sigma_{k+1}^0\text{-CA})$ holds.

The following theorem states not only that properties like completeness (3) and sequential compactness (2) of real numbers cannot be proved in RCA_0 , but also that they are equivalent to ACA_0 .

Theorem 3.4

Each of the following propositions is equivalent in RCA_0 .

- (1) ACA_0 ,
- (2) Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence,
- (3) Every Cauchy sequence converges,
- (4) Every bounded sequence of real numbers has a supremum,
- (5) Monotone Convergence Theorem: Every bounded increasing sequence converges.

Proof (1) \Rightarrow (2), (3), (4), (5) follow the usual proofs found in calculus textbooks and are omitted here. (We use Bisection method. Note that conditions like each interval containing at least one or infinitely many points from the sequence can be written in arithmetical formulas) Also, (5) immediately follows from (2), (3), or (4), so it suffices to show (5) implies (1).

Monotone Convergence Theorem \Rightarrow ACA_0

Assuming (5) and utilizing Lemma 3.3(3), let $f : \mathbb{N} \rightarrow \mathbb{N}$ be any one-to-one function. Define the bounded increasing sequence of rational numbers $\{c_n\}$ by:

$$c_n = \sum_{i=0}^n 2^{-f(i)}$$

Then, by the Monotone Convergence Theorem, the limit

$$c = \lim_{n \rightarrow \infty} c_n = \sum_{i=0}^{\infty} 2^{-f(i)}$$

exists. Consequently, for any n ,

$$n \in \text{ran } f \leftrightarrow \exists m f(m) = n \leftrightarrow \forall k (|c_k - c| < 2^{-n} \rightarrow \exists m \leq k (f(m) = n))$$

Thus, by $(\Delta_1^0\text{-CA})$, the range $\text{ran } f$ exists, and the above lemma implies ACA_0 . \square

Thank you for your attention!