Topics in Applied Math:

Logic and Foundations of Mathematics

Part 5. Models of first-order arithmetic

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Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
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Part 5. Schedule

- Nov. 5, (0) Nonstandard models and Overspill
- Nov. 7, (1) The omitting type theorem
- Nov.12, (2) Recursively saturated models
- Nov.14, (3) Friedman's theorem
- Nov.19, (4) Resplendency

Theorem 3.11 (Friedman's self-embedding theorem)

Let $n > 0$, $\mathfrak{A}$ be a countable non-standard model of $\mathrm{I}\Sigma_n$, and take $a \in A$ arbitrarily. Then there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $a \in A'$ but $A' \subsetneq A$, and $\mathfrak{A} \cong \mathfrak{A}'$ and for any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{a'} \in A'^{<\omega}$,

$$\mathfrak{A}'_{A'} \models \varphi(\vec{a'}) \Leftrightarrow \mathfrak{A}_{A'} \models \varphi(\vec{a'}).$$

- The essence of this theorem is that a countable non-standard model of $\mathrm{I}\Sigma_1$ has an initial segment that is isomorphic to itself.
- Friedman first proved this theorem for a countable non-standard model of Peano arithmetic, and several researchers sophisticated it to the above form.
- The same theorem does not hold for non-countable models, and also it does not hold in general for countable non-standard models of $\mathrm{I}\Sigma_0$.
- Furthermore, an important result related to this is McAloon's theorem, which states that a countable non-standard model of $\mathrm{I}\Sigma_0$ has an initial segment that is a model of Peano arithmetic PA.

§4. Resplendency

- Recursive saturation of a structure means that it contains many "elements" that satisfy recursive conditions, but by generalizing this property to relations and functions, we introduce a new concept.
- By saying that a structure $\mathfrak{A}$ in the language $\mathcal{L}$ has "resplendency", we mean that if a formula $\varphi(\vec{\mathrm{R}})$ with new relation symbols $\vec{\mathrm{R}} \notin \mathcal{L}$ consistent with $\mathrm{Th}(\mathfrak{A}_A)$, $\varphi(\vec{\mathrm{R}})$ can hold in $\mathfrak{A}$ by appropriate interpretation of $\vec{\mathrm{R}}$.
- In a resplendent model of arithmetic, hidden properties of the structure can be found by using new relation symbols for an initial segment and satisfaction relation.

Definition 4.1

The $\mathcal{L}$-structure $\mathfrak{A}$ is said to be **resplendent**, if for a sentence φ in a language $\mathcal{L}^+ \supseteq \mathcal{L}_A$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ such that $\mathfrak{A}^+ \models \varphi$.

- The statement that $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is consistent is equivalent to that φ is true in the $\mathcal{L}^+$-extension of an elementary extension of $\mathfrak{A}$.

  In other words, resplendent structures are considered to potentially possess the properties of relations and functions manifested in their elementary extensions.

- The consistency of $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is that of $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \varphi$ where $\vec{a}$ denotes the elements of A contained in φ.

  $\because$ Suppose $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is inconsistent. Then there exists a formula $\psi(\vec{a}, \vec{b})$ in $\mathrm{Th}(\mathfrak{A}_A)$ such that $\vdash \psi(\vec{a}, \vec{b}) \to \neg\varphi$. Thus we also have $\vdash \exists y \psi(\vec{a}, \vec{y}) \to \neg\varphi$. Since $\exists y \psi(\vec{a}, \vec{y}) \in \mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}})$, it follows that $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \{\varphi\}$ is inconsistent. The reverse implication is trivial.

- Every finite structure is resplendent because its elementary extension is only itself.

Since "resplendency" does not imply "recursive saturation" in general, we introduce the following stronger notion which implies both.

Definition 4.2

An $\mathcal{L}$-structure $\mathfrak{A}$ is **strongly resplendent**, if for any recursive type $\Phi(\vec{x})$ in a language $\mathcal{L}^+ = \mathcal{L} \cup \{\text{finitely many additional symbols}\}$ and $\vec{a} \in A^{<\omega}$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ which is a model of $\Phi(\vec{a})$.

- In the definition of **strongly resplendent**, if we restrict the type $\Phi(\vec{x})$ to be a single formula, we obtain the definition of **resplendent**, and if we let $\mathcal{L}^+ = \mathcal{L} \cup \{c\}$, it becomes the definition of **recursive saturation**. Hence, strongly resplendent structures are both resplendent and recursively saturated.

- Furthermore, similar to the case of resplendent structures, it is worth noting that the consistency of $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ coincides with the consistency of $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \Phi(\vec{a})$.

We will now demonstrate that under certain natural assumptions, the above three properties coincide.

Theorem 4.3 (Barwise-Ressayre)

Countable recursively saturated structures are strongly resplendent.

Proof

- Let $\mathfrak{A}$ be a countable structure in a countable language $\mathcal{L}$ and assume it is recursively saturated. Furthermore, suppose we are given a recursive type $\Phi(\vec{x})$ in a finitely extended language $\mathcal{L}^+$ of $\mathcal{L}$ and $\vec{a} \in A^\omega$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent.
- Then, we want to construct a model $\mathfrak{A}^+$ of this theory without expanding the domain $|\mathfrak{A}|$. The key idea of the construction is that by utilizing the recursively saturated nature of $\mathfrak{A}$, we can select Henkin constants from elements of A.

Now, let's look into the details of construction of $\mathfrak{A}^+$.

- First, we enumerate the formulas in $\mathcal{L}_A$ with only one free variable x, denoted by $\{\varphi_n(x) : n \in \omega\}$.

- We construct a sequence of finite subsets of A and that of recursive theories in $\mathcal{L}_A^+$,

$$A_0 = \{\vec{a}\} \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad T_0 = \Phi(\vec{a}) \subseteq T_1 \subseteq T_2 \subseteq \cdots,$$

satisfying the following conditions: for each n
(1) T_n is a recursive set of sentences in $\mathcal{L}_{A_n}^+$, and $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent.
(2) either $\varphi_n(a) \in T_{n+1}$ for some $a \in A$ or $\neg\exists x \varphi_n(x) \in T_{n+1}$.
- Once the construction is completed, letting $T_\omega = \bigcup_n T_n$, we will show T_ω is a complete Henkin theory.
- Let σ be a sentence in $\mathcal{L}_A^+$ such that $T_\omega \nvdash \sigma$. Suppose σ is φ_k (with no free variable) for some k. Then we have $\sigma \notin T_{k+1}$, since $T_\omega \nvdash \sigma$. Thus, by condition (2), we have $\neg\exists x \sigma \in T_{k+1}$, and so $T_\omega \vdash \neg\sigma$. Therefore, T_ω is complete, and so $\mathrm{Th}(\mathfrak{A}_A) \subseteq T_\omega$ since $T_\omega \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent by condition (1).
- If $T_\omega \vdash \exists x \varphi_n(x, \vec{a})$, then by (2), there exists some $a \in A$ such that $\varphi_n(a) \in T_\omega$.
- Then T_ω is a complete Henkin theory. By Henkin method, we can construct a structure $\mathfrak{A}^+$ over the domain A, such that $T_\omega = \mathrm{Th}(\mathfrak{A}_A^+)$, and therefore $\mathfrak{A}^+ \models \Phi(\vec{a})$.

Finally, we will construct the sequences $\{A_n\}$ and $\{T_n\}$ by induction.

- Assuming that the constructions up to A_n and T_n have been done. Take $\varphi_n(x)$.
- Let $B = A_n \cup \{$elements of A occurring in $\varphi_n(x)\}$, and define

$$\Psi(x) = \{\psi(x) : \psi(x) \text{ is a one-variable formula in } \mathcal{L}_B, \text{ and } T_n \vdash \varphi_n(x) \to \psi(x)\}.$$

- Although $\Psi(x)$ is Σ_1 as it is, it can be treated as a recursive type by Craig's method.
- Since the structure $\mathfrak{A}$ is recursively saturated, we can either find an $a \in A$ realizing $\Psi(x)$ or find a finite subset $\{\psi_i(x) : i \leq j\}$ of $\Psi(x)$ such that

$$\mathfrak{A}_A \models \neg \exists x \bigwedge_{i \leq j} \psi_i(x).$$

- In the former case, we let $A_{n+1} = B \cup \{a\}, \quad T_{n+1} = T_n \cup \{\varphi_n(a)\}$.
- To check the consistency of $T_{n+1} \cup \mathrm{Th}(\mathfrak{A}_A)$, we show that any $\mathcal{L}_{A_{n+1}}$ sentence provable in T_{n+1} is true in $\mathfrak{A}_A$. So, let $\psi(x)$ be a formula in $\mathcal{L}_B$ such that $T_{n+1} \vdash \psi(a)$. If $a \notin B$, $T_n \vdash \varphi_n(a) \to \psi(a)$ implies $T_n \vdash \varphi_n(x) \to \psi(x)$ and so $\psi(x) \in \Psi(x)$. Since a realizes $\Psi(x)$, $\psi(a)$ holds in $\mathfrak{A}_A$. On the other hand, if $a \in B$, then by $T_n \vdash \varphi_n(x) \to (x = a \to \psi(x))$, we get $(x = a \to \psi(x)) \in \Psi(x)$, which implies $(a = a \to \psi(a)) \in \mathrm{Th}(\mathfrak{A}_A)$. Thus, $\psi(a)$ holds in $\mathfrak{A}_A$.

- Next, we consider the case that $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\le j} \psi_i(x)$. In this case, we can simply set

$$A_{n+1} = A_n, \quad T_{n+1} = T_n \cup \{\neg\exists x \varphi_n(x)\}.$$

- Since $T_n \vdash \neg\exists x \bigwedge_{i\le j} \psi_i(x) \to \neg\exists x \varphi_n(x)$, we may show the consistency of

$$T_n \cup \{\neg\exists x \bigwedge_{i\le j} \psi_i(x)\} \cup \mathrm{Th}(\mathfrak{A}_A).$$

- Let ψ be a sentence in $\mathcal{L}_B$ such that $T_n \vdash \neg\exists x \bigwedge_{i\le j} \psi_i(x) \to \psi$. By the induction hypothesis, $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent, so $\neg\exists x \bigwedge_{i\le j} \psi_i(x) \to \psi$ holds in $\mathfrak{A}_A$.
- Moreover, since we have the premise $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\le j} \psi_i(x)$, it follows that ψ also holds in $\mathfrak{A}_A$. This completes the proof. □

Recall Problem 5 of Lec05-02

Let $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ be a non-standard model of $\mathrm{I}\Sigma_1$. Show that $\mathfrak{A}' = (A, +, 0, 1, <)$ is recursively saturated.

Example 5

- In the above problem 5, it was shown that if $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ is a nonstandard model of $\mathrm{I}\Sigma_1$, then $\mathfrak{A}' = (A, +, 0, 1, <)$ becomes recursively saturated.
- Conversely, suppose $\mathfrak{A}' = (A, +, 0, 1, <)$ is a recursively saturated model of Presburger arithmetic and is countable. Then, by the previous theorem, $\mathfrak{A}'$ is strongly resplendent.
- On the other hand, Presburger arithmetic is complete, and the set of its theorems coincides with $\mathrm{Th}(\mathfrak{A}')$. Therefore, $\mathrm{Th}(\mathfrak{A}') \cup \mathrm{PA}$ is nothing but PA, which is a recursive consistent set.
- Hence, there exists a suitable interpretation of $\bullet$ such that $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ becomes a model of PA. In summary, a countable model $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ of $\mathrm{I}\Sigma_1$ can be turned into a model $\mathfrak{A}' = (A, +, \bullet', 0, 1, <)$ of PA by changing the interpretation of multiplication (the "misbuttoning theorem").

Now, when $\mathcal{L}$ is finite, the equivalence of resplendency and strong resplendency can be derived from the following Kleene's theorem.

Theorem 4.4 (Kleene)

Let $\mathcal{L}$ be finite, and $\Phi(\vec{v})$ be its recursive type. Then, there exists a formula $\varphi(\vec{v})$ in some finite extension language $\mathcal{L}^+ \supseteq \mathcal{L}$ such that,
(1) If a structure $\mathfrak{A}^+$ in $\mathcal{L}^+$ satisfies $\varphi(\vec{a})$, then its reduct $\mathfrak{A}$ to $\mathcal{L}$ satisfies $\Phi(\vec{a})$.
(2) If an infinite structure $\mathfrak{A}$ in $\mathcal{L}$ satisfies $\Phi(\vec{a})$, then there exists an expansion $\mathfrak{A}^+$ in $\mathcal{L}^+$ that satisfies $\varphi(\vec{a})$.

In part 4, we show that in weak arithmetic such as $\mathsf{Q}_<$ (or Q), all recursive sets are representable, and hence ample meta-mathematical arguments of arithmetic can be developed. Here, we aim to formalize meta-mathematics of general $\mathcal{L}$-structures, and this can also be done in $\mathsf{Q}_<$, so by extending the language to include $\mathsf{Q}_<$, recursive types of $\mathcal{L}$-structures can be represented by a single formula.

Proof. The basic idea is to transform meta-mathematical arguments about $\mathcal{L}$-structures into mathematical (object-language) arguments by utilizing the language of $Q_<$. The crucial point is that instead of creating the natural numbers outside of $\mathcal{L}$-structures, we will incorporate the arithmetical structure with part of the domain.

- Let $\mathcal{L}^+$ be an extended language of $\mathcal{L}$ obtained by adding the following symbols:

$$(x),\ +,\ \bullet,\ 0,\ 1,\ <,\ (n,x),\ \mathrm{Sat}(n,x),\ \pi(x,i).$$

  Here, (x) represents the domain of arithmetic, (n,x) is a function to evaluate terms in $\mathcal{L}$, $\mathrm{Sat}(n,x)$ the satisfaction relation of $\mathcal{L}$-structures, and $\pi(x,i)=x_i$ the projection function extracting the i-th component x_i from the code x of an infinite sequence $(x_0,x_1,\cdots)$.

- We want to express the recursive type $\Phi(\vec{v})$ in $\mathcal{L}^+$ as a formula $\varphi(\vec{v})$, which we will define in six components σ_i $(i=1,\cdots,6)$. Each σ_i $(i=1,\cdots,5)$ is a sentence, and σ_6 is a formula with free variables $\vec{v}$, and $\varphi(\vec{v})$ is defined by

$$\varphi(\vec{v}) \equiv \sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_6.$$

1. σ_1 expresses the basic properties of (x) as follows:

$$(0) \wedge (1) \wedge \forall x \forall y((x) \wedge (y) \to (x+y) \wedge (x \bullet y)).$$

2. σ_2 represents $(N, +, \bullet, 0, 1, <) \models \mathsf{Q}_<$, i.e., σ_2 is the conjunction of the eight axioms of $\mathsf{Q}_<$ with quantifiers restricted to N. For example, A10 (predecessor) is expressed as

$$\forall x((x) \to (x \neq 0 \to \exists y((y) \wedge y + 1 = x))).$$

Since all primitive recursive functions over N are representable in $\mathsf{Q}_<$, Gödel numbers of terms and formulas in $\mathcal{L}$ can be handled as elements of N.

3. σ_3 is the following sentence which stipulates a projection function $\pi(x, i)$: assuming variables i, j ranges over N for simplicity,

$$\forall x \forall i \forall z \exists y (\forall j \neq i (\pi(y, j) = \pi(x, j)) \wedge \pi(y, i) = z).$$

Here, y is the code of a sequence obtained by replacing the i-th element of $x = (x_0, x_1, \cdots)$ with z. We write this y as $x[z/i]$. Note that σ_3 does not assert the existence of infinite sequences in general, but it says that finite parts can be specified arbitrarily.
In fact, we will treat an infinite sequence as a finite sequence followed by infinitely many 0's. More strictly, letting $0 = (0, 0, 0, \dots)$, $0[u_0/\overline{0}][u_1/\overline{1}] \cdots [u_{k-1}/\overline{k-1}]$ denotes $\vec{u} = (u_0, u_1, \dots, u_{k-1})$.

4. σ_4 describes the function (n,x) that evaluates terms in $\mathcal{L}$. It is defined as the conjunction of the following sentences: For variables $v_0, v_1, \cdots$,

$$\forall i(\in N)\forall a((\ulcorner v_i \urcorner, a) = \pi(a,i)).$$

For each m-ary function symbol $\mathtt{f}$ in $\mathcal{L}$,

$$\forall t_0, \cdots, t_{m-1}(\in N)\forall a((\ulcorner \mathtt{f}(t_0, \cdots, t_{m-1})\urcorner, a)$$

$$= \mathtt{f}((\ulcorner t_0 \urcorner, a), \cdots, (\ulcorner t_{m-1}\urcorner, a))).$$

5. σ_5 describes the satisfaction relation $\mathrm{Sat}(n,x)$ of $\mathcal{L}$-structures. It consists of the following sentences. For each n-ary relation symbol $\mathtt{R}$ of $\mathcal{L}$ (including equality), we have

$$\forall t_0, \cdots, t_{n-1}\forall a(\mathrm{Sat}(\ulcorner \mathtt{R}(t_0, \cdots, t_{n-1})\urcorner, a) \leftrightarrow \mathtt{R}((\ulcorner t_0 \urcorner, a), \cdots, (\ulcorner t_{m-1}\urcorner, a))).$$

For each logical symbol, we have

$$\forall a(\mathrm{Sat}(\ulcorner \psi_0 \land \psi_1 \urcorner, a) \leftrightarrow (\mathrm{Sat}(\ulcorner \psi_0 \urcorner, a) \land \mathrm{Sat}(\ulcorner \psi_1 \urcorner, a))),$$

$$\forall a(\mathrm{Sat}(\ulcorner \exists x_i \psi \urcorner, a) \leftrightarrow \exists b\, \mathrm{Sat}(\ulcorner \psi \urcorner, a[b/i]))$$

and so on.

6. σ_6 is a formula expressing $\Phi(\vec{v})$ using Sat. For a recursive type $\Phi(\vec{v})$, let $\gamma(n)$ be a formula expressing the set of Gödel numbers of $\Phi(\vec{v})$ in $Q_<$, and define σ_6 as follows:

$$\forall n \in N(((N, +, \bullet, 0, 1, <) \models \gamma(\overline{n})) \to \mathrm{Sat}(n, \vec{v})).$$

In this way, we have obtained $\varphi(\vec{x})$, and we will now verify that it satisfies the conditions of the theorem. First, to prove condition (1), suppose that in a structure $\mathfrak{A}^+$ in $\mathcal{L}^+$, $a = (a_0, \overline{\cdots, a_{l-1}})$ realizes $\varphi(\vec{v})$. Let $\mathfrak{A}$ be its reduct to $\mathcal{L}$. For each $\psi(\vec{v})$ in $\Phi(\vec{v})$, we have $Q_< \vdash \gamma(\ulcorner\psi(\vec{v})\urcorner)$, and then by σ_2 and σ_6, we have:

$$\mathfrak{A}^+ \models \mathrm{Sat}(\ulcorner\psi\urcorner, a)$$

Furthermore, by meta-induction on the construction of the formula ψ, we can prove by σ_4 and σ_5 that

$$\mathfrak{A}^+ \models \mathrm{Sat}(\ulcorner\psi\urcorner, a) \leftrightarrow \psi(a_0, \cdots, a_{l-1})$$

Therefore, we have

$$\mathfrak{A}^+ \models \psi(a_0, \cdots, a_{l-1}),$$

which implies that $\psi(a_0, \cdots, a_{l-1})$ holds in $\mathfrak{A}$. Since $\psi(\vec{v}) \in \Phi(\vec{v})$ is arbitrary, $\mathfrak{A}$ realizes $\Phi(\vec{v})$ by $\vec{a}$, which proves condition (1).

Next, to prove (2), suppose conversely that an infinite structure $\mathfrak{A}$ in $\mathcal{L}$ realizes $\Phi(\vec{v})$ by $\vec{a}$.

- Choose a countably infinite subset N of $|\mathfrak{A}|$ and define $+, \bullet, 0, 1, <$ on N so that $(N, +, \bullet, 0, 1, <)$ is isomorphic to the standard structure of arithmetic. And extend $+, \bullet$ to total functions on A in an arbitrary way. Then, σ_1 and σ_2 clearly hold.
- Since A is infinite, there exists a bijection between A and $A^{<\omega}$. Let $B \subset A^\omega$ be the set of infinite sequences with all but finitely many elements being 0. Then, we can take a surjection $h : A \to B$. Now, define $\pi(a, i)$ to be the i-th element b_i of $h(a) = (b_0, b_1, \cdots)$. Then, σ_3 holds.
- Furthermore, by defining $(\ulcorner t \urcorner, a)$ as the value of a term t at a, and the satisfaction relation $\mathrm{Sat}(n, x)$ as

$$\mathrm{Sat}(\ulcorner \psi \urcorner, a) \Leftrightarrow \mathfrak{A} \models \psi(a_0, \cdots, a_{l-1}),$$

  we establish σ_4 and σ_5.
- Finally, for σ_6, we have:

$$(N, +, \bullet, 0, 1, <) \models \gamma(\overline{\ulcorner \psi \urcorner}) \Leftrightarrow \psi(\vec{v}) \in \Phi(\vec{v}) \Leftrightarrow \psi(a_0, \cdots, a_{l-1}) \Leftrightarrow \mathrm{Sat}(\overline{\ulcorner \psi \urcorner}, a)).$$

Thus, condition (2) is also satisfied. □

Corollary 4.5 (Barwise)

A resplendent structure in a finite language $\mathcal{L}$ is strongly resplendent, and so recursively saturated.

Proof.

- Let $\mathcal{L}$ be a finite language, and $\mathfrak{A}$ be a resplendent structure in $\mathcal{L}$. If $\mathfrak{A}$ is finite, then it is already recursively saturated and so strongly resplendent (by Barwise-Ressayre). Thus, we may assume that $\mathfrak{A}$ is infinite.
- To show that $\mathfrak{A}$ is strongly resplendent, suppose a recursive type $\Phi(\vec{v})$ in $\mathcal{L}'(\supset \mathcal{L})$ is given so that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent.
- Then, we can construct $\varphi(\vec{v})$ in $\mathcal{L}'^{+}$ to satisfy Kleene's Theorem.
- Let $\mathfrak{A}'$ be an $\mathcal{L}'$-expansion of an elementary extension of $\mathfrak{A}$ which satisfies $\Phi(\vec{a})$. Then, by Kleene's Theorem (2), $\mathfrak{A}'$ has an $\mathcal{L}'^{+}$-expansion $\mathfrak{A}'^{+}$ which satisfies $\phi(\vec{a})$. Thus by the resplendency of $\mathfrak{A}$, $\mathfrak{A}$ also has an $\mathcal{L}'^{+}$-expansion which satisfies $\phi(\vec{a})$.
- Finally, by Kleene's Theorem (1), $\Phi(\vec{a})$ holds in $\mathfrak{A}$. This proves that $\mathfrak{A}$ is strongly resplendent. □

Next we consider Kleene's Theorem for an arithmetic structure $\mathfrak{A}$.

- If $\mathcal{L}$ already includes the language of arithmetic $\mathcal{L}_{\mathrm{OR}}$, and a $\mathcal{L}$-structure $\mathfrak{A}$ is an expansion of a model of $\mathrm{Q}_<$, there is no need to introduce $+$, $\bullet$, 0, 1, $<$, (n, x), $\pi(x, i)$ separately. To prove Kleene's theorem, it suffices to use (x) and $\mathrm{Sat}(n, x)$.
- If $\mathfrak{A}$ is resplendent, we can introduce (x) and $\mathrm{Sat}(n, x)$ as relations in $\mathfrak{A}$, and then we can derive various properties of $\mathfrak{A}$ by adding various conditions to them.
- We start with a representative application.

Theorem 4.6

For any countable resplendent model $\mathfrak{A}$ of Peano Arithmetic PA, there exists a (proper) initial segment that is isomorphic to $\mathfrak{A}$, and $\mathfrak{A}$ is an elementary extension of this initial segment.

Proof. To the language of arithmetic $\mathcal{L}_{\mathrm{OR}}$, add (x), $\mathrm{Sat}(n, x)$, as well as $\mathrm{Sat}_N(n, x)$ to represent the satisfaction relation for N, and $\mathtt{f}(x)$ to represent an isomorphism.
Now, consider a recursive type claiming that N is an initial segment isomorphic to the whole $\mathfrak{A}$, and is also an elementary substructure. This type is consistent with Th$(\mathfrak{A}_A)$ by Friedman's theorem. By resplendency, N can be realized as an initial segment of $\mathfrak{A}$. □

Theorem 4.7

For a resplendent model $\mathfrak{A}$ of Peano Arithmetic PA, there exists a satisfaction relation Sat, such that for any $\mathcal{L}_{\mathrm{OR}}$ formula ψ,

$$(\mathfrak{A}, Sat) \models \forall a(\mathrm{Sat}(\ulcorner\psi\urcorner, a) \leftrightarrow \psi(a_0, \cdots, a_{l-1}))$$

and $(\mathfrak{A}, Sat)$ satisfies induction for formulas in $\mathcal{L}_{\mathrm{OR}} \cup \{\mathrm{Sat}\}$. Conversely, if a model $\mathfrak{A}$ of Peano Arithmetic PA has such a relation Sat, then $\mathfrak{A}$ is recursively saturated, and hence, if countable, it is resplendent.

Proof. The existence of Sat follows from the resplendency and the definition of Sat in Kleene's theorem. To show that $(\mathfrak{A}, Sat)$ satisfies induction, it is enough to see that the recursive set of sentences representing the induction for $\mathcal{L}_{\mathrm{OR}} \cup \{\mathrm{Sat}\}$ is consistent with $\mathrm{Th}(\mathfrak{A}_A)$. The second part is obvious from the following lemma.

Lemma (revisited)

For each $n > 0$, a non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$ is Σ_n-**recursively saturated**.

□

§5. Applications

Theorem 5.1 (Robinson's Joint Consistency Theorem)

Let $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$, and let T be a complete theory in the language $\mathcal{L}$, with T_1 and T_2 being extensions of T in the languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then, the necessary and sufficient condition for $T_1 \cup T_2$ to be consistent is that T_1 and T_2 are separately consistent.

Proof. The necessity is clear, so we will prove the sufficiency. Assume T_1 and T_2 are consistent, but $T_1 \cup T_2$ is inconsistent.

- Since $T_1 \cup T_2$ is inconsistent, there exist finite subsets $S_1 \subseteq T_1$ and $S_2 \subseteq T_2$ such that $S_1 \cup S_2$ also leads to a contradiction.
- Suppose S_1 and S_2 are theories in finite languages $\mathcal{L}'_1$ and $\mathcal{L}'_2$, respectively. Define $\mathcal{L}' = \mathcal{L}'_1 \cap \mathcal{L}'_2$, and let T' be the set of $\mathcal{L}'$-sentences that can be deduced from T. Then, T' is a complete and consistent set in the language $\mathcal{L}'$, since T is a complete and consistent set in $\mathcal{L}$
- Moreover, let $S'_1 = S_1 \cup T'$ and $S'_2 = S_2 \cup T'$. Since S'_1 and S'_2 are subsets of T_1 and T_2, respectively, they are separately consistent.

- Consider a countable saturated model $\mathfrak{A}$ of T'. Since T' is complete, $T' = \mathrm{Th}(\mathfrak{A})$.
- Since $S_1' = S_1 \cup \mathrm{Th}(\mathfrak{A})$ is consistent, by resplendency of $\mathfrak{A}$, $\mathfrak{A}$ can be extended to a model $\mathfrak{A}_1$ of S_1 in $\mathcal{L}_1'$.
- Similarly, $\mathfrak{A}$ can be extended to a model $\mathfrak{A}_2$ of S_2 in $\mathcal{L}_2'$. Therefore, by defining the interpretation of symbols in $\mathcal{L}_1' - \mathcal{L}'$ to be the same as in $\mathfrak{A}_1$ and in $\mathcal{L}_2' - \mathcal{L}'$ to be the same as in $\mathfrak{A}_2$, we extend $\mathfrak{A}$ to a structure $\mathfrak{A}'$ in $\mathcal{L}_1' \cup \mathcal{L}_2'$.
- Then, $\mathfrak{A}'$ is a model of $S_1 \cup S_2$, which contradicts our assumption. Thus, we complete the proof.

□

Corollary 5.2 (Craig's Interpolation Theorem)

If a formula $\varphi \to \psi$ is provable from logical axioms ($\vdash \varphi \to \psi$), then there exists a formula θ consisting of mathematical symbols commonly appearing both in φ and ψ besides logical symbols and $=$, such that $\vdash \varphi \to \theta$ and $\vdash \theta \to \psi$.

The formula θ satisfying the above theorem is called an **interpolant** for φ and ψ.

Proof

- Assume $\vdash \varphi \to \psi$ with no interpolant θ. Let $\mathcal{L}$ be the language consisting of symbols common to φ and ψ. Let T_0 be the set of formulas ξ in $\mathcal{L}$ such that $\vdash \varphi \to \xi$.
- Since no finite subset of T_0 implies ψ, $T_0 \cup \{\neg\psi\}$ is consistent.
- Consider a model $\mathfrak{A}$ of $T_0 \cup \{\neg\psi\}$, and let T be the set of all $\mathcal{L}$ formulas contained in $\mathrm{Th}(\mathfrak{A})$. Clearly, $T \cup \{\neg\psi\}$ is consistent.
- To show that $T \cup \{\varphi\}$ is also consistent, assume otherwise. Then there exists a formula σ in T such that $\vdash \varphi \to \neg\sigma$. Thus, $\neg\sigma \in T_0 \subseteq T$, which implies the inconsistency of T.
- By Robinson's joint consistency theorem, $T \cup \{\varphi, \neg\psi\}$ is also consistent, contradicting the assumption $\vdash \varphi \to \psi$. □

Logic and Foundations

- Part 5. Models of first-order arithmetic
- **Part 6. Reverse mathematics**
- Part 7. Second order arithmetic and non-standard methods

Part 6. Schedule

- Nov. 19, (0) Introduction
- Nov. 21, (1) Defining real numbers in RCA_0
- Nov. 26, (2) Completeness of the reals and ACA_0
- Nov. 28, (3) Continuous functions and WKL_0

Part 6. From Hilbert's program to reverse mathematics

- **Second-order arithmetic** is a formal theory targeting natural numbers and sets of natural numbers. It was D. Hilbert who first drew attention to its importance as foundations of mathematics. He formulated a deductive system of second-order arithmetic Z_2 around 1920, which can also encompass real numbers, sequences of real numbers, continuous functions and much more.
- The second problem of Hilbert's 23 problems was to show the consistency of basic arithmetic of reals. This problem was then conceived as **Hilbert's program** whose aim is to establish the consistency of Z_2 finitistically. As known well, Gödel's second incompleteness theorem blocked its progress.
- However, it is also known that a considerable breadth of mathematics can be developed within weak subsystems of Z_2, whose consistency is secured finitistically. From the mid-1970's, H. Friedman, S. Simpson, and others started research to investigate which subsystem is needed to prove a popular theorem of mathematics in the framework of second order arithmetic. This research program has evolved into a significant field known as **reverse mathematics**.

Reverse Mathematics Program

Reverse Mathematics

Which axioms are needed to prove a theorem?

Big Five subsystems in order of increasing strength: RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0

- RCA_0 stands for the Recursive Comprehension Axiom, and it only guarantees the existence of recursive (computable) sets. The subscript 0 indicates a restriction on induction, which will be discussed later.

- $\mathsf{WKL}_0 = \mathsf{RCA}_0 + \overbrace{\text{any infinite binary tree has an infinite path}}^{\text{Weak König Lemma}}$
$= \mathsf{RCA}_0 + \Sigma^0_1$-SP

Σ^0_1-SP (Σ^0_1 *separation*):

$$\neg\exists x(\varphi_0(x) \land \varphi_1(x)) \to \exists X \forall x((\varphi_0(x) \to x \in X) \land (\varphi_1(x) \to x \notin X)),$$

where $\varphi_0(x)$ and $\varphi_1(x)$ are Σ^0_1 formulas.

- $\text{ACA}_0 = \text{RCA}_0 + \overbrace{\exists X \forall n \big(n \in X \leftrightarrow \varphi(n) \big) \text{ for all arithmetical } \varphi(n)}^{\text{Arithmetical Comprehension}}$
$= \text{RCA}_0 + \Sigma^0_1\text{-CA}$

- $\text{ATR}_0 = \text{RCA}_0 + \overbrace{\text{the existence of a transfinite hierarchy produced}}^{\text{Arithmetical Transfinite Recursion}}$
by interating arithemetic comprehension along a given well order

- $\Pi^1_1\text{-CA}_0 = \text{RCA}_0 + \overbrace{\exists X \forall n \big(n \in X \leftrightarrow \varphi(n) \big) \text{ for all } \Pi^1_1\ \varphi(n)}^{\Pi^1_1 \text{ Comprehension}}$
A formula in the form $\forall X \psi$ with ψ arithmetical is called a Π^1_1 formula.

The Reverse Mathematics Phenomenon

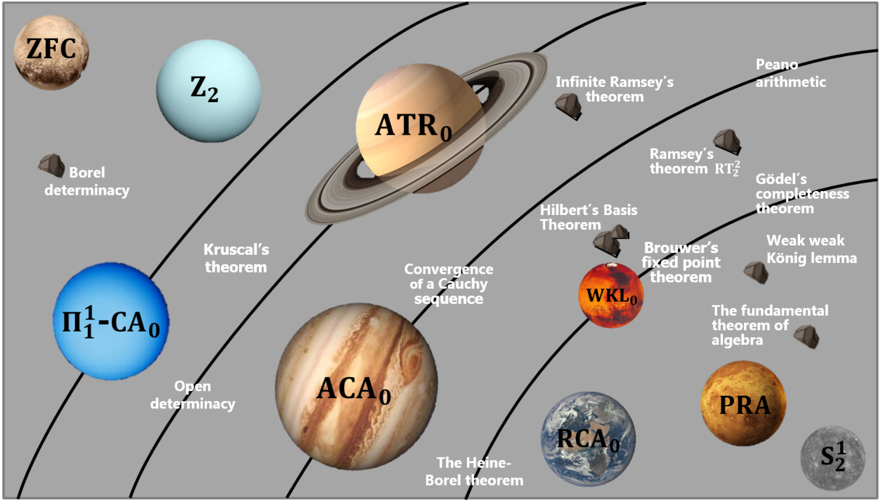
Many theorems of mathematics are either provable in RCA_0*, or logically equivalent (over* RCA_0*) to one of the other four systems mentioned above.*

$\mathsf{RCA}_0 \Rightarrow$ the intermediate value theorem
$\Rightarrow$ fundamental theorem of algebra
$\mathsf{WKL}_0 \leftrightarrow$ the maximum principle $\leftrightarrow$ the Cauchy-Peano theorem
$\leftrightarrow$ Brouwer's fixed point theorem
$\mathsf{ACA}_0 \leftrightarrow$ the Bolzano-Weierstrass theorem $\leftrightarrow$ the Ascoli-Arzela lemma
$\mathsf{ATR}_0 \leftrightarrow$ the Luzin separation theorem $\leftrightarrow$ Open-determinacy
$\Pi^1_1\text{-}\mathsf{CA}_0 \leftrightarrow$ the Cantor-Bendixson theorem $\leftrightarrow$ (Open $\wedge$ Closed)-determinacy

Friedman's conservation result

$\mathsf{WKL}_0 \vdash \sigma \Rightarrow$ Primitive Recursive Arithmetic $\vdash \sigma$ for $\sigma \in \Pi^0_2$.

Planets of Reverse Mathematics



§1. Second-order arithmetic RCA_0

- The language $\mathcal{L}^2_{\mathrm{OR}}$ of second-order arithmetic is the language of first-order arithmetic $\mathcal{L}_{\mathrm{OR}} = \{+, \cdot, 0, 1, <\}$ plus a symbol $\in$ for the membership relation.
- The **formulas** of second-order arithmetic are constructed from atomic formulas $(t_1 = t_2,\ t_1 < t_2,\ t \in X)$ by propositional connectives such as $\neg$, $\vee$, etc., and quantifiers over arithmetic $\forall x$, $\exists x$, as well as over sets $\forall X$, $\exists X$.
- A formula can be rewritten in the prenex normal form by shifting quantifiers to the head of formula. Moreover, all second-order quantifiers can be placed outside of the scopes of any first-order quantifier. The following transformation is possible even in a very weak theory,

$$\forall x \exists Y \varphi(x, Y) \Leftrightarrow \forall X \exists Y (\exists! x (x \in X) \to \forall x (x \in X \to \varphi(x, Y))).$$

If the axiom of choice is available, the places of quantifiers are exchanged as follows:

$$\forall x \exists Y \varphi(x, Y) \Leftrightarrow \exists Y' \forall x \varphi(x, Y'_x),$$

where Y' is a set-valued choice function, that is, $Y'(x) = Y'_x = \{y : (x, y) \in Y'\}$.

Hierarchy of formulas

We inductively define the hierarchy of $\mathcal{L}^2_{\mathrm{OR}}$-formulas, Σ^i_j and Π^i_j $(i = 0, 1,\ j \in \mathbb{N})$.

Definition 1.1

- The **bounded** formulas are constructed from atomic formulas $t_1 = t_2, t_1 < t_2, t \in X$ by propositional connectives and bounded quantifiers $\forall x < t$, $\exists x < t$.
  The class of such formulas is written as Π^0_0 or Σ^0_0.
- For each $j \geq 0$, if $\varphi \in \Sigma^0_j$, then $\forall x_1 \cdots \forall x_k \varphi \in \Pi^0_{j+1}$;
  if $\varphi \in \Pi^0_j$, then $\exists x_1 \cdots \exists x_k \varphi \in \Sigma^0_{j+1}$.
  All formulas in Σ^0_j and Π^0_j are called **arithmetical**.
  The class of arithmetical formulas is also denoted as Π^1_0 or Σ^1_0.
- For each $j \geq 0$, if $\varphi \in \Sigma^1_j$, then $\forall X_1 \cdots \forall X_k \varphi \in \Pi^1_{j+1}$;
  if $\varphi \in \Pi^1_j$ then $\exists X_1 \cdots \exists X_k \varphi \in \Sigma^1_{j+1}$.
  All formulas in Σ^1_j and Π^1_j are called **analytical**.

- Formulas belonging to Σ^i_j or Π^i_j are referred to as Σ^i_j or Π^i_j formulas, respectively.
- Σ^0_i (or Π^0_i) formulas without set variables are nothing but Σ_i (or Π_i) formulas of first-order arithmetic.
- A formula that is equivalent to a Σ^i_j (or Π^i_j) formula on a given base system is also called Σ^i_j (or Π^i_j).
- Furthermore, if a Σ^i_j formula is equivalent to a Π^i_j formula, each of them is called a Δ^i_j formula. Since the equivalence of formulas depends on a base theory T, Δ^i_j is strictly expressed as $(\Delta^i_j)^T$.
- When dealing with arithmetical hierarchies Σ^0_i Π^0_i, a system of second-order arithmetic RCA_0 is often assumed as a base theory. While dealing with analytical hierarchies, a stronger system ACA_0 is often needed.

Examples:

- "X is an infinite set" is represented by a Π^0_2 formula $\forall x \exists y(x < y \land y \in X)$.
- "A linear order $\preceq$ is a well-ordering", that is, "every non-empty set has the least element", can be represented by the following Π^1_1 formula
  $\forall X(\exists z(z \in X) \to \exists x(x \in X \land \forall y \in X(x \preceq y)))$,
  or rewritten as $\forall X \forall z \exists x(z \notin X \lor (x \in X \land \forall y \in X(x \preceq y)))$.

The system of recursive comprehension axioms (RCA_0) is a weak base system of second-order arithmetic, which serves as the foundation for our subsequent observation.

Definition 1.2

The system of **recursive comprehension axioms** (RCA_0) consists of the following axioms:

(0) Axioms and inference rules of first-order logic with axioms of equality for numbers. Equality between sets $X = Y$ is defined as $\forall n(n \in X \leftrightarrow n \in Y)$.

(1) Basic arithmetic axioms: Same as $\mathsf{Q}_<$ (Part 4).

(2) Δ^0_1 comprehension axiom (Δ^0_1-CA_0):

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \to \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is a Σ^0_1 formula, $\psi(n)$ is a Π^0_1 formula, and neither includes X as a free variable. This axiom ensures the existence of set $X = \{n : \varphi(n)\}$.

(3) Σ^0_1 induction: $\varphi(0) \land \forall n(\varphi(n) \to \varphi(n+1)) \to \forall n \varphi(n)$, for any Σ^0_1 formula $\varphi(n)$.

- Since the Δ^0_1 comprehension axiom asserts the existence of recursive sets (=computable sets) in the standard model $\mathbb{N}$, it is called the **recursive comprehension axiom**.

- More precisely, since $\psi(x)$ and $\varphi(x)$ in the axiom may include set variables (other than X) as parameters, this axiom indeed asserts that there exists a set that can be computed in a parameter set as an oracle. But notice that it does not assert the non-existence of a non-recursive set.

- RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$. That is, a sentence of $\mathcal{L}_{\mathrm{OR}}$ that is provable in RCA_0 is already provable in $\mathsf{I}\Sigma_1$, as shown in the next lemma.

Definition (preliminary). The system of **arithmetical comprehension axioms** ACA_0 is obtained from RCA_0 by replacing the Δ^0_1 comprehension with the Σ^0_1 comprehension [1].

- ACA_0 is a conservative extension of first-order arithmetic PA.

[1] Arithmetical (Σ^1_0) comprehension can be achieved by repeatedly applying the Σ^0_1 comprehension axiom to the parameters.

Lemma 1.3

RCA_0 is a conservative extension of first-order arithmetic $\mathsf{I}\Sigma_1$, that is, any theorem of $\mathsf{I}\Sigma_1$ is provable in RCA_0, and any sentence in $\mathcal{L}_{\mathrm{OR}}$ provable in RCA_0 is already provable in $\mathsf{I}\Sigma_1$.

Proof: It is obvious that any theorem of $\mathsf{I}\Sigma_1$ can be proved in RCA_0, since all axioms of $\mathsf{I}\Sigma_1$ are included in RCA_0.

To prove the converse, consider a sentence σ in $\mathcal{L}_{\mathrm{OR}}$ such that $\mathsf{I}\Sigma_1 \nvdash \sigma$. By the completeness theorem, there exists a model $\mathfrak{M} = (M, +, \cdot, 0, 1, <)$ of $\mathsf{I}\Sigma_1$ where $\mathfrak{M} \models \neg\sigma$.

For a Σ_1 formula $\varphi(x, y_1, \dots, y_k)$, a Π_1 formula $\psi(x, y_1, \dots, y_k)$ and $b_1, \dots, b_k \in M$, if

$$\mathfrak{M} \models \forall x(\varphi(x, b_1, \dots, b_k) \leftrightarrow \psi(x, b_1, \dots, b_k))$$

holds, then we put

$$A_{\varphi,\psi,b_1,\dots,b_k} = \{a \in M : \mathfrak{M} \models \varphi(a, b_1, \dots, b_k)\}.$$

Otherwise, we let $A_{\varphi,\psi,b_1,\dots,b_k} = \emptyset$. Finally, let S be the set of Δ_1 definable subsets of M,

$$S = \{A_{\varphi,\psi,b_1,\dots,b_k} : \varphi \in \Sigma_1, \psi \in \Pi_1, \text{ and } b_1, \dots, b_k \in M\}.$$

- To show that $(\mathfrak{M}, S) = (M \cup S, +, \cdot, 0, 1, <, \in)$ forms a model of RCA_0, it suffices to prove that any Σ^0_1 formula with set parameters from S can be rewritten as an equivalent Σ^0_1 formula without set parameters. If so, Σ^0_1 induction of $(\mathfrak{M}, S)$ can be derived from Σ_1 induction of $\mathfrak{M}$. Also, $(\mathfrak{M}, S)$ satisfies Δ^0_1 comprehension, since any set Δ^0_1 (i.e., Σ^0_1 and Π^0_1) definable with set parameters can be Δ^0_1 definable without set parameters, and so already belongs to S.

- Now, consider a Σ^0_1 formula $\theta(x, b_1, \dots, b_k, A_{\varphi_1,\psi_1,\bar{c}}, \dots, A_{\varphi_l,\psi_l,\bar{c}})$ with $b_i \in M$ and $A_{\varphi_j,\psi_j,\bar{c}} \in S$. In the formula, replace $t \in A_{\varphi_j,\psi_j,\bar{c}}$ with either $\varphi_i(t, \bar{c})$ or $\psi_i(t, \bar{c})$ so that the whole formula keeps in Σ^0_1. Thus, we obtain a Σ^0_1 formula $\theta'(x, b_1, \dots, b_k, \bar{c})$, which is equivalent to $\theta(x, b_1, \dots, b_k, A_{\varphi_1,\psi_1,\bar{c}}, \dots, A_{\varphi_l,\psi_l,\bar{c}})$. The same for Π^0_1 formulas. Thus, $(\mathfrak{M}, S)$ is a model of RCA_0.

- Finally, since σ does not contain set variables, its truth value is independent of S, and hence $(\mathfrak{M}, S) \models \neg\sigma$. Therefore, $\mathsf{RCA}_0 + \neg\sigma$ is consistent, which implies $\mathsf{RCA}_0 \not\vdash \sigma$. This completes the proof. $\square$

The various properties of $\mathrm{I}\Sigma_1$ demonstrated in Part 4 also hold true in RCA_0. In particular, the following fact is frequently used.

Lemma 1.4

In RCA_0, the following holds:
(1) Π^0_1 induction.
(2) The class of Σ^0_1 formulas is closed under bounded quantification.

Proof ideas. (1) Let $\varphi(x)$ be a Π^0_1 formula and assume $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1))$. By way of contradiction, we assume $\neg\varphi(c)$. Use induction for a Σ^0_1 formula $\neg\varphi(c-x)$. Then, $\neg\varphi(c-0)$ and $\neg\ \varphi(c-x) \to \neg\varphi(c-(x+1))$ imply $\neg\varphi(0)$, a contradiction.

(2) Suppose $\forall x < u \exists y \varphi(x,y)$ with $\varphi(x,y)$ bounded. Let $\psi(w)$ be a Σ^0_1 formula $\exists v \forall x < w \exists y < v \varphi(x,y) \vee u < w$. By Σ^0_1 induction, we have $\forall w \psi(w)$, in particular, $\exists v \forall x < u \exists y < v \varphi(x,y)$.

Let X, Y be sets of natural numbers. $X \subseteq Y$ is an abbreviation for $\forall n(n \in X \to n \in Y)$, and $X = Y$ is defined as $X \subseteq Y \wedge Y \subseteq X$. The equality of terms $t_1 = t_2$ is a Π^0_0 formula, but the equality of sets $X = Y$ is a Π^0_1 formula.

The **pair** of natural numbers (m,n) is coded by a natural number $\frac{(m+n)(m+n+1)}{2}+m$.

The **product** $X \times Y$ is the set of pairs (codes) of one from X and the other from Y. Thus,

$$n \in X \times Y \leftrightarrow \exists x \leq n \exists y \leq n(x \in X \wedge y \in Y \wedge (x,y)=n).$$

Since the above formula is Σ^0_0, the existence of $X \times Y$ is guaranteed in RCA_0.

A **function** $f : X \to Y$ is a subset $F \subseteq X \times Y$ such that

$$\forall x \forall y_0 \forall y_1((x,y_0) \in F \wedge (x,y_1) \in F \to y_0 = y_1) \text{ and } \forall x \in X \exists y \in Y (x,y) \in F.$$

When $(x,y) \in F$, we write $f(x)=y$. The set X of $f : X \to Y$ is called the domain of f.

A function whose domain is $\mathbb{N}$ or $\mathbb{N}^k$ is called a **total function**.

Furthermore, a function f whose domain is $X = \{i : i < n\}$ is called a **finite sequence** with **length** n. In RCA_0, a finite sequence can be coded by a natural number, and this code (Gödel number) is often identified with the sequence itself.

Lemma 1.5

In RCA_0, it is provable that the set of total functions is closed under primitive recursion.

Proof In Part 4, we proved that a function defined by primitive recursion is Δ_1 definable in $\mathsf{I}\Sigma_1$, thus by Δ^0_1 comprehension, it exists as a set. □

Moreover, we have

Lemma 1.6

In RCA_0, it is provable that the set of (partial) functions is closed under minimization μ.

Proof Expressing $g(x_1, \cdots, x_n) = \mu y(f(x_1, \cdots, x_n, y) = 0)$ in a formula, we have

$$((x_1, \cdots, x_n), y) \in g \Leftrightarrow ((x_1, \cdots, x_n, y), 0) \in f \land \forall z < y((x_1, \cdots, x_n, z), 0) \notin f.$$

The right side is a Σ^0_0 formula, so the existence of g and its totality can be shown in RCA_0. □

Note For a recursive function defined using μ-operator, if its totality is provable in RCA_0, it can be defined by primitive recursion without using μ (by Friedman's Theorem in the next chapter).

Lemma 1.7

In RCA_0, for any Σ^0_1 formula $\varphi(x)$, there exists a finite set X such that $\forall x(x \in X \leftrightarrow \varphi(x))$, or there exists a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall y(\exists x f(x) = y \leftrightarrow \varphi(y))$.

Proof

- Let $\varphi(x)$ be a Σ^0_1 formula. So, there exists a Σ^0_0 formula $\theta(x, y)$ such that $\varphi(x) \leftrightarrow \exists y\theta(x, y)$. By ($\Sigma^0_0$-CA),

$$Y = \{(x, y) : \theta(x, y) \wedge \forall y' < y \neg\theta(x, y')\}$$

  exists. Note that $\exists y\theta(x, y) \leftrightarrow \exists y(x, y) \in Y \leftrightarrow \exists! y(x, y) \in Y$ for all x.
- If Y is bounded, there exist u, v such that $\varphi(x) \leftrightarrow (x < u \wedge \exists y < v\theta(x, y))$. Then, by ($\Sigma^0_0$-CA), there exists a finite set X such that $\forall x(x \in X \leftrightarrow \varphi(x))$.
- Next, suppose that Y is unbounded. By Lemmas1.5 and 1.6, we can define a function which enumerates the elements of Y, and a function which extracts the first component x from (x, y). Combining them, we can create a one-to-one function f such that $\forall y(\exists x f(x) = y \leftrightarrow \varphi(y))$. This proves the lemma. □

Lemma 1.8

In RCA_0, the following form of set existence axiom is provable:

$$(\text{Bounded } \Sigma^0_1\text{-CA}) : \forall x \exists X \forall y (y \in X \leftrightarrow (y < x \wedge \varphi(y))),$$

where $\varphi(y)$ is a Σ^0_1 formula, not containing X as a free variable.

Proof For a fixed x, if there exist no finite set X such that

$$\forall y (y \in X \leftrightarrow (y < x \wedge \varphi(y))),$$

then by the previous lemma, there must exist a one-to-one function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall y (\exists z f(z) = y \leftrightarrow (y < x \wedge \varphi(y))),$$

which is absurd. □

HW # 5-1 (Strong Σ^0_1 Collection Axiom)

Prove in RCA_0: for a Σ^0_1 formula $\varphi(i, j)$ (not containing n as a free variable),

$$(\mathsf{S}\Sigma^0_1) : \forall m \exists n \forall i < m (\exists j \varphi(i, j) \to \exists j < n \varphi(i, j)).$$

Thank you for your attention!