Topics in Applied Math: Logic and Foundations of Mathematics

Part 5. Models of first-order arithmetic

Kazuyuki Tanaka

BIMSA

November 14, 2025

清华大学 求真书院
Qiuzhen College, Tsinghua University

Logic and Foundations

- **Part 1. Equational theory**
- **Part 2. First order theory**
- **Part 3. Basic Model theory**
- **Part 4. First order arithmetic and incompleteness theorems**
- **Part 5. Models of first-order arithmetic**
- **Part 6. Second order arithmetic and reverse mathematics**

Part 5. Schedule

- Nov. 5, (0) Nonstandard models and Overspill
- Nov. 7, (1) The omitting type theorem
- Nov.12, (2) Recursively saturated models
- Nov.14, (3) Friedman's theorem
- Nov.19, (4) Resplendency

Recap

- PA^- is the theory of discrete ordered semirings.
- $\mathrm{I}\Sigma_n$ is PA^-+ induction for Σ_n formulas. $\mathsf{PA} \supset \mathrm{I}\Sigma_1 \supset \mathrm{I}\Sigma_0 \supset \mathrm{IOpen} \supset \mathsf{PA}^- \supset \mathsf{Q}_<$.

Theorem 1.3 (Overspill principle)

Let $\mathfrak{A}$ be any non-standard model of $\mathrm{I}\Sigma_n$, and $\varphi(x)$ be any Σ_n formula. If $\mathfrak{A}_A \models \varphi(i)$ holds for infinitely many $i \in \mathbb{N}$, then there exists $a \notin \mathbb{N}$ such that $\mathfrak{A}_A \models \varphi(a)$.

- A type $\Phi(\vec{x})$ is a **type of a theory** T if $T \cup \Phi(\vec{c})$ ($\vec{c}$ new constants) is consistent. That is, there exists a model of T that realizes $\Phi(\vec{x})$.
- A **type on $C \subset A$ in $\mathfrak{A}$** is a type of theory $\mathrm{Th}(\mathfrak{A}_C)$.
- A type $\Phi(\vec{x})$ is a **principal** type of theory T, if there exists a formula $\psi(\vec{x})$ such that $T \cup \{\exists \vec{x} \psi(\vec{x})\}$ is consistent, and for any $\varphi(\vec{x}) \in \Phi(\vec{x})$, $T \vdash \forall \vec{x}(\psi(\vec{x}) \to \varphi(\vec{x}))$.

Theorem 2.5 (The omitting type theorem)

Let T be a consistent theory in a countable language $\mathcal{L}$. Given countably many non-principal types $\Phi_i(\vec{x}_i)$ of T, then there is a countable model of T that omits all Φ_i.

- A countable model of Peano arithmetic PA has a proper elementary end-extension.

§3. Recursively saturated models

Definition 3.1

Let $\mathcal{L}$ be a countable language. An $\mathcal{L}$-structure $\mathfrak{A}$ is **recursively saturated** if for any recursive type $\{\varphi_i(x, x_1, \cdots, x_n) \mid i \in \mathbb{N}\}$ and any $a_1, \cdots, a_n \in A$,

$$\forall j \exists a \in A \forall i < j\, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n) \Rightarrow \exists a \in A \forall i\, \mathfrak{A}_A \models \varphi_i(a, a_1, \cdots, a_n).$$

- A countable structure in a countable language has a countable elementary extension which is recursively saturated.

Lemma 3.4

For each $n > 0$, there exist formulas $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$ in language $\mathcal{L}_{\mathrm{OR}}$ such that for any Σ_n formula $\varphi(v_1, \cdots, v_k)$ and Π_n formula $\psi(v_1, \cdots, v_k)$,

$$\mathrm{I}\Sigma_1 \vdash \forall s(\mathrm{Sat}_{\Sigma_n}(\overline{\ulcorner \varphi \urcorner}, s) \leftrightarrow \varphi(s_1, \cdots, s_k)),$$
$$\mathrm{I}\Sigma_1 \vdash \forall s(\mathrm{Sat}_{\Pi_n}(\overline{\ulcorner \psi \urcorner}, s) \leftrightarrow \psi(s_1, \cdots, s_k)),$$

where s is the code of $(s_1, \cdots, s_k)$. When $n > 0$, $\mathrm{Sat}_{\Sigma_n} \in \Sigma_n$ and $\mathrm{Sat}_{\Pi_n} \in \Pi_n$.

Lemma 3.5

For $n > 0$, a non-standard model $\mathfrak{A}$ of $\mathrm{I}\Sigma_n$ is **Σ_n-recursively saturated**, i.e., it realizes any (finitely satisfiable) recursive 1-type on a finite subset of A consisting of only Σ_n formulas.

Proof. Let $\Phi(x, \vec{x})$ be a recursive type consisting only of Σ_n formulas. Then, the Gödel numbers of formulas in Φ can be expressed by a Δ_1 formula $\theta(i)$. Thereby,

- The finite satisfiability of $\Phi(x, \vec{a})$ is expressed as: for each natural number j,

  $$\exists x \forall i < \bar{j}(\theta(i) \to \mathrm{Sat}_{\Sigma_n}(i, (x, \vec{a}))),$$

  which is shown to be Σ_n in $\mathrm{B}\Sigma_n(\subseteq \mathrm{I}\Sigma_n)$.
- Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. By the overspill principle, the above formula holds for some infinite element j'. Suppose $x = a$ satisfies the formula for this j'.
- Then, we have $\theta(\bar{i}) \to \mathrm{Sat}_{\Sigma_n}(\bar{i}, (a, \vec{a}))$ for any natural number i. Namely, all Σ_n formulas in $\Phi(x, \vec{a})$ are realized by a in $\mathfrak{A}_A$. □

By the above lemma, any non-standard model of PA is Σ_n-recursively saturated for each $n > 0$, but there is a non-standard model of PA which is not recursively saturated.

Definition 3.6

Let $\mathfrak{A}$ be a model of $\mathrm{I}\Sigma_1$, and $a \in A$. The set

$$\{n \in \mathbb{N} : \mathfrak{A} \models \overline{p(n)}|a\}$$

is called the set **coded by a** in $\mathfrak{A}$, where $p(n)$ is a primitive recursive function representing the $n+1$-th prime number, and $u|v \equiv \exists w \leq v(u \bullet w = v)$. The collection of all the sets encoded by an element in $\mathfrak{A}$ is called the **standard system** of $\mathfrak{A}$, denoted as $\mathrm{SSy}(\mathfrak{A})$.

Lemma 3.7 (D. Scott)

Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_1$. Given two disjoint Σ_1 sets, there exists a set in $\mathrm{SSy}(\mathfrak{A})$ which separates them. In particular, any recursive set belongs to $\mathrm{SSy}(\mathfrak{A})$.

Note that in general, a set that separates two Σ_1 sets cannot be obtained recursively. That is, $\mathrm{SSy}(\mathfrak{A})$ is properly larger than the class of recursive sets.

Homework Problem #3-4

For any non-recursive set S, there exists a non-standard model of PA in which S cannot be coded by any element.

Lemma 3.8

Let $n > 0$ and $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. If a type $\Phi(\vec{x})$ of Σ_n formulas on a finite subset of A is coded in $\mathfrak{A}$, then $\mathfrak{A}$ realizes $\Phi(\vec{x})$.

The proof is exactly the same as that of lemma in Page 5. The converse holds as follows.

Lemma 3.9

Let $n > 0$ and $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$. Fix $\vec{a} \in A^k$ arbitrarily. Then the following k-types can be coded.

$$\Phi(\vec{x}) = \{\varphi(\vec{x}) : \varphi(\vec{x}) \in \Sigma_n \wedge \mathfrak{A} \models \varphi(\vec{a})\},$$
$$\Psi(\vec{x}) = \{\psi(\vec{x}) : \psi(\vec{x}) \in \Pi_n \wedge \mathfrak{A} \models \psi(\vec{a})\}$$

Proof. In $\mathrm{I}\Sigma_1$, $\mathrm{Sat}_{\Sigma_n}(x, y)$ and $\mathrm{Sat}_{\Pi_n}(x, y)$ can be defined. Let $\mathfrak{A}$ be a non-standard model of $\mathrm{I}\Sigma_n$ and c be any nonstandard element in A. By the least number principle for a Π_n formula (equivalent to $\mathrm{I}\Sigma_n$), there exists the minimal number $d \in A$ such that $\forall x < c(\mathrm{Sat}_{\Pi_n}(x, \vec{a}) \to p(x)|d)$. Then, d codes $\Psi(\vec{x})$ in $\mathfrak{A}$. Similarly for $\Phi(\vec{x})$. □

With the above preparations, we will prove Friedman's self-embedding theorem. The following is a key lemma, and also used in several variations of the theorem.

Lemma 3.10

Assuming $n > 0$, let $\mathfrak{A}$, $\mathfrak{B}$ be countable non-standard models of $\mathrm{I}\Sigma_n$. Take $a_0 \in A$ and $b_0, c \in B$ arbitrarily. Then the following two conditions are equivalent.

(1) There exists $\mathfrak{B}' \subseteq_e \mathfrak{B}$ such that $c \notin B'$. There is an isomorphism h between $\mathfrak{A}$ and $\mathfrak{B}'$ such that $h(a_0) = b_0$. For any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{b} \in B'^{<\omega}$,

$$\mathfrak{B}'_{\{\vec{b}\}} \models \varphi(\vec{b}) \Leftrightarrow \mathfrak{B}_{\{\vec{b}\}} \models \varphi(\vec{b}).$$

(2) $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$, and for any Π_{n-1} formula $\varphi(\vec{v}, u)$,

$$\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0) \Rightarrow \mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0),$$

where $\vec{v} = (v_1, \ldots, v_k)$ and $\exists \vec{v} < c$ means $\exists v_1 < c \cdots \exists v_k < c$.

Proof. Assume (1) and we show the first half of (2).

- By $\mathfrak{A} \cong \mathfrak{B}'$, $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B}')$ is obvious.
- Since $\mathfrak{B}' \subseteq_e \mathfrak{B}$, it is also clear that $\mathrm{SSy}(\mathfrak{B}') \subseteq \mathrm{SSy}(\mathfrak{B})$.
- Assume that $R \in \mathrm{SSy}(\mathfrak{B})$, i.e, R is coded by r in $\mathfrak{B}$. We will show that R is also coded in $\mathfrak{B}'$.
- Take any non-standard element l of B'. Since $\mathfrak{B}'$ is also a model of $\mathrm{I}\Sigma_1$, the $l+1$-th prime $p(l)$ belongs to B', and so $p(l)! \in B'$.
- Now, letting m be the greatest common divisor of r and $p(l)!$ in $\mathfrak{B}$, we have $m \in B'$ since $\mathfrak{B}'$ is an initial segment of $\mathfrak{B}$. Then, it is clear that m also encodes R.
- From the above, we obtain $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$.

Next we show the second half of (2).

- Let $\varphi(\vec{v}, u)$ be a Π_{n-1} formula, and $\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0)$.
- By the isomorphism between $\mathfrak{A}$ and $\mathfrak{B}'$, $\mathfrak{B}'_{B'} \models \exists \vec{v} \varphi(\vec{v}, b_0)$.
- Then, since there exists $\vec{d} \in B'$ such that $\mathfrak{B}'_{B'} \models \varphi(\vec{d}, b_0)$, from the assumption (1), $\mathfrak{B}_B \models \varphi(\vec{d}, b_0)$. Therefore, $\mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0)$.

Next, assuming (2), we show (1).

- This is an application of the so-called **back-and-forth argument**. We alternately produce a list $a_0, a_1, \cdots$ of the elements of A and a list $b_0, b_1, \cdots$ of the elements of B', so that an isomorphism h between $\mathfrak{A}$ and $\mathfrak{B}'$ is obtained by $h(a_i) = b_i$.
- Now, suppose $a_0, a_1, \cdots, a_{2k}$ and $b_0, b_1, \cdots, b_{2k}$ have been chosen, and for any Π_{n-1} formula $\varphi(\vec{v}, \vec{u})$,

$$\mathfrak{A}_A \models \exists \vec{v} \varphi(\vec{v}, a_0, \cdots, a_{2k}) \Rightarrow \mathfrak{B}_B \models \exists \vec{v} < c\, \varphi(\vec{v}, b_0, \cdots, b_{2k}) \qquad (\sharp)$$

  holds.
- We next choose a_{2k+1}, a_{2k+2} and b_{2k+1}, b_{2k+2} such that this condition is preserved. We will explain later that (1) can be obtained by this.
- Since A is countable, each member can be assigned by a natural number uniquely. Then choose one with the smallest number among the elements that do not appear in $a_0, a_1, \cdots, a_{2k}$ and denote it as a_{2k+1}. This procedure guarantees that $\{a_i : i \in \mathbb{N}\}$ lists all the members of A.

- Now we will search for b_{2k+1} such that ($\sharp$) holds.
- Let $\Phi(\vec{x})$ be the set of Σ_n formulas $\exists \vec{v} \varphi(\vec{v}, x_0, \cdots, x_{2k+1})$ ($\varphi \in \Pi_{n-1}$) which holds for $a_0, \cdots, a_{2k}, a_{2k+1}$ in $\mathfrak{A}$. By Lemma 3.9 on page 7, $\Phi(\vec{x})$ is coded in $\mathfrak{A}$. Since $\mathrm{SSy}(\mathfrak{A}) = \mathrm{SSy}(\mathfrak{B})$, so it is also coded in $\mathfrak{B}$.
- Furthermore, we let

$$\begin{aligned}&\Phi'(x_0, \cdots, x_{2k+1}, x_{2k+2})\\&= \{\exists \vec{v} < x_{2k+2}\, \varphi(\vec{v}, x_0, \cdots, x_{2k+1}) : \exists \vec{v} \varphi(\vec{v}, x_0, \cdots, x_{2k+1}) \in \Phi\}.\end{aligned}$$

  Since there is a primitive recursive transformation between Φ and Φ', Φ' is also coded in $\mathfrak{B}$.
- Then, if $\Phi'(b_0, \cdots, b_{2k}, x, c)$ is shown to be finitely satisfiable in $\mathfrak{B}$, then by Lemma 3.8 on page 7, we can find an element $x = b$ that realizes $\Phi'(b_0, \cdots, b_{2k}, x, c)$, and letting b_{2k+1} be such a b, ($\sharp$) holds.
- Now, let $\exists \vec{v} < c\, \varphi_i(\vec{v}, b_0, \cdots, b_{2k}, x)$ $(i \le j)$ be any finite set of formulas from $\Phi'(b_0, \cdots, b_{2k}, x, c)$.

- From the definition of Φ', for each $i \leq j$, $\exists \vec{v} \varphi_i(\vec{v}, a_0, \cdots, a_{2k}, a_{2k+1})$ holds in $\mathfrak{A}$, so

$$\mathfrak{A}_A \models \exists \vec{v}_0 \cdots \exists \vec{v}_j \exists x \bigwedge_{i \leq j} \varphi_i(\vec{v}_i, a_0, \cdots, a_{2k}, x).$$

- On the other hand, using ($\sharp$),

$$\mathfrak{B}_B \models \exists \vec{v}_0 < c \cdots \exists \vec{v}_j < c \exists x < c \bigwedge_{i \leq j} \varphi_i(\vec{v}_i, b_0, \cdots, b_{2k}, x).$$

- Therefore, by simple transformation,

$$\mathfrak{B}_B \models \exists x \bigwedge_{i \leq j} \exists \vec{v} < c\, \varphi_i(\vec{v}, b_0, \cdots, b_{2k}, x).$$

- In other words, $\Phi'(b_0, \cdots, b_{2k}, x, c)$ is finitely satisfiable, and b_{2k+1} is obtained.

- Next, we first select b_{2k+2} and we search for a corresponding a_{2k+2}. If $\{b_0, \cdots, b_{2k}, b_{2k+1}\}$ is an initial segment of $\mathfrak{B}$, then $b_{2k+2} = b_{2k+1}$, $a_{2k+2} = a_{2k+1}$, and ($\sharp$) holds.
- Otherwise, there exists a $b < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\}$ such that b does not appear in $b_0, \cdots, b_{2k}, b_{2k+1}$. Then among such, let b_{2k+2} be one with the minimal number assigned in advance to the members of B. This finally produces $\{b_i : i \in \mathbb{N}\}$ as an initial segment of $\mathfrak{B}$.
- Then we will find a_{2k+2} corresponding to b_{2k+2}.
- Let $\Psi(\vec{x})$ be the set of Σ_n formulas $\forall \vec{v} < x_{2k+3}\, \psi(\vec{v}, x_0, \cdots, x_{2k+2})$ holds for $b_0, \cdots, b_{2k+1}, b_{2k+2}, c$ in $\mathfrak{B}$. This can be coded in $\mathfrak{B}$.
- Therefore, if we define

$$\begin{aligned}&\Psi'(x_0, \cdots, x_{2k+1}, x_{2k+2})\\&= \{\forall \vec{v} \psi(\vec{v}, x_0, \cdots, x_{2k+2}) : \forall \vec{v} < x_{2k+3}\, \psi(\vec{v}, x_0, \cdots, x_{2k+2}) \in \Psi\}\end{aligned}$$

  then Ψ' is coded in $\mathfrak{A}$ by the same argument as above.
- All that remains is to show $\Psi'(a_0, \cdots, a_{2k+1}, x)$ is finitely satisfiable in $\mathfrak{A}$. So, let $\forall \vec{v} \psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x)$ $(i \leq j)$ be a finite subset of $\Psi'(a_0, \cdots, a_{2k+1}, x)$.

- We will show that these formulas are realized by $x = a$ such that $a < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\}$.
- By way of contradiction, assume

$$\mathfrak{A}_A \models \forall x < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\} \exists \vec{v} \bigvee_{i \leq j} \neg\psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x).$$

- By the Σ_n collection principle that follows from Σ_n induction,

$$\mathfrak{A}_A \models \exists y \forall x < \max\{a_0, \cdots, a_{2k}, a_{2k+1}\} \exists \vec{v} < y \bigvee_{i \leq j} \neg\psi_i(\vec{v}, a_0, \cdots, a_{2k+1}, x).$$

- On the other hand, using ($\sharp$),

$$\mathfrak{B}_B \models \exists y < c \forall x < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\} \exists \vec{v} < y \bigvee_{i \leq j} \neg\psi_i(\vec{v}, b_0, \cdots, b_{2k+1}, x).$$

- Therefore, by simple transformation,

$$\mathfrak{B}_B \models \forall x < \max\{b_0, \cdots, b_{2k}, b_{2k+1}\} \exists \vec{v} < c \bigvee_{i \leq j} \neg\psi_i(\vec{v}, b_0, \cdots, b_{2k+1}, x)$$

  This contradicts with the assumption that $b_0, \cdots, b_{2k+1}, b_{2k+2}, c$ realize $\Psi(\vec{x})$.
- Thus, $\Psi'(a_0, \cdots, a_{2k+1}, x)$ is finitely satisfiable, and so the desired a_{2k+2} exists.

- Suppose that we have completed the construction of a list $a_0, a_1, \cdots$, and a list $b_0, b_1, \cdots$. As described above, $A = \{a_i : i \in \mathbb{N}\}$ and $B' = \{b_i : i \in \mathbb{N}\}$ is an initial segment of $\mathfrak{B}$. It is also obvious that $c \notin B'$.
- Next, we define a function h between $\mathfrak{A}$ and $\mathfrak{B}'$ by $h(a_i) = b_i$. Then, h is an isomorphism, since by (♯), for an atomic formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Rightarrow \mathfrak{B}_B \models \varphi(b_0, \cdots, b_k),$$

which implies h preserves operations and $<$.
- Moreover, by (♯), we can show that for any Π_{n-1} formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Leftrightarrow \mathfrak{B}_B \models \varphi(b_0, \cdots, b_k).$$

$\Rightarrow$ is clear. For $\Leftarrow$, let $\mathfrak{A}_A \not\models \varphi(a_0, \cdots, a_k)$. Then $\mathfrak{A}_A \models \neg\varphi(a_0, \cdots, a_k)$, and $\neg\varphi(a_0, \cdots, a_k)$ is Σ_{n-1}, so by (♯), $\mathfrak{B}_B \models \neg\varphi(b_0, \cdots, b_k)$, and $\mathfrak{B}_B \not\models \varphi(b_0, \cdots, b_k)$.
- On the other hand, since h is isomorphic, for any formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{A}_A \models \varphi(a_0, \cdots, a_k) \Leftrightarrow \mathfrak{B}'_{B'} \models \varphi(b_0, \cdots, b_k).$$

So for any Π_{n-1} formula $\varphi(x_0, \cdots, x_k)$,

$$\mathfrak{B}'_{B'} \models \varphi(b_0, \cdots, b_k) \Leftrightarrow \mathfrak{B}_{B'} \models \varphi(b_0, \cdots, b_k),$$

and thus (1) is obtained. □

Theorem 3.11 (Friedman's self-embedding theorem)

Let $n > 0$, $\mathfrak{A}$ be a countable non-standard model of $\mathrm{I}\Sigma_n$, and take $a \in A$ arbitrarily. Then there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ such that $a \in A'$ but $A' \subsetneq A$, and $\mathfrak{A} \cong \mathfrak{A}'$ and for any Π_{n-1} formula $\varphi(\vec{x})$ and any $\vec{a'} \in A'^{<\omega}$,

$$\mathfrak{A}'_{A'} \models \varphi(\vec{a'}) \Leftrightarrow \mathfrak{A}_{A'} \models \varphi(\vec{a'}).$$

Proof.

- In last lemma, we consider the case $\mathfrak{A} = \mathfrak{B}$. In order to satisfy the condition (2) of the last lemma, for any Π_{n-1} formula $\varphi(\vec{v}, u)$, it is sufficient to find c such that

$$\mathfrak{A}_{\{a\}} \models \exists \vec{v} \varphi(\vec{v}, a) \Rightarrow \mathfrak{A}_{\{a,c\}} \models \exists \vec{v} < c\, \varphi(\vec{v}, a).$$

- Now, let

$$\Phi(x) = \{\exists \vec{v} \varphi(\vec{v}, a) \to \exists \vec{v} < x\, \varphi(\vec{v}, a) : \varphi(\vec{v}, u) \in \Pi_{n-1}\}.$$

  This is a recursive type consisting only of Π_n formulas, and is clearly finitely satisfiable.
- Therefore, there exists c that realizes $\Phi(x)$. Therefore, by the last lemma, there exists an initial segment $\mathfrak{A}'$ of $\mathfrak{A}$ which satisfies the conditions of the theorem. □

Remarks

- The essence of this theorem is that a countable non-standard model of $\mathrm{I}\Sigma_1$ has an initial segment that is isomorphic to itself.
- Friedman first proved this theorem for a countable non-standard model of Peano arithmetic, and several researchers sophisticated it to the above form.
- The same theorem does not hold for non-countable models, and also it does not hold in general for countable non-standard models of $\mathrm{I}\Sigma_0$.
- Furthermore, an important result related to this is McAloon's theorem, which states that a countable non-standard model of $\mathrm{I}\Sigma_0$ has an initial segment that is a model of Peano arithmetic PA.

§4. Introduction to Resplendency

- Recursive saturation of a structure means that it contains many "elements" that satisfy recursive conditions, but by generalizing this property to relations and functions, we introduce a new concept.
- By saying that a structure $\mathfrak{A}$ in the language $\mathcal{L}$ has "resplendency", we mean that if a formula $\varphi(\vec{\mathrm{R}})$ with new relation symbols $\vec{\mathrm{R}} \notin \mathcal{L}$ consistent with $\mathrm{Th}(\mathfrak{A}_A)$, $\varphi(\vec{\mathrm{R}})$ can hold in $\mathfrak{A}$ by appropriate interpretation of $\vec{\mathrm{R}}$.
- In a resplendent model of arithmetic, hidden properties of the structure can be found by using new relation symbols for an initial segment and satisfaction relation.

Definition 4.1

The $\mathcal{L}$-structure $\mathfrak{A}$ is said to be **resplendent**, if for a sentence φ in a language $\mathcal{L}^+ \supseteq \mathcal{L}_A$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ such that $\mathfrak{A}^+ \models \varphi$.

- The statement that $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is consistent is equivalent to that φ is true in the $\mathcal{L}^+$-extension of an elementary extension of $\mathfrak{A}$.

- In other words, resplendent structures are considered to potentially possess the properties of relations and functions manifested in their elementary extensions.

- We remark that if we denote the elements of A contained in φ (shown as constants) as $\vec{a}$, then this condition is equivalent to the consistency of $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \varphi$.

  $\because$ Suppose $\mathrm{Th}(\mathfrak{A}_A) \cup \{\varphi\}$ is inconsistent. Then there exists a formula $\psi(\vec{a}, \vec{b})$ in $\mathrm{Th}(\mathfrak{A}_A)$ such that $\vdash \psi(\vec{a}, \vec{b}) \to \neg\varphi$. Thus we also have $\vdash \exists y \psi(\vec{a}, \vec{y}) \to \neg\varphi$. Since $\exists y \psi(\vec{a}, \vec{y}) \in \mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}})$, it follows that $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \{\varphi\}$ is inconsistent. The reverse implication is trivial.

- Every finite structure is resplendent because its elementary extension is only itself.

Since "resplendency" does not imply "recursive saturation" in general, we introduce the following stronger notion which implies both.

Definition 4.2

An $\mathcal{L}$-structure $\mathfrak{A}$ is **strongly resplendent**, if for any recursive type $\Phi(\vec{x})$ in a language $\mathcal{L}^+ = \mathcal{L} \cup \{$finitely many additional symbols$\}$ and $\vec{a} \in A^{<\omega}$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent, there exists an $\mathcal{L}^+$-expansion $\mathfrak{A}^+$ of $\mathfrak{A}$ which is a model of $\Phi(\vec{a})$.

- In the definition of **strongly resplendent**, if we restrict the type $\Phi(\vec{x})$ to be a single formula, we obtain the definition of **resplendent**, and if we let $\mathcal{L}^+ = \mathcal{L} \cup \{c\}$, it becomes the definition of **recursive saturation**. Hence, strongly resplendent structures are both resplendent and recursively saturated.
- Furthermore, similar to the case of resplendent structures, it is worth noting that the consistency of $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ coincides with the consistency of $\mathrm{Th}(\mathfrak{A}_{\{\vec{a}\}}) \cup \Phi(\vec{a})$.

We will now demonstrate that under certain natural assumptions, the above three properties coincide.

Theorem 4.3 (Barwise-Ressayre)

Countable recursively saturated structures are strongly resplendent.

Proof

- Let $\mathfrak{A}$ be a countable structure in a countable language $\mathcal{L}$ and assume it is recursively saturated. Furthermore, suppose we are given a recursive type $\Phi(\vec{x})$ in a finitely extended language $\mathcal{L}^+$ of $\mathcal{L}$ and $\vec{a} \in A^\omega$ such that $\mathrm{Th}(\mathfrak{A}_A) \cup \Phi(\vec{a})$ is consistent.
- Then, we want to construct a model $\mathfrak{A}^+$ of this theory without expanding the domain $|\mathfrak{A}|$. The key idea of the construction is that by utilizing the recursively saturated nature of $\mathfrak{A}$, we can select Henkin constants from elements of A.

Now, let's look into the details of construction of $\mathfrak{A}^+$.

- First, we enumerate the formulas in $\mathcal{L}_A$ with only one free variable x, denoted by $\{\varphi_n(x) : n \in \omega\}$.

- We construct a sequence of finite subsets of A and that of recursive theories in $\mathcal{L}_A^+$,

$$A_0 = \{\vec{a}\} \subseteq A_1 \subseteq A_2 \subseteq \cdots, \quad T_0 = \Phi(\vec{a}) \subseteq T_1 \subseteq T_2 \subseteq \cdots,$$

satisfying the following conditions: for each n
(1) T_n is a recursive set of sentences in $\mathcal{L}_{A_n}^+$, and $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent.
(2) either $\varphi_n(a) \in T_{n+1}$ for some $a \in A$ or $\neg\exists x \varphi_n(x) \in T_{n+1}$.
- Once the construction is completed, letting $T_\omega = \bigcup_n T_n$, we will show T_ω is a complete Henkin theory.
- Let σ be a sentence in $\mathcal{L}_A^+$ such that $T_\omega \nvdash \sigma$. Suppose σ is φ_k (with no occurrence of x) for some k. Then we have $\sigma \notin T_{k+1}$, since $T_\omega \nvdash \sigma$. Thus, by condition (2), we have $\neg\exists x \sigma \in T_{k+1}$, and so $T_\omega \vdash \neg\sigma$. Therefore, T_ω is complete, and so $\mathrm{Th}(\mathfrak{A}_A) \subseteq T_\omega$ since $T_\omega \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent by condition (1).
- If $T_\omega \vdash \exists x \varphi_n(x, \vec{a})$, then by (2), there exists some $a \in A$ such that $\varphi_n(a) \in T_\omega$.
- Then T_ω is a complete Henkin theory. By Henkin method, we can construct a structure $\mathfrak{A}^+$ over the domain A, such that $T_\omega = \mathrm{Th}(\mathfrak{A}_A^+)$, and therefore $\mathfrak{A}^+ \models \Phi(\vec{a})$.

Finally, we will construct the sequences $\{A_n\}$ and $\{T_n\}$ by induction.

- Assuming that the constructions up to A_n and T_n have been done. Take $\varphi_n(x)$.
- Let $B = A_n \cup \{$elements of A occurring in $\varphi_n(x)\}$, and define

$$\Psi(x) = \{\psi(x) : \psi(x) \text{ is a one-variable formula in } \mathcal{L}_B, \text{ and } T_n \vdash \varphi_n(x) \to \psi(x)\}.$$

- Although $\Psi(x)$ is Σ_1 as it is, it can be treated as a recursive type by Craig's method.
- Since the structure $\mathfrak{A}$ is recursively saturated, we can either find an $a \in A$ realizing $\Psi(x)$ or find a finite subset $\{\psi_i(x) : i \leq j\}$ of $\Psi(x)$ such that

$$\mathfrak{A}_A \models \neg\exists x \bigwedge_{i \leq j} \psi_i(x).$$

- In the former case, we let $A_{n+1} = B \cup \{a\}, \quad T_{n+1} = T_n \cup \{\varphi_n(a)\}$.
- To check the consistency of $T_{n+1} \cup \mathrm{Th}(\mathfrak{A}_A)$, we will show that any $\mathcal{L}_{A_{n+1}}$ sentence provable in T_{n+1} is true in $\mathfrak{A}_A$. Now, let $\psi(x)$ be a formula in $\mathcal{L}_B$ and assume $T_{n+1} \vdash \psi(a)$. If $a \notin B$, $T_n \vdash \varphi_n(a) \to \psi(a)$ implies $T_n \vdash \varphi_n(x) \to \psi(x)$ and so $\psi(x) \in \Psi(x)$. Since a realizes $\Psi(x)$, $\psi(a)$ holds in $\mathfrak{A}_A$. On the other hand, if $a \in B$, then by $T_n \vdash \varphi_n(x) \to (x = a \to \psi(x))$, we get $(x = a \to \psi(x)) \in \Psi(x)$, which implies $(a = a \to \psi(a)) \in \mathrm{Th}(\mathfrak{A}_A)$. Thus, $\psi(a)$ holds in $\mathfrak{A}_A$.

- Next, we consider the case that $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\leq j} \psi_i(x)$. In this case, we can simply set

$$A_{n+1} = A_n, \quad T_{n+1} = T_n \cup \{\neg\exists x \varphi_n(x)\}.$$

- Since $T_n \vdash \neg\exists x \bigwedge_{i\leq j} \psi_i(x) \to \neg\exists x \varphi_n(x)$, we may show the consistency of

$$T_n \cup \{\neg\exists x \bigwedge_{i\leq j} \psi_i(x)\} \cup \mathrm{Th}(\mathfrak{A}_A).$$

- Let ψ be a sentence in $\mathcal{L}_B$ such that $T_n \vdash \neg\exists x \bigwedge_{i\leq j} \psi_i(x) \to \psi$. By the induction hypothesis, $T_n \cup \mathrm{Th}(\mathfrak{A}_A)$ is consistent, so $\neg\exists x \bigwedge_{i\leq j} \psi_i(x) \to \psi$ holds in $\mathfrak{A}_A$.
- Moreover, since we have the premise $\mathfrak{A}_A \models \neg\exists x \bigwedge_{i\leq j} \psi_i(x)$, it follows that ψ also holds in $\mathfrak{A}_A$. This completes the proof. □

Recall Problem 5 of Lec05-02

Let $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ be a non-standard model of $\mathrm{I}\Sigma_1$. Show that $\mathfrak{A}' = (A, +, 0, 1, <)$ is recursively saturated.

Example 5

- In the above problem 5, it was shown that if $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ is a nonstandard model of $\mathrm{I}\Sigma_1$, then $\mathfrak{A}' = (A, +, 0, 1, <)$ becomes recursively saturated.
- Conversely, suppose $\mathfrak{A}' = (A, +, 0, 1, <)$ is a recursively saturated model of Presburger arithmetic and is countable. Then, by the previous theorem, $\mathfrak{A}'$ is strongly resplendent.
- On the other hand, Presburger arithmetic is complete, and the set of its theorems coincides with $\mathrm{Th}(\mathfrak{A}')$. Therefore, $\mathrm{Th}(\mathfrak{A}') \cup$ PA is nothing but PA, which is a recursive consistent set.
- Hence, there exists a suitable interpretation of $\bullet$ such that $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ becomes a model of PA. In summary, a countable model $\mathfrak{A} = (A, +, \bullet, 0, 1, <)$ of $\mathrm{I}\Sigma_1$ can be turned into a model $\mathfrak{A}' = (A, +, \bullet', 0, 1, <)$ of PA by changing the interpretation of multiplication (the "misbuttoning theorem").

Thank you for your attention!